

# Temas de Computação Visual

## Modelagem Geométrica

Luiz Henrique de Figueiredo



## Formas geométricas em mundos virtuais



PIXAR  
PRESENTS

# Geris's game

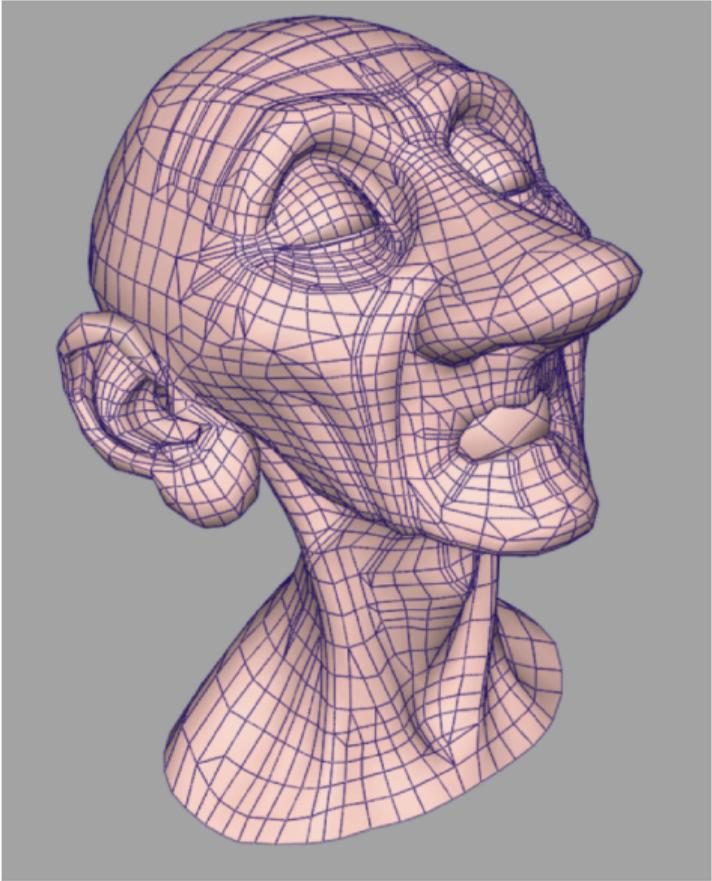
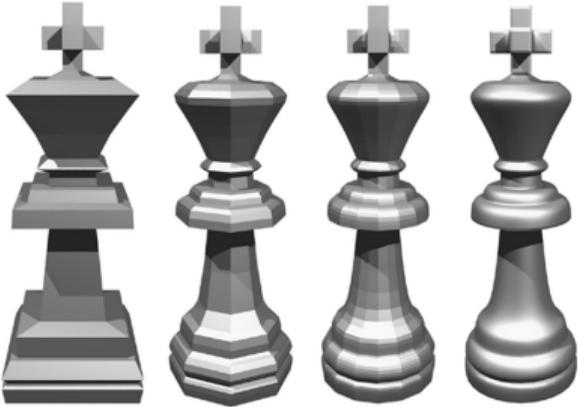


1997

## Formas geométricas em mundos virtuais

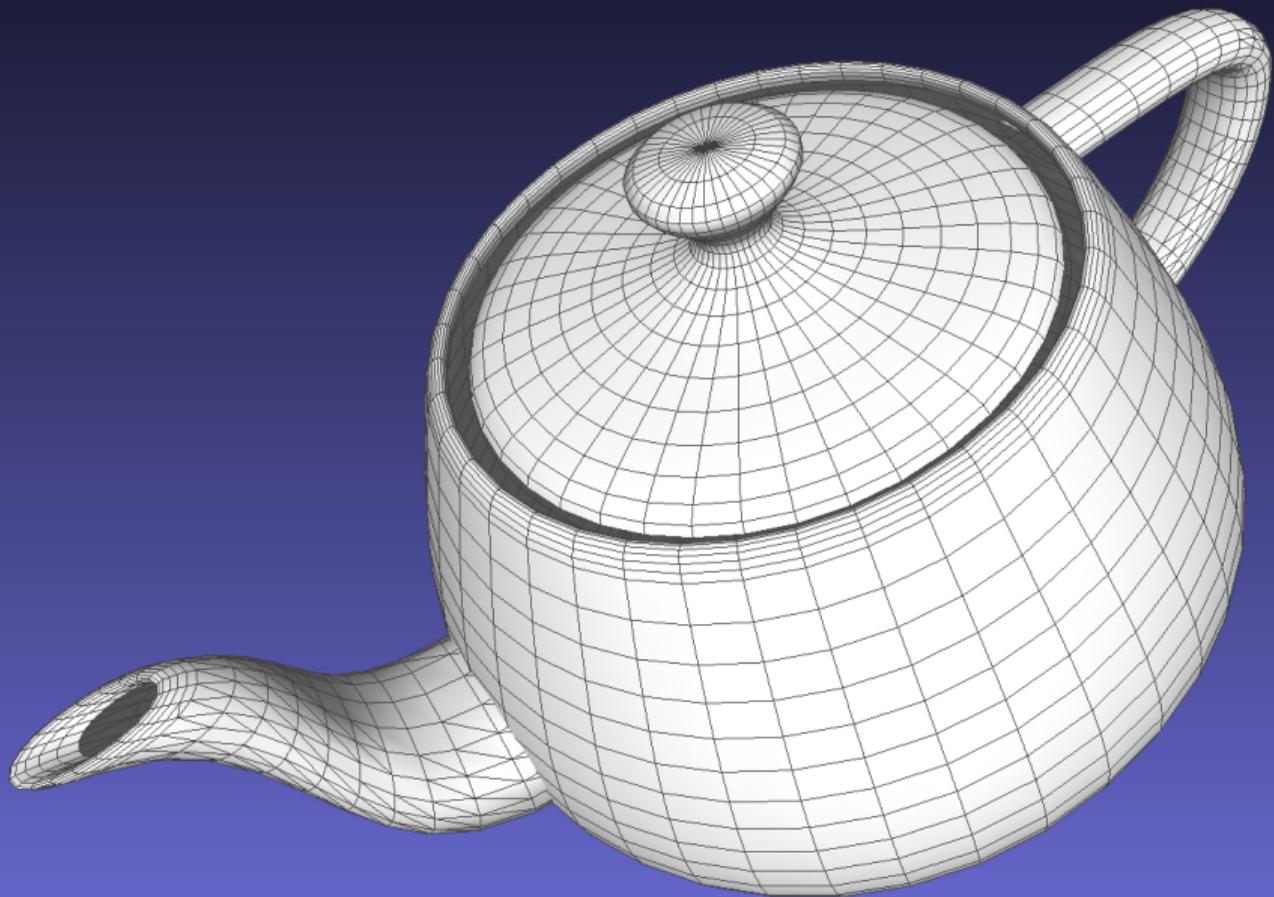


# Anatomia das formas virtuais

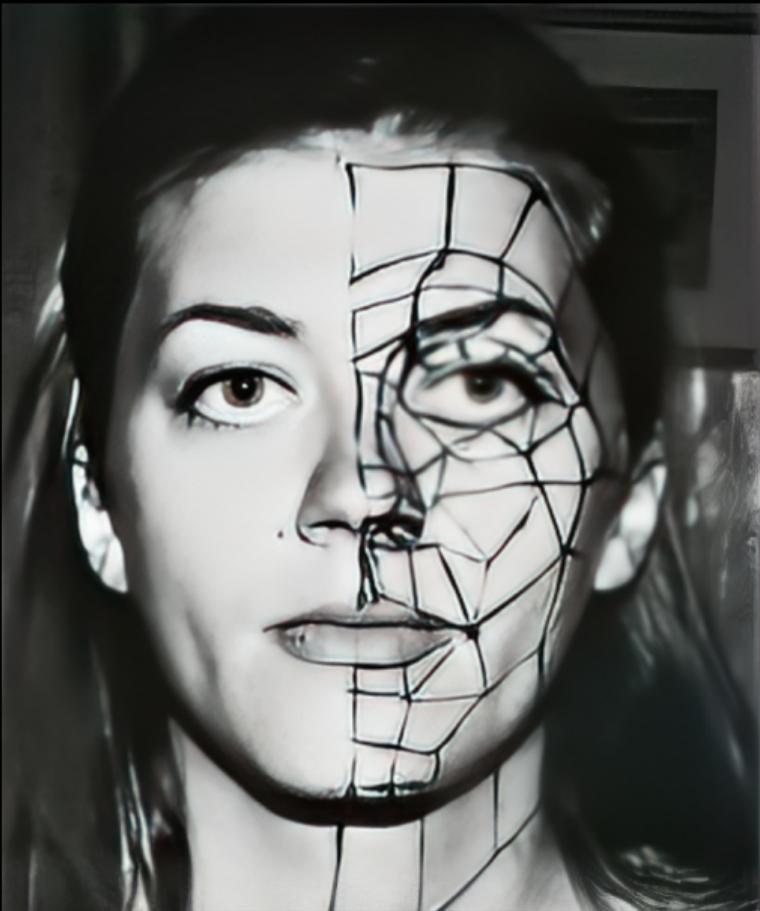




Newell (1975) / Morrical (2019)







Anatomia das formas virtuais

abc

Anatomia das formas virtuais



Anatomia das formas virtuais



fundamentos

Jonas Gomes  
Luiz Velho

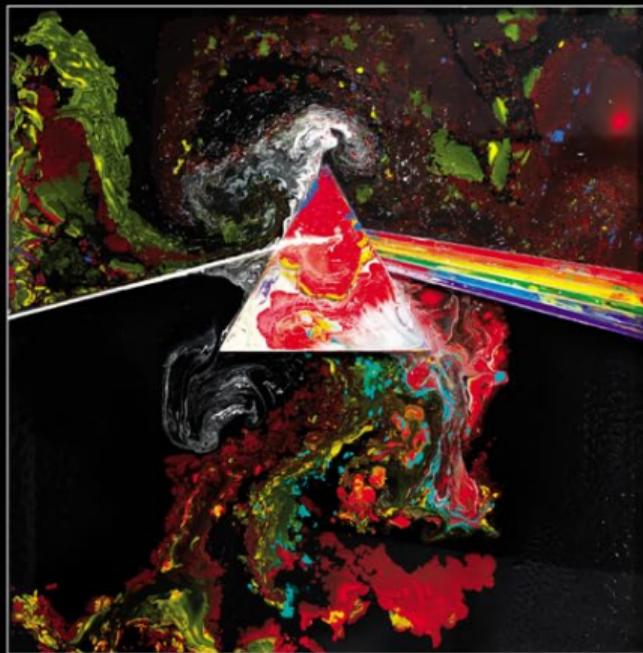
Fundamentos da Computação Gráfica



Instituto Nacional de Matemática Pura e Aplicada - IMPA  
Série de Computação e Matemática

2003

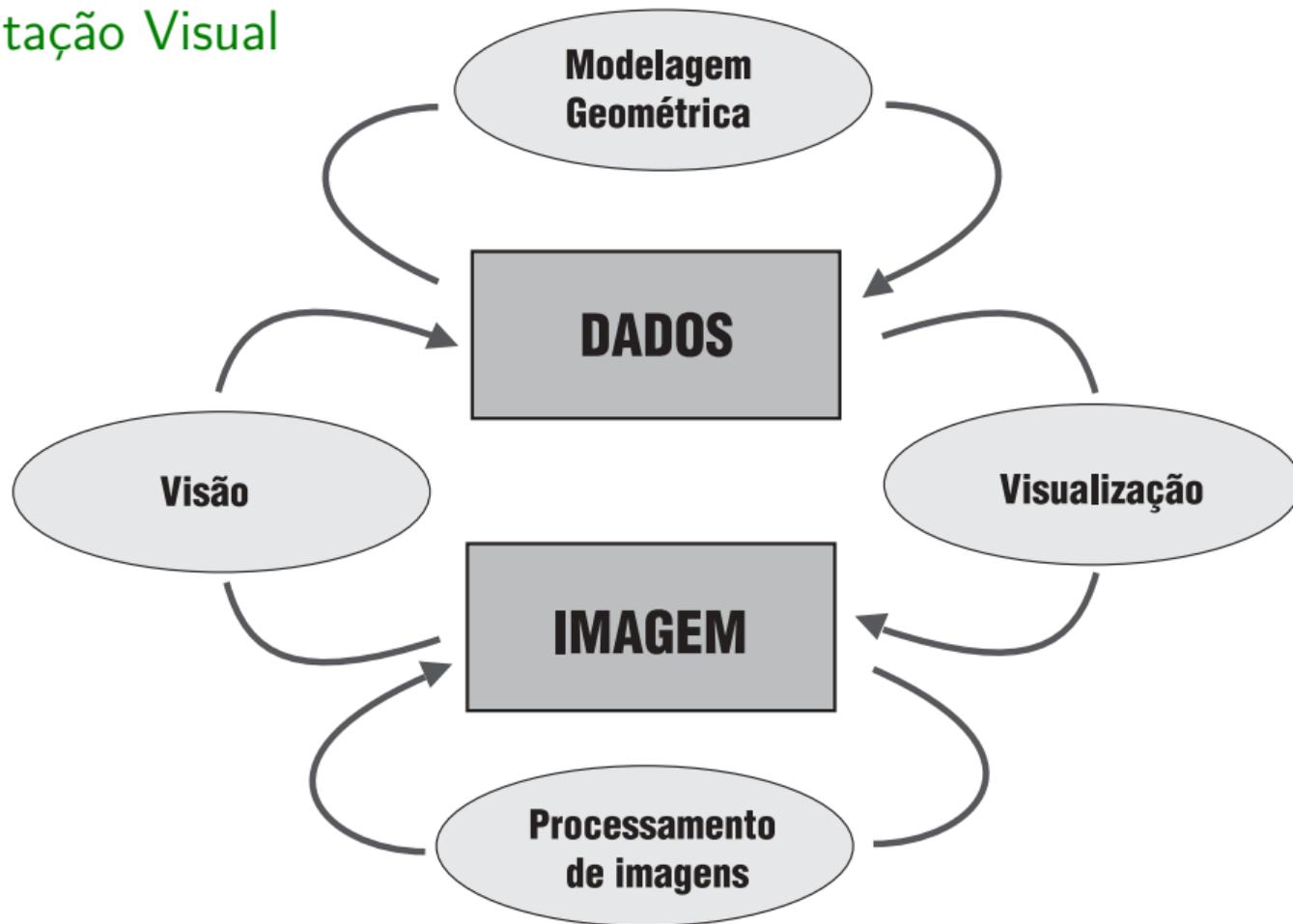
COMPUTER GRAPHICS  
THEORY AND PRACTICE

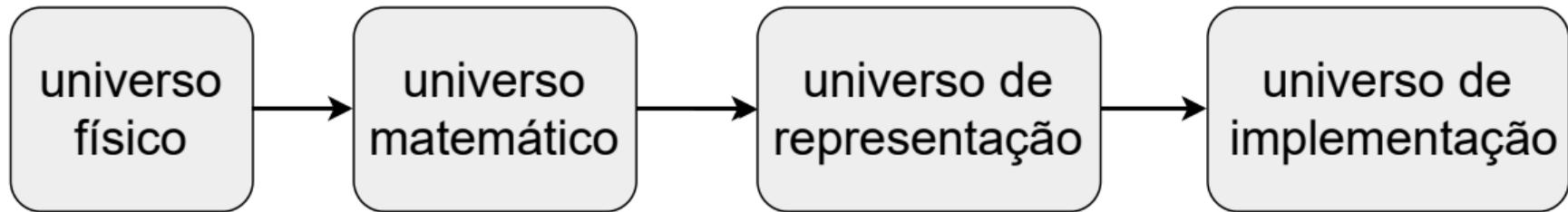


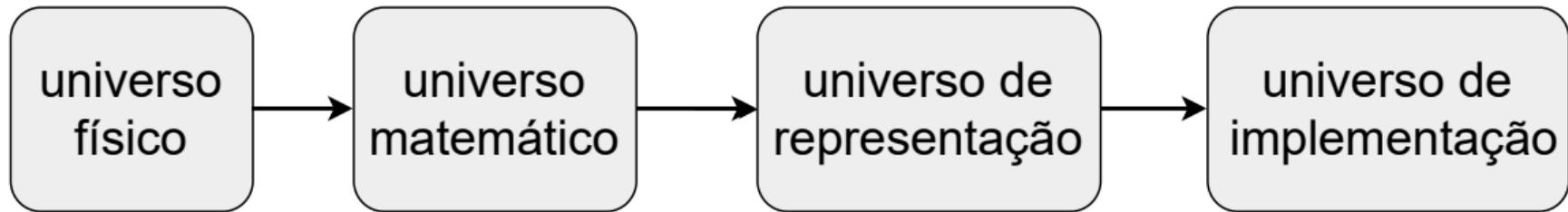
Jonas Gomes • Luiz Velho • Mario Costa Sousa

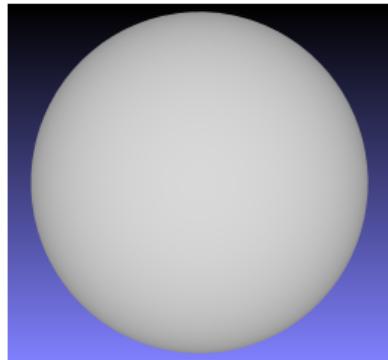
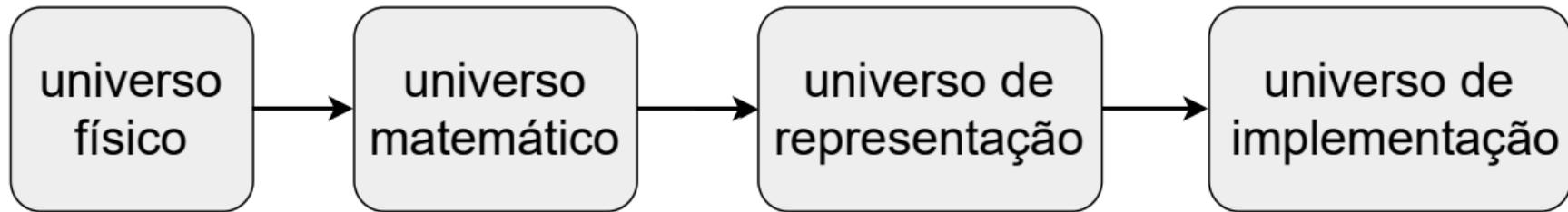
2012

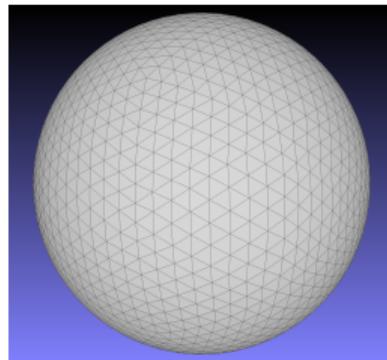
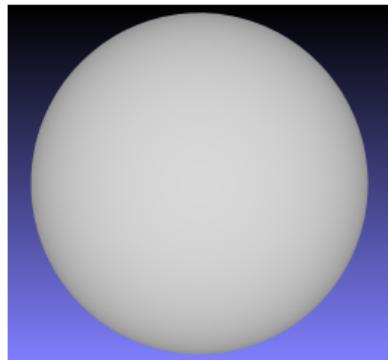
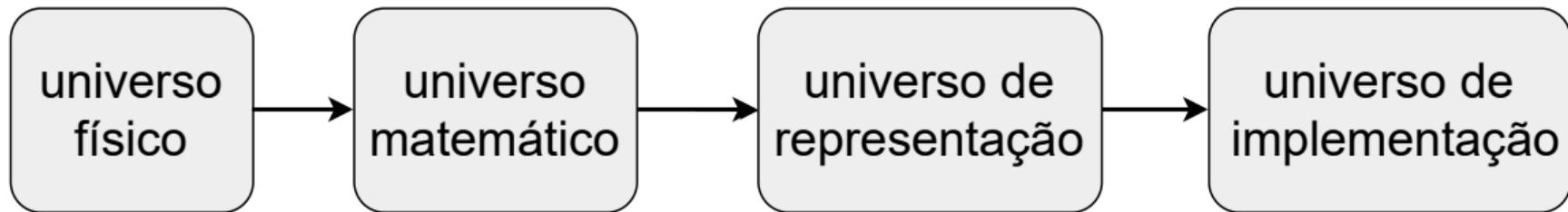
# Computação Visual

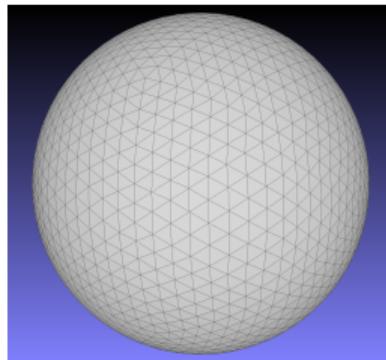
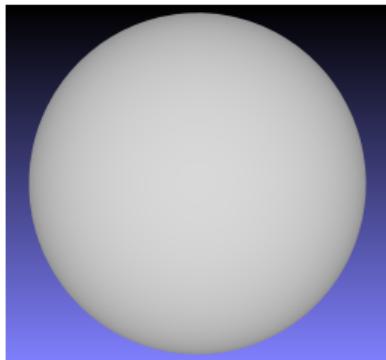
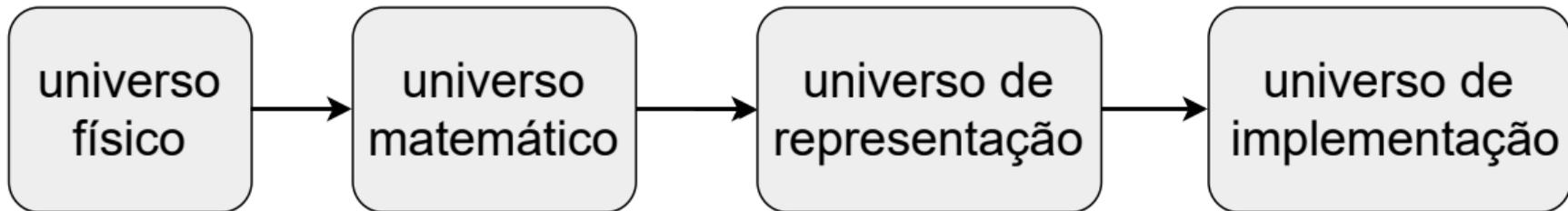










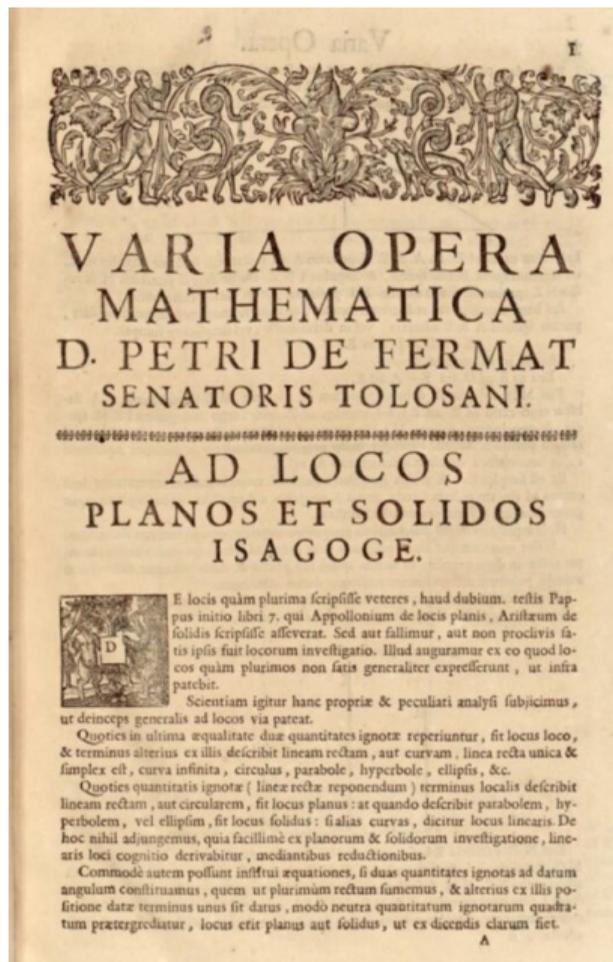


```
v 0.525731 0 -0.850651
v 0.525731 0 0.850651
v -0.525731 0 0.850651
v -0.850651 -0.525731 0
v -0.850651 0.525731 0
v 0.850651 0.525731 0
v 0.850651 -0.525731 0
v 0 -0.850651 0.525731
v 0 -0.850651 -0.525731
v 0 0.850651 -0.525731
...
f 23 24 25
f 25 24 20
f 25 21 23
f 22 24 23
f 37 45 23
f 17 45 37
f 23 21 37
f 23 45 22
f 31 24 47
f 20 24 31
```

universo matemático

# Geometria analítica

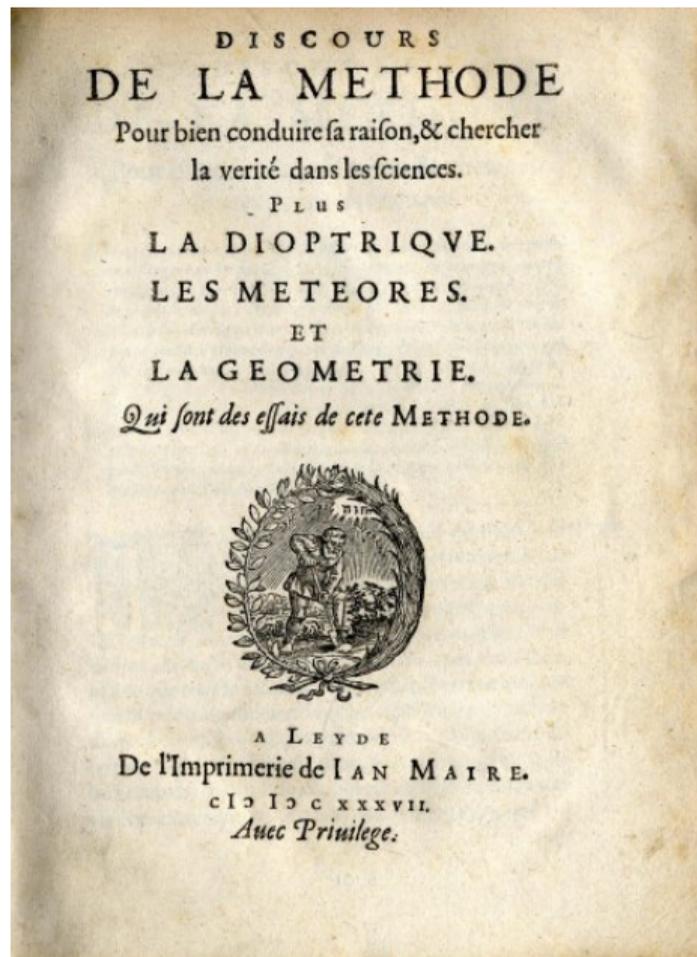
Fermat (1636, 1679)



## Geometria analítica

Fermat (1636, 1679)

Descartes (1637)



# Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

L A

## G E O M E T R I E.

LIVRE PREMIER.

*Des problemes qu'on peut construire sans  
y employer que des cercles & des  
lignes droites.*



Ou s les Problemes de Geometrie se  
peuvent facilement reduire a tels termes,  
qu'il n'est befoin par après que de connoi-  
stre la longueur de quelques lignes droites,  
pour les construire.

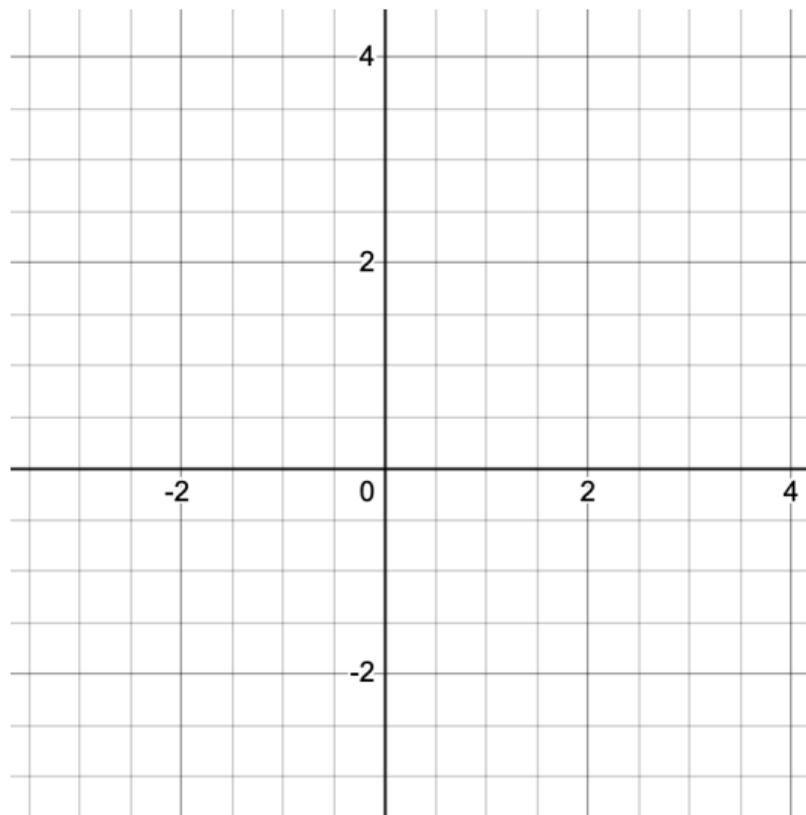
Et comme toute l'Arithmetique n'est composée, que Comme  
de quatre ou cinq operations, qui sont l'Addition, la le calcul  
Soustraction, la Multiplication, la Division, & l'Extra- d'Arithmeti-  
ction des racines, qu'on peut prendre pour vne espece que se  
de Division : Ainsi n'at'on autre chose a faire en Geo- rappelle  
metrie touchant les lignes qu'on cherche, pour les pre- aux ope-  
parer a estre connus, que leur en adiouster d'autres, ou rations de  
en oster, Oubien en ayant vne, que ie nommeray l'vnité Geome-  
pour la rapporter d'autant mieux aux nombres, & qui trie.  
peut ordinairement estre prise a discretion, puis en ayant  
encore deux autres, en trouuer vne quatriesme, qui soit  
à l'vne de ces deux, comme l'autre est à l'vnité, ce qui est  
le mesme que la Multiplication, oubien en trouuer vne  
quatriesme, qui soit à l'vne de ces deux, comme l'vnité

# Geometría analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas	álgebra	+	geometría
	fórmulas	$\longleftrightarrow$	formas
	finito	$\longleftrightarrow$	infinito

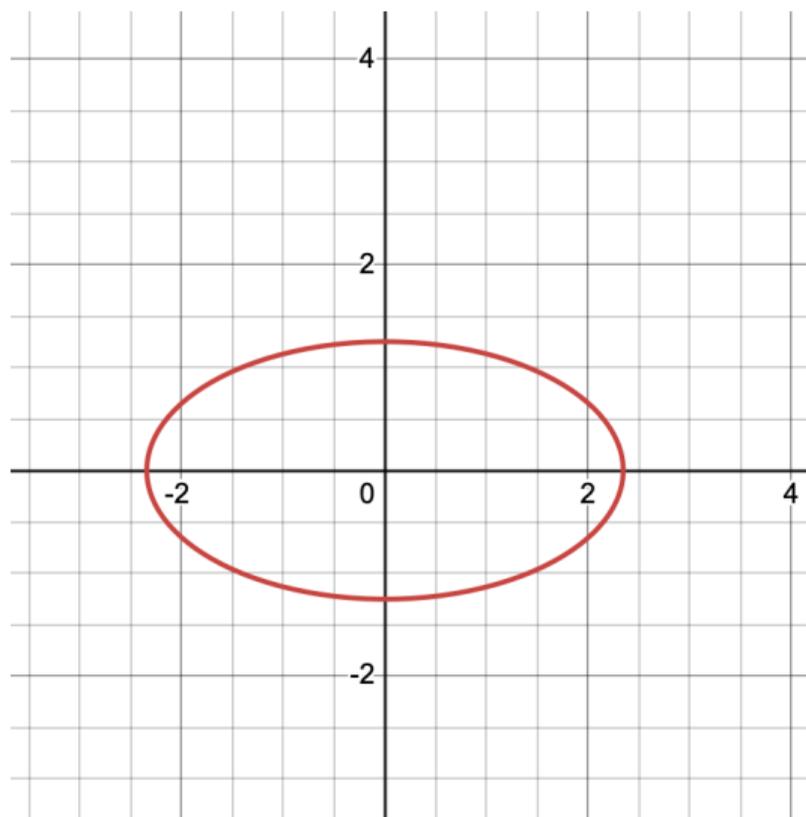


# Geometría analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas	álgebra	+	geometría
	fórmulas	↔	formas
	finito	↔	infinito



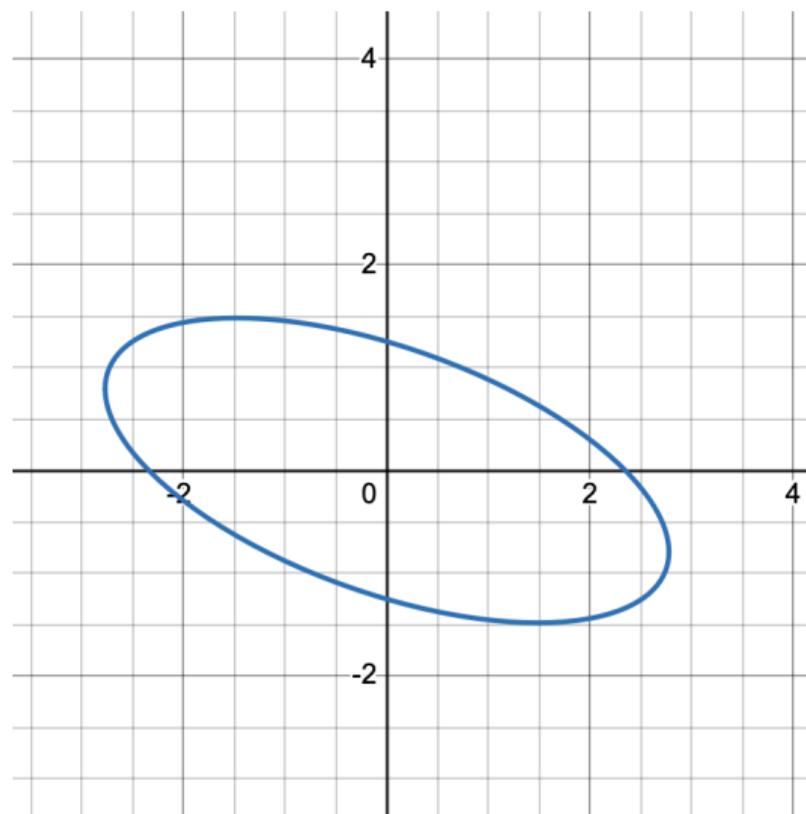
$$2x^2 + 7y^2 = 11$$

# Geometría analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas    álgebra    +    geometría  
fórmulas     $\longleftrightarrow$     formas  
finito     $\longleftrightarrow$     infinito



$$2x^2 + 4xy + 7y^2 = 11$$

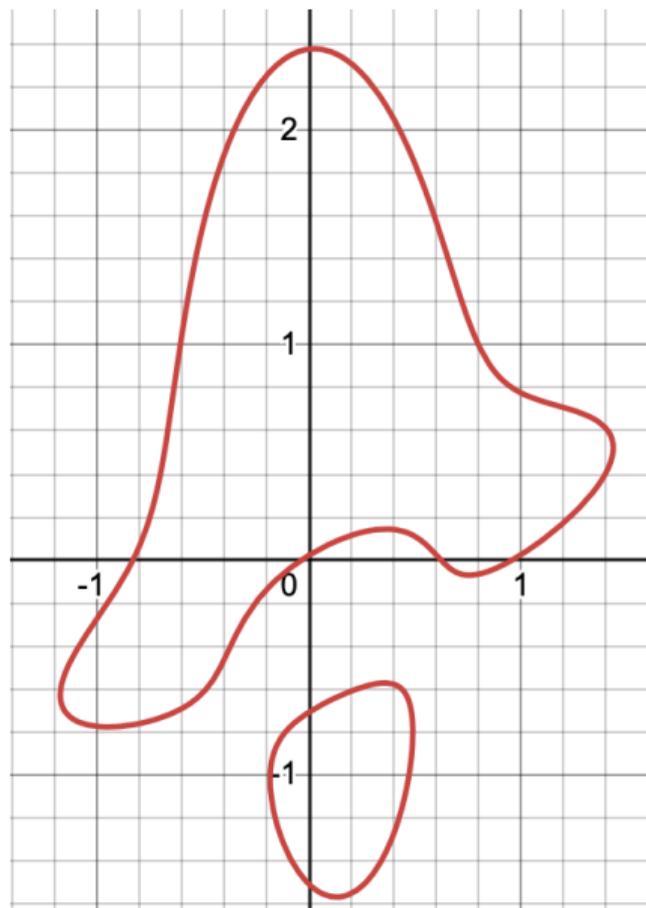
# Geometría analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas    álgebra    +    geometría  
                  fórmulas     $\longleftrightarrow$     formas  
                  finito         $\longleftrightarrow$     infinito

$$0.004 + 0.110x - 0.177y - 0.174x^2 + 0.224xy - 0.303y^2 \\ - 0.168x^3 + 0.327x^2y - 0.087xy^2 - 0.013y^3 + 0.235x^4 \\ - 0.667x^3y + 0.745x^2y^2 - 0.029xy^3 + 0.072y^4 = 0$$

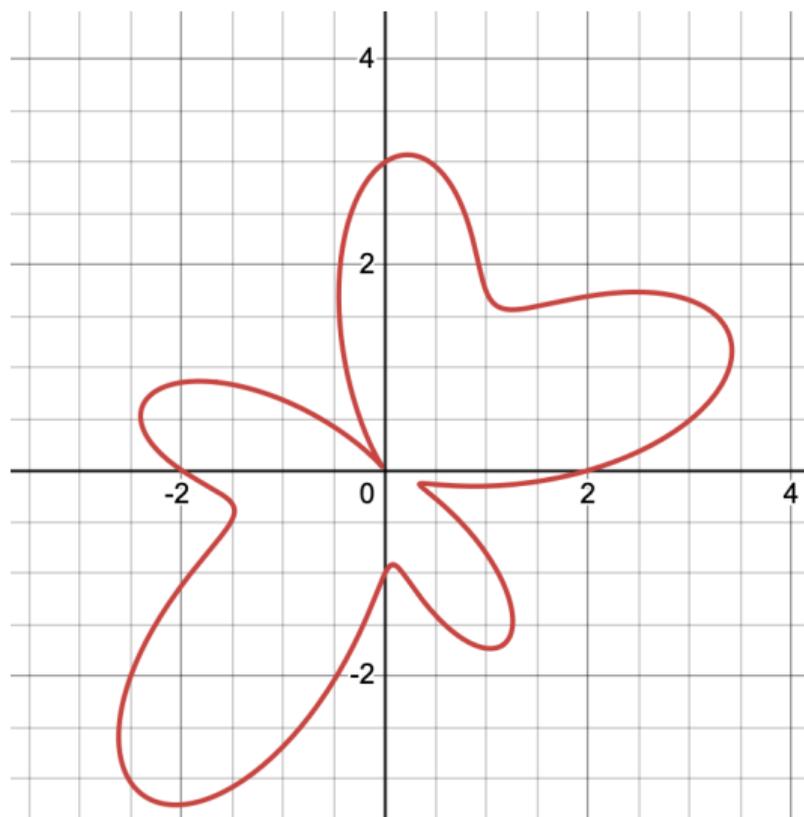


# Geometría analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas    álgebra    +    geometría  
fórmulas     $\longleftrightarrow$     formas  
finito     $\longleftrightarrow$     infinito



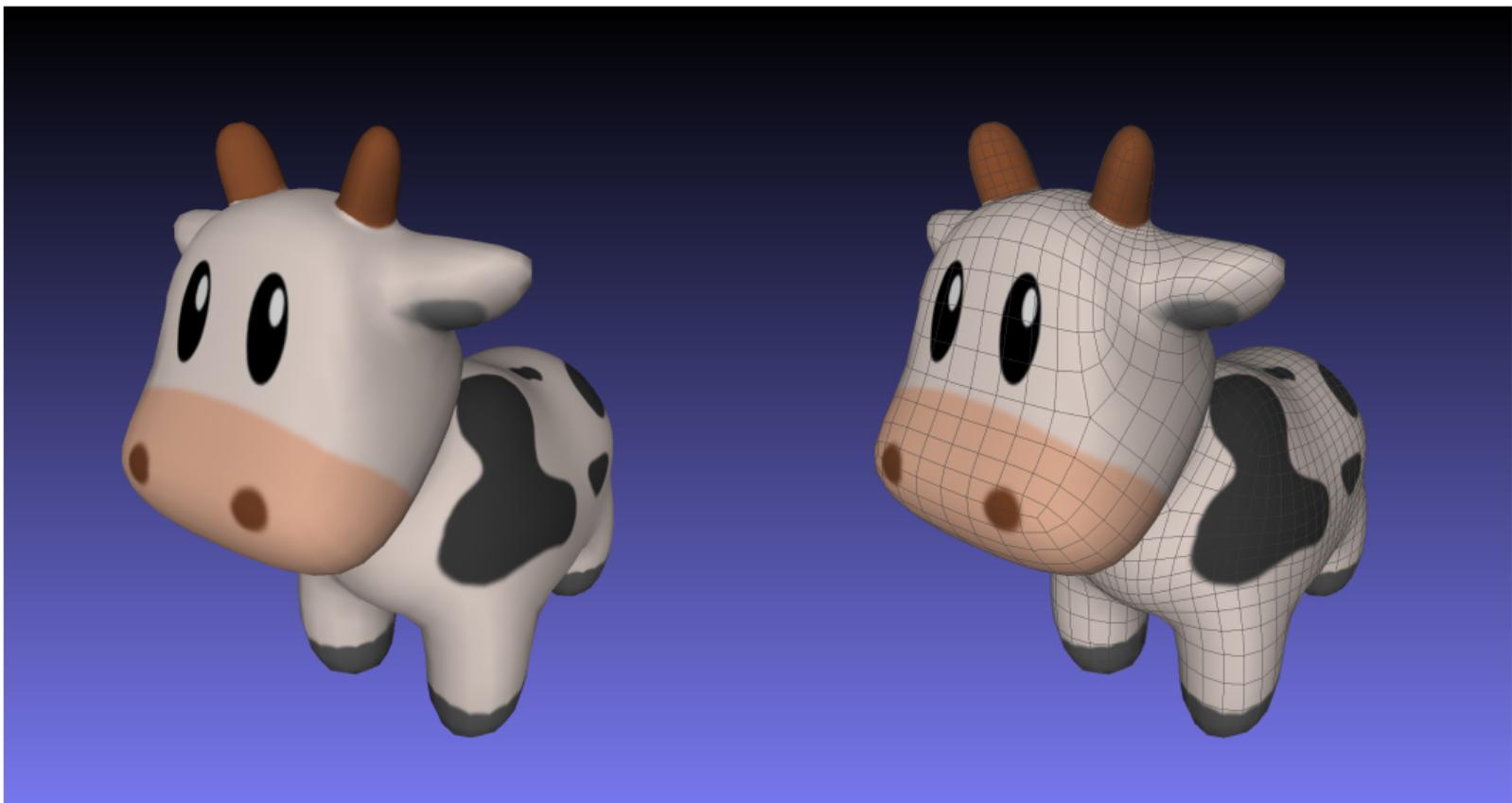
$$r = \sin(2\theta) + \sin(5\theta) + 2 \quad \begin{aligned} x &= r \cos(\theta) \\ y &= r \sin(\theta) \end{aligned}$$

# Geometria analítica



Keenan Crane

# Geometria analítica

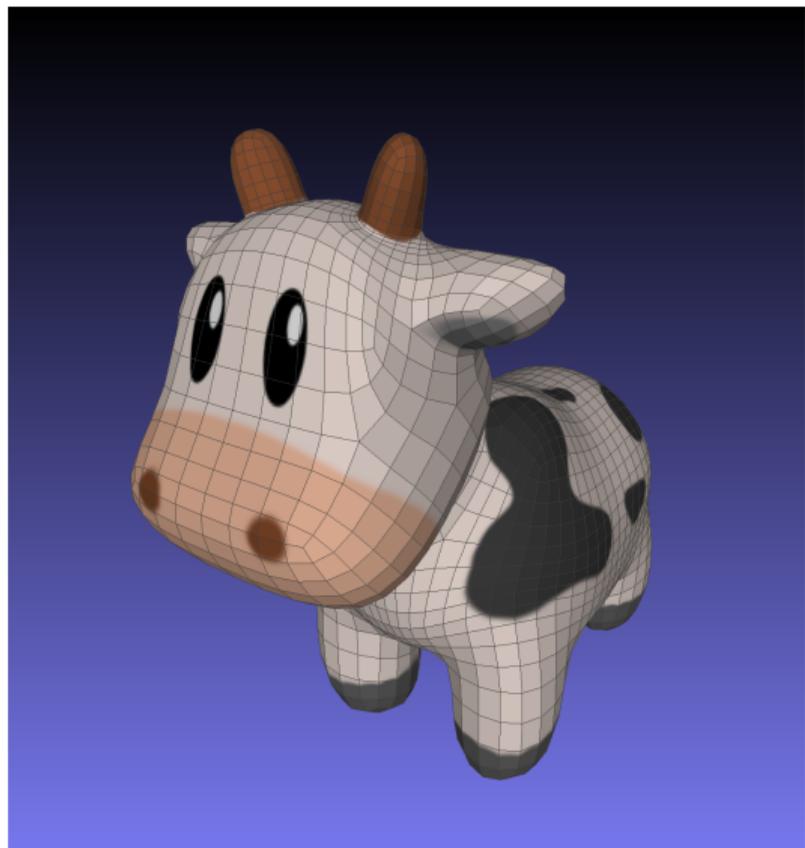


# Geometria analítica



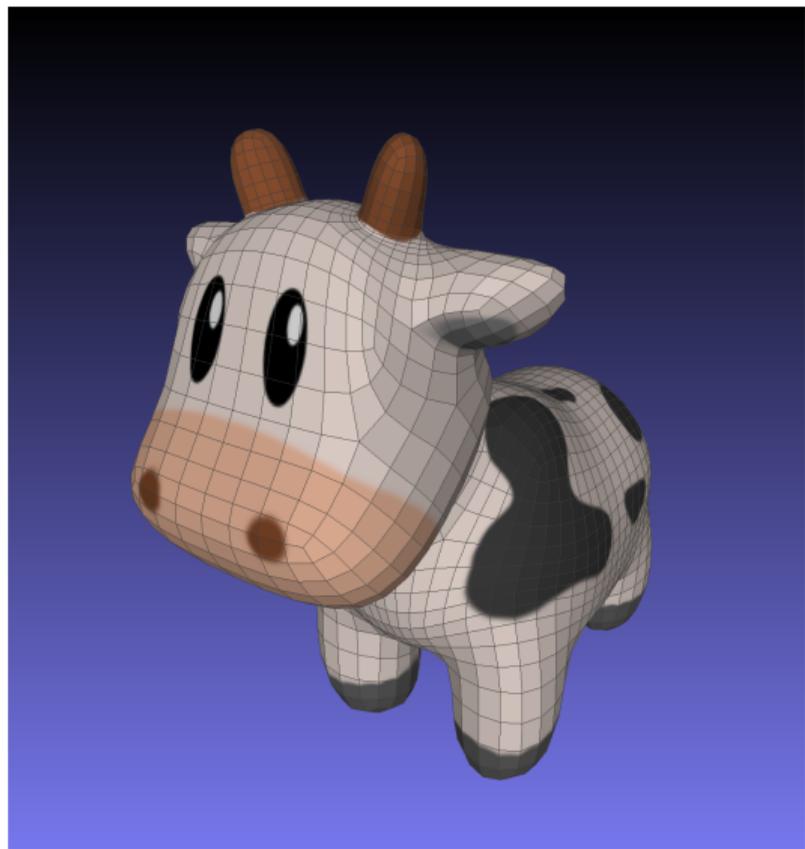
## Geometria analítica

```
v 0.348799 -0.334989 -0.0832331  
v 0.313132 -0.399051 0.881192  
v 0.266758 0.181628 0.122726  
v 0.229555 0.0663178 0.828702  
v 0.353623 -0.0486456 0.443969  
v 0.335878 -0.384495 0.425693  
v 0.305362 0.0307983 -0.00655663  
v 0.279611 -0.0552387 0.858547  
v 0.266719 0.10578 0.46681  
v 0.149341 -0.451522 0.166423  
v 0.12606 -0.162036 -0.185668  
v 0.114009 -0.358339 -0.0759521  
v 0.101546 -0.475382 0.663157  
v 0.240757 -0.435635 0.421981  
v 0.385825 -0.174501 0.180766
```

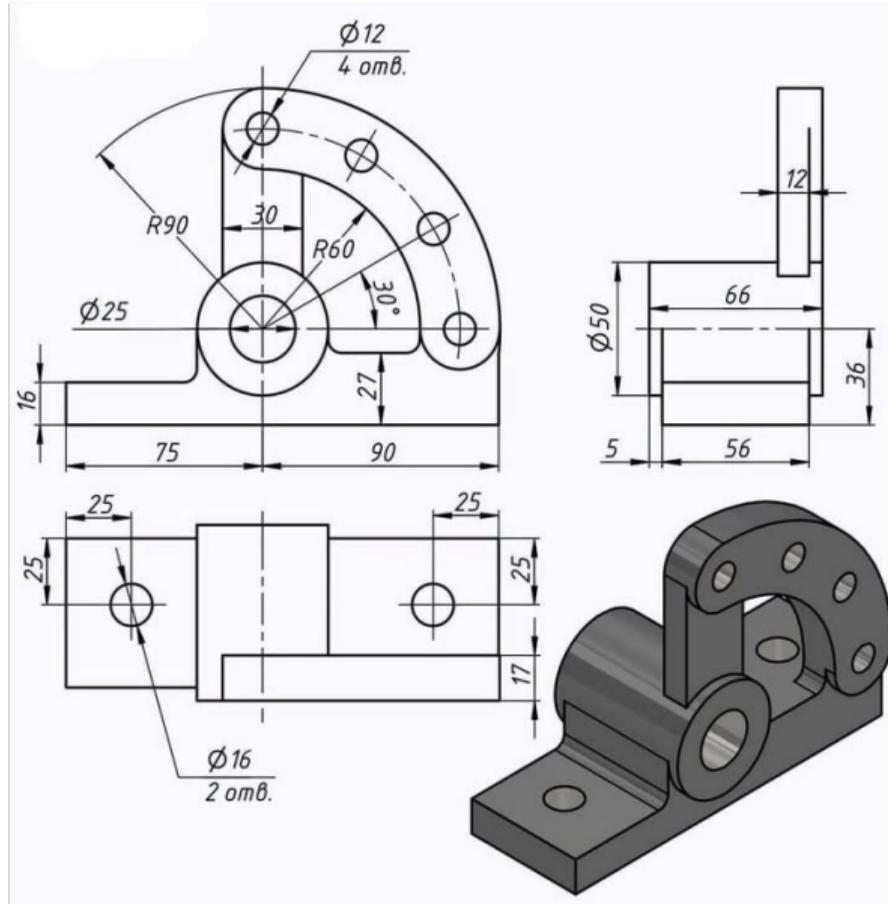


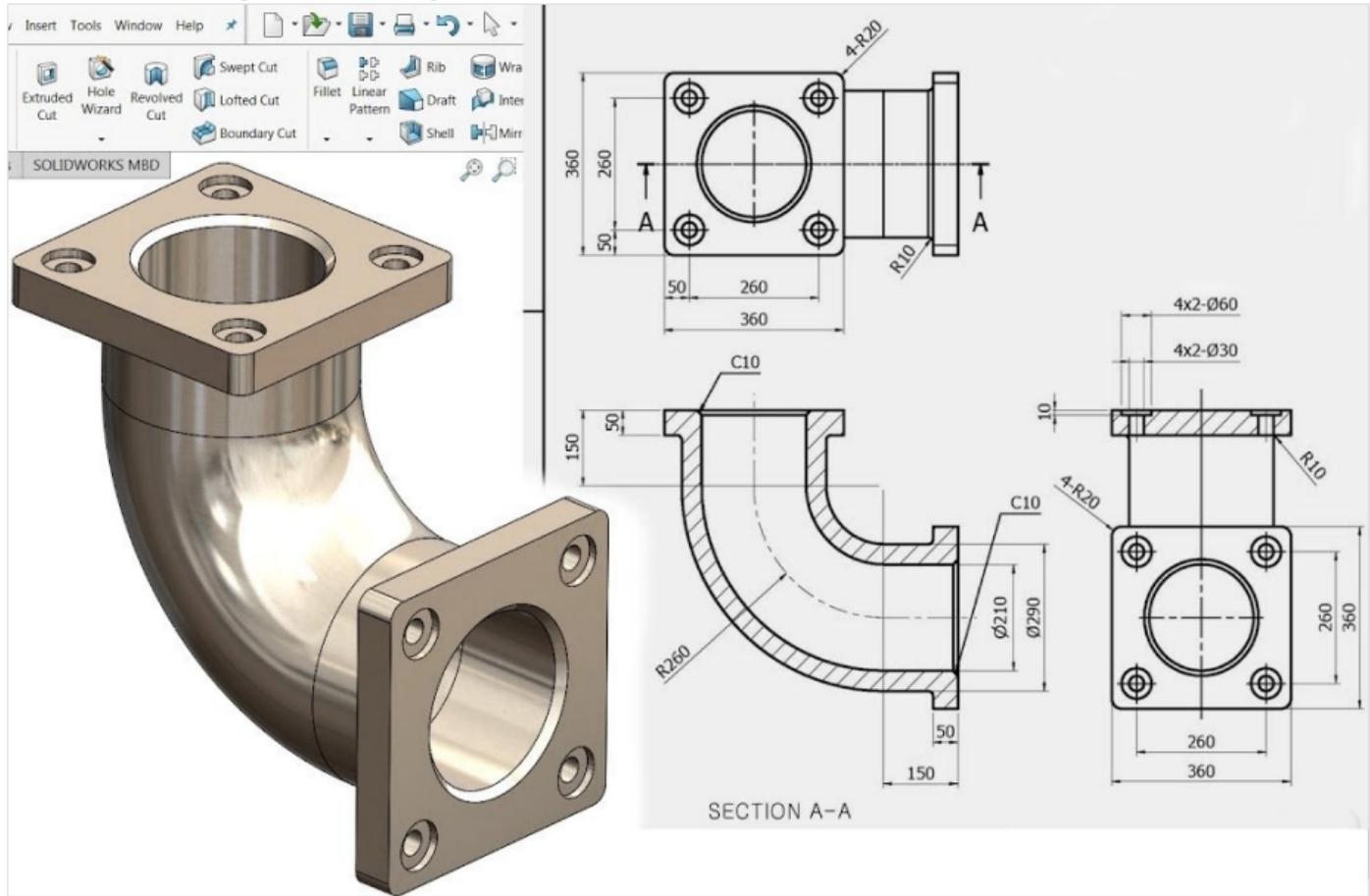
## Geometria analítica

```
f 6 735 739 738  
f 189 736 739 735  
f 193 737 739 736  
f 192 738 739 737  
f 14 740 743 742  
f 190 741 743 740  
f 193 736 743 741  
f 189 742 743 736  
f 10 744 747 746  
f 191 745 747 744  
f 193 741 747 745  
f 190 746 747 741  
f 16 748 750 749  
f 192 737 750 748  
f 193 745 750 737
```



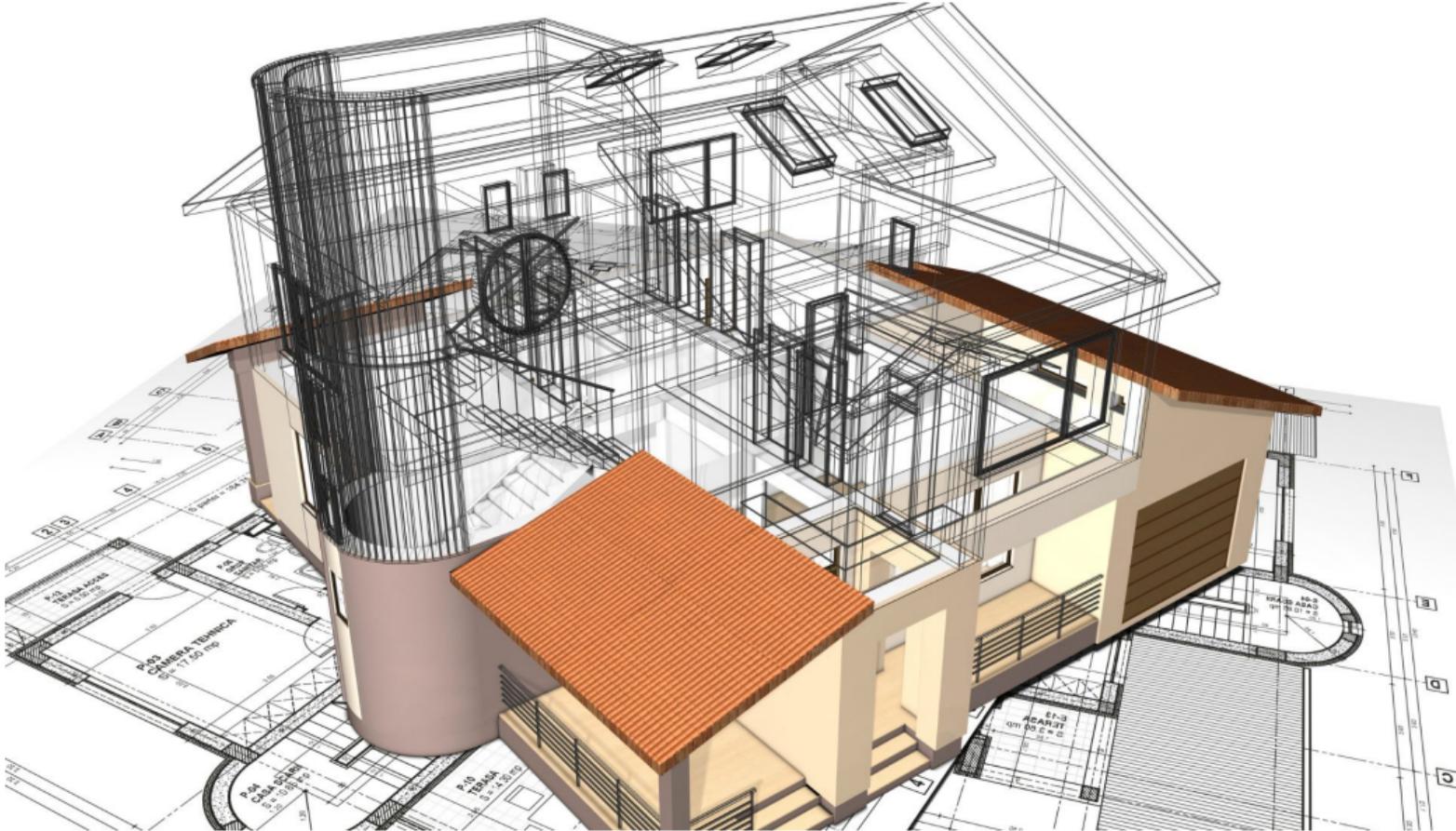
métodos de modelagem

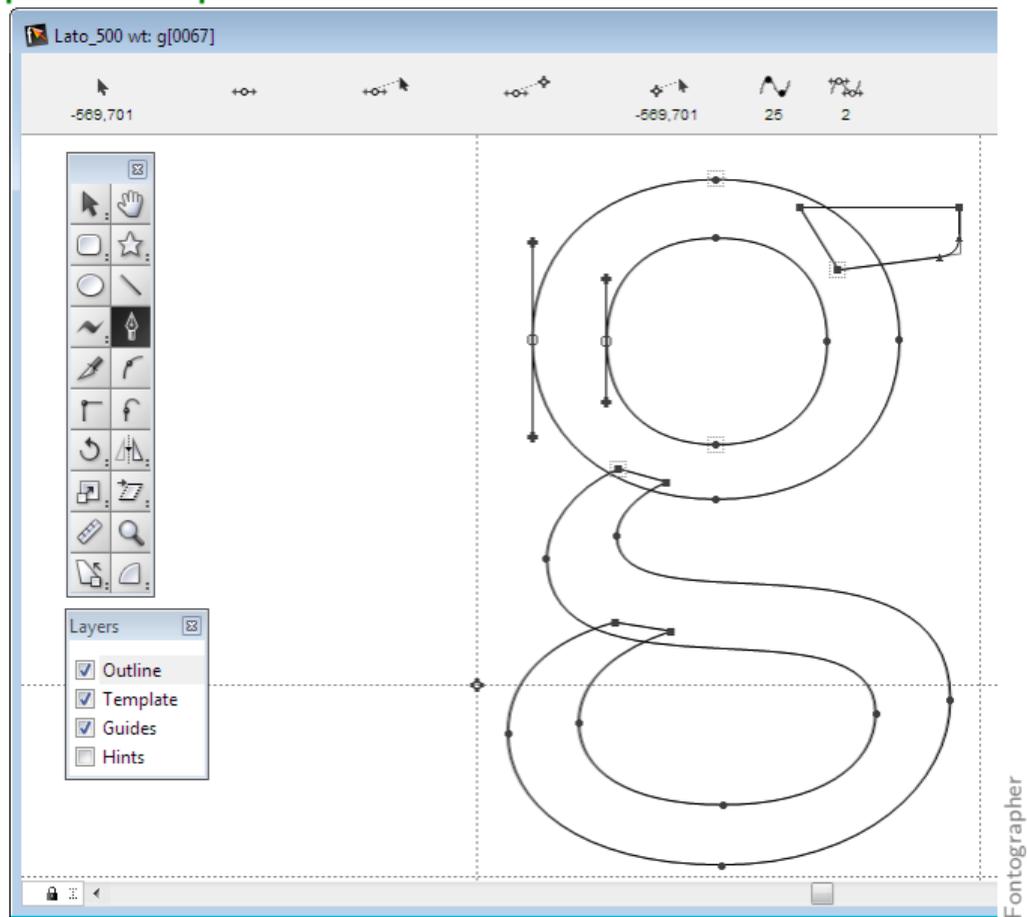


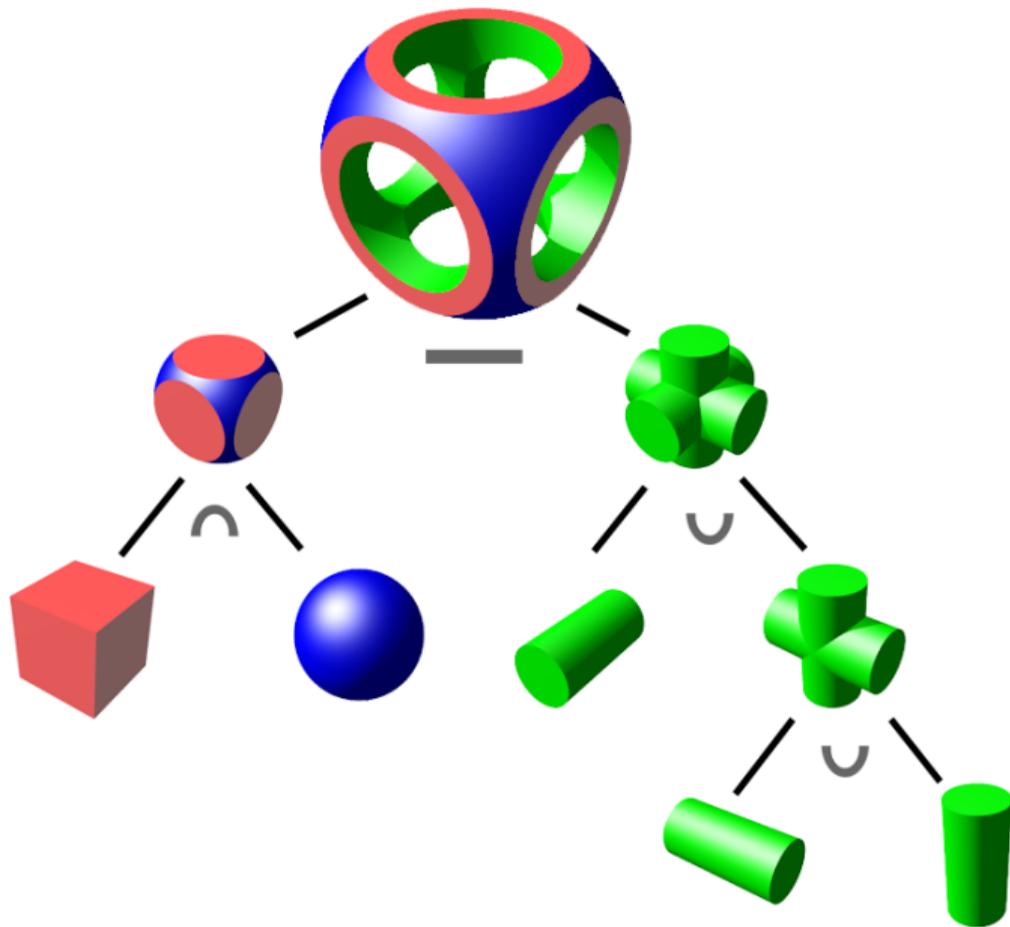


# Projeto assistido por computador

CAD

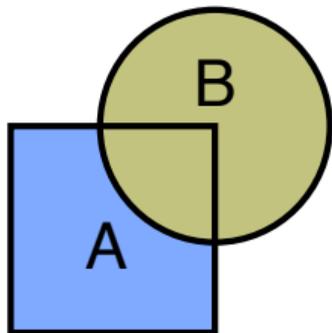




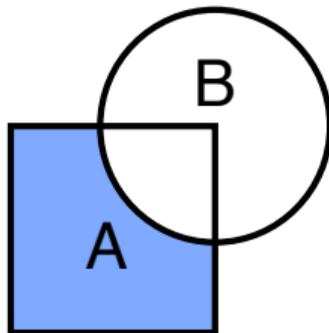


# Operações booleanas

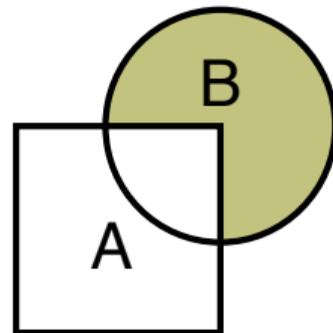
A e B



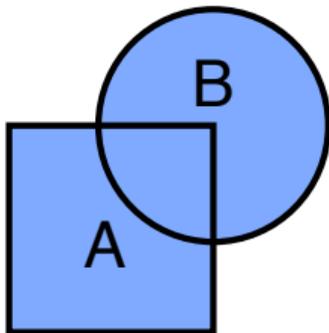
A menos B



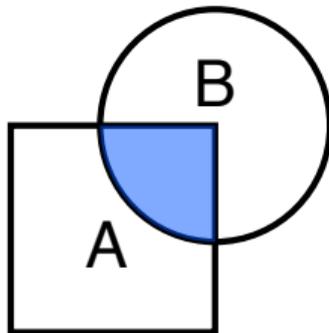
B menos A



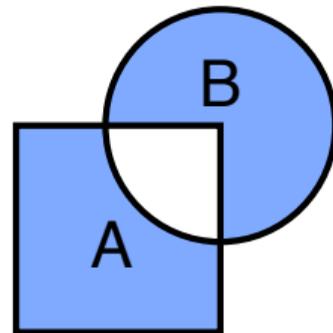
A união B



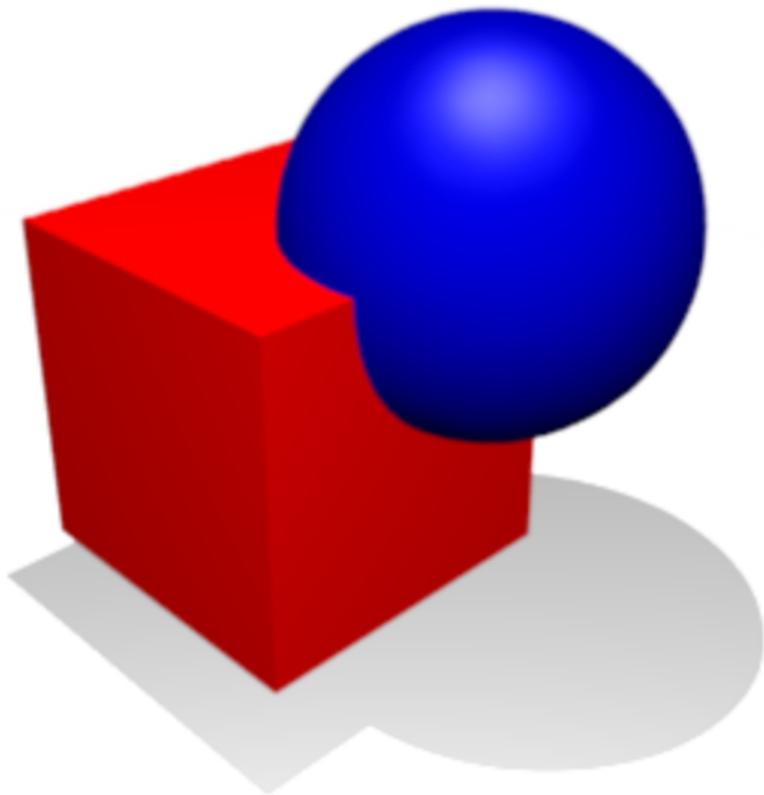
A interseção B



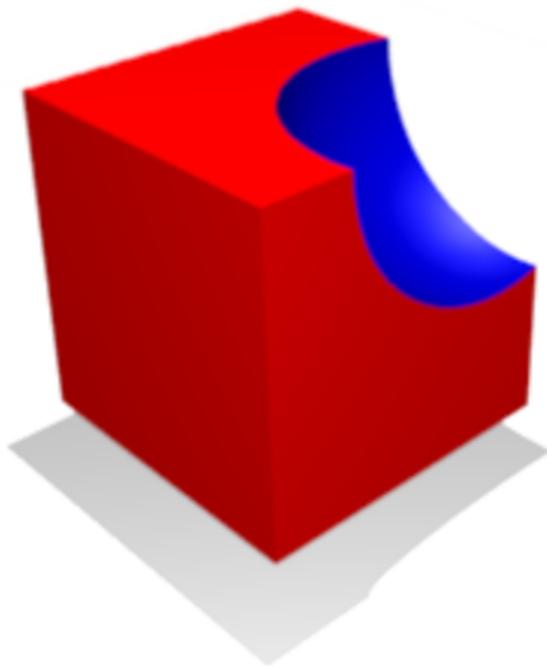
A diff B



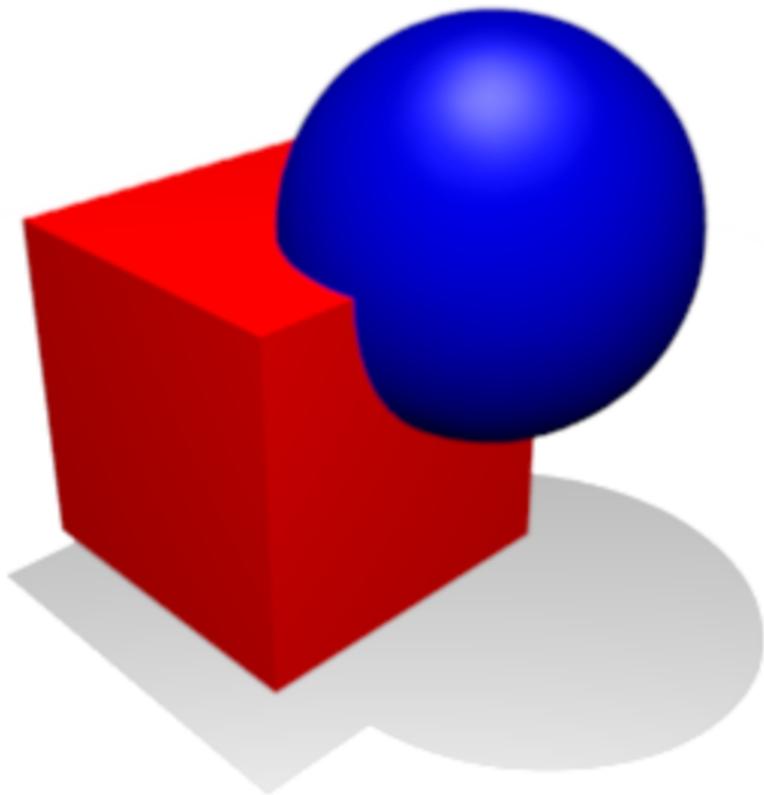
## Operações booleanas



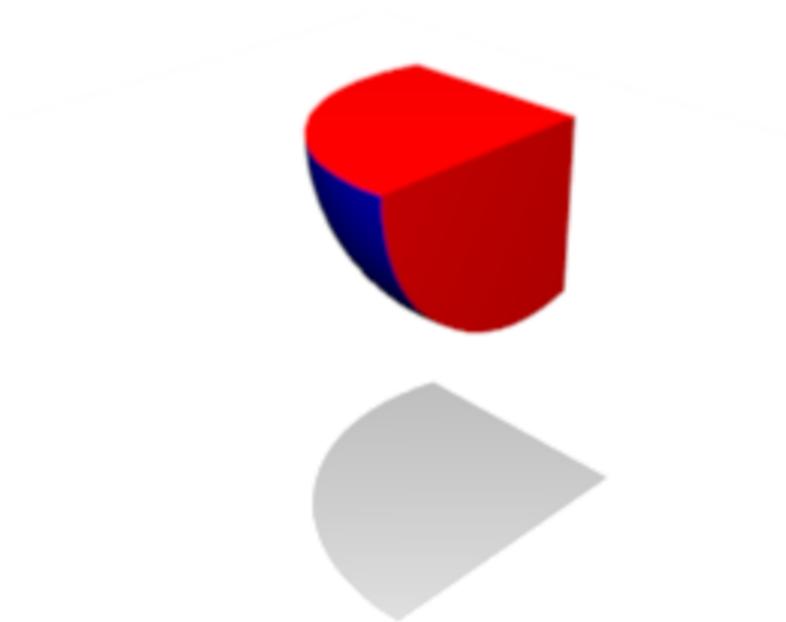
## Operações booleanas



## Operações booleanas



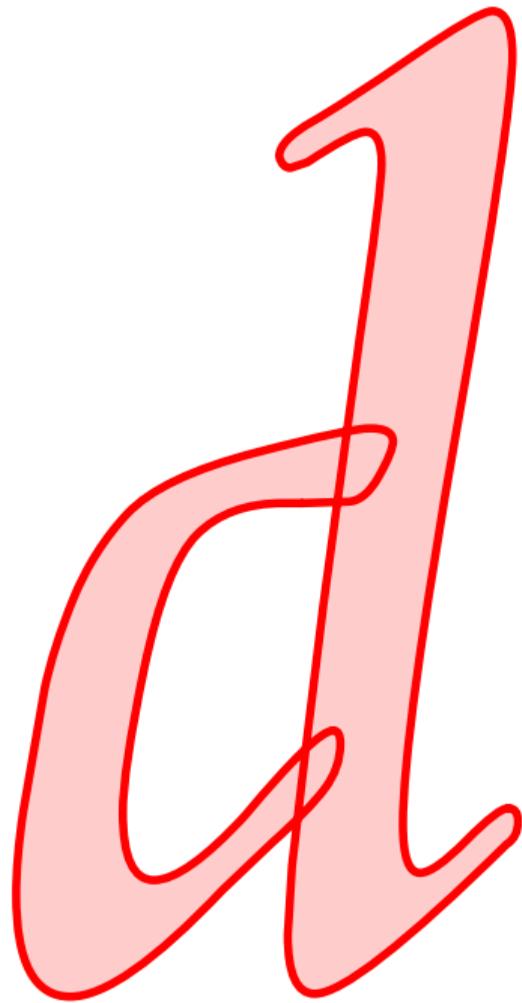
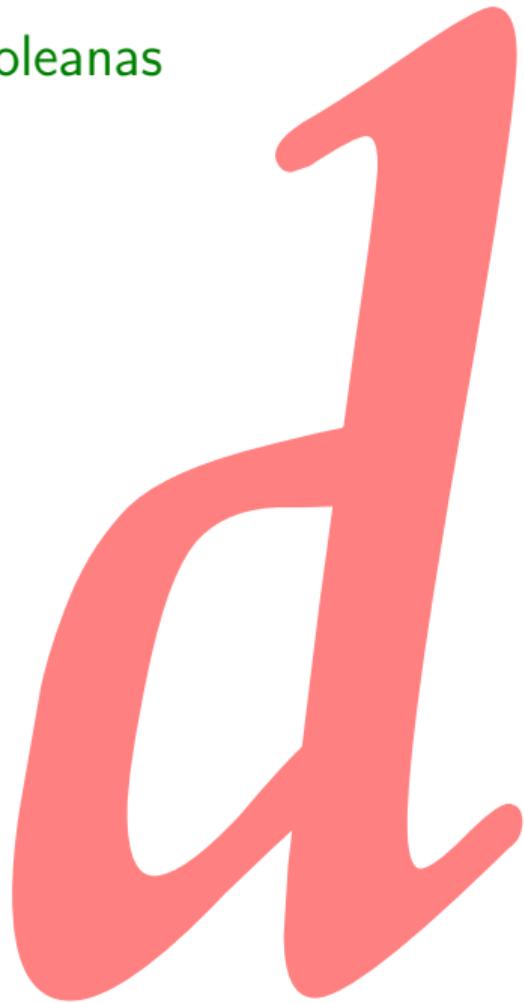
# Operações booleanas



Operações booleanas

*a*

## Operações booleanas



Operações booleanas

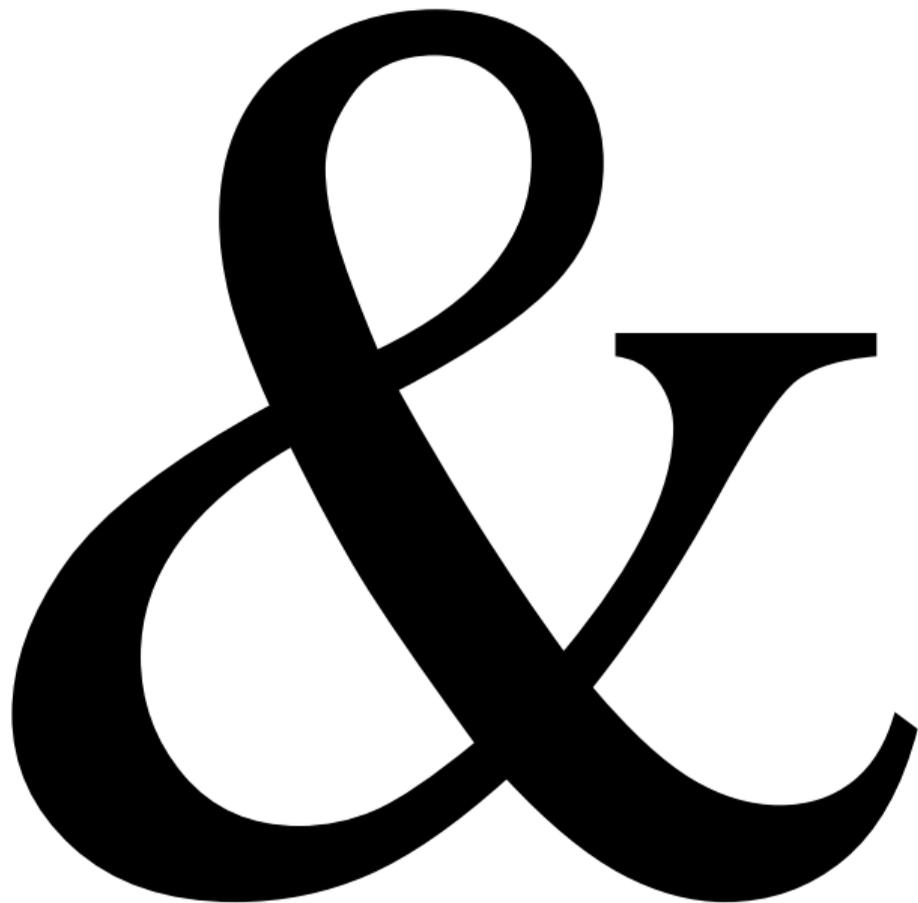
*d*

Operações booleanas

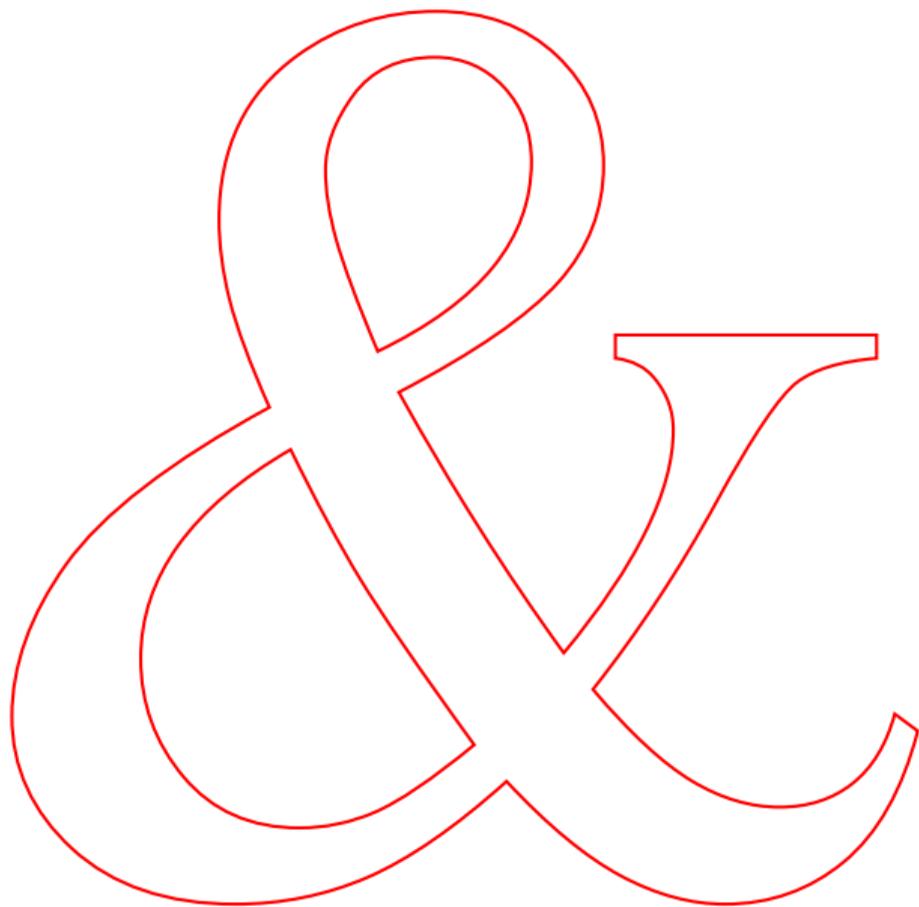
*d*

*d*

Representação por fronteira

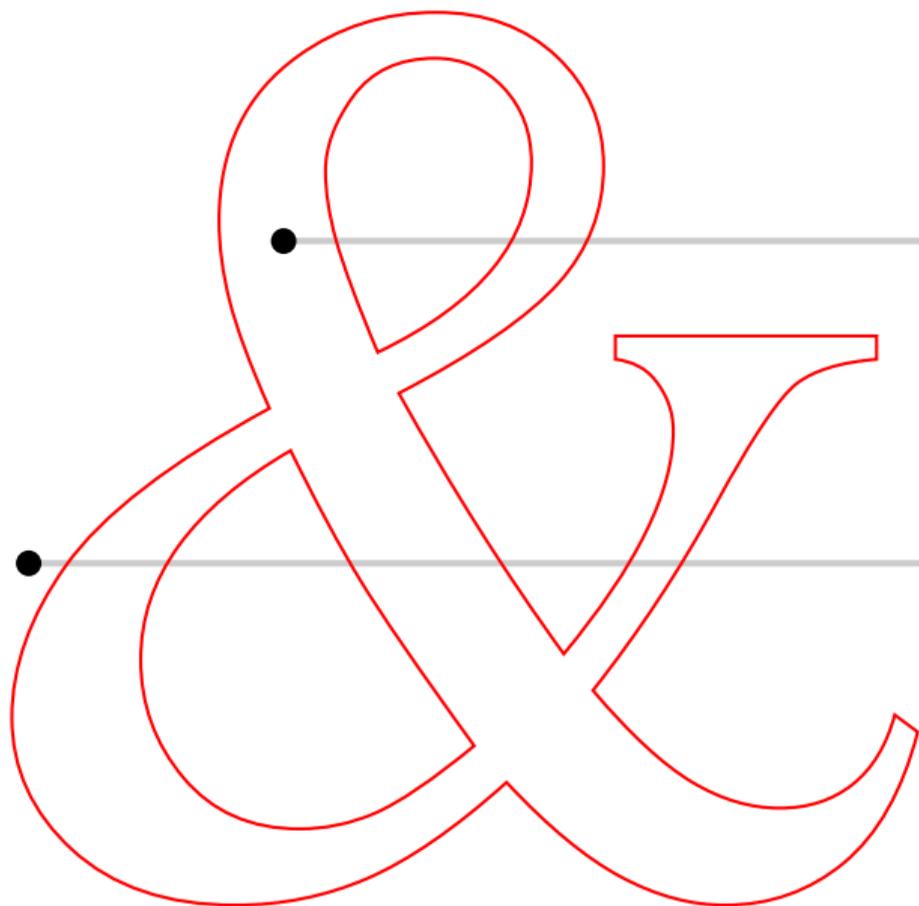


Representação por fronteira



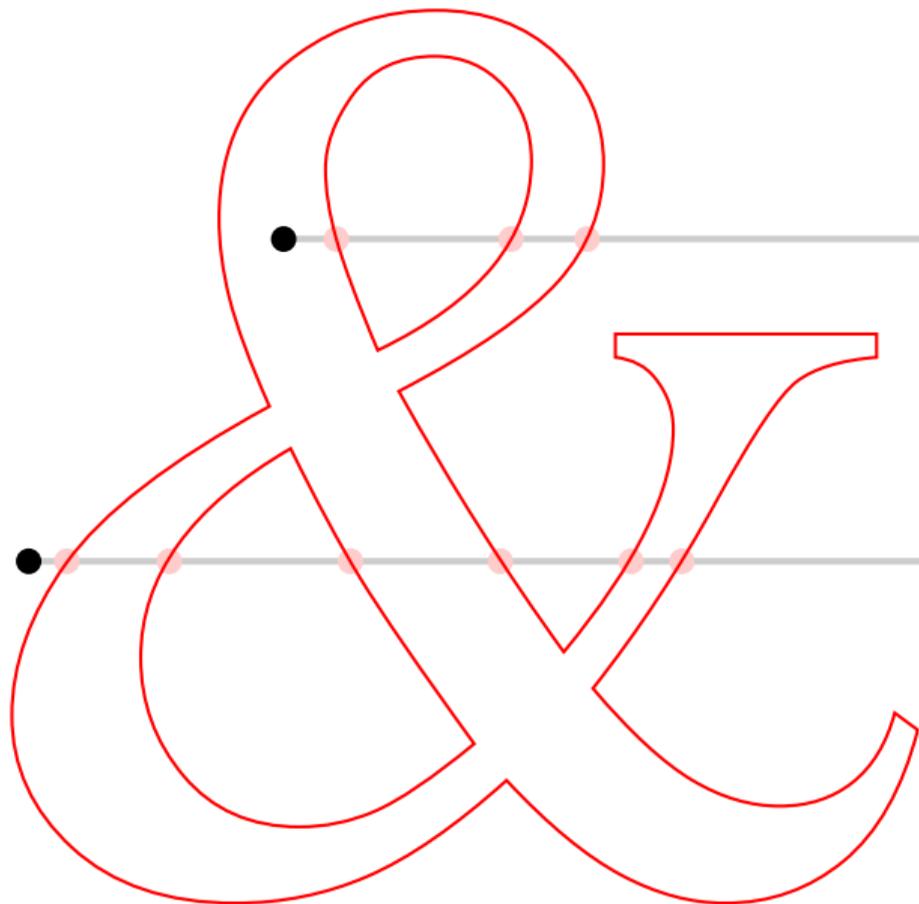
Representação por fronteira

Jordan (1887)

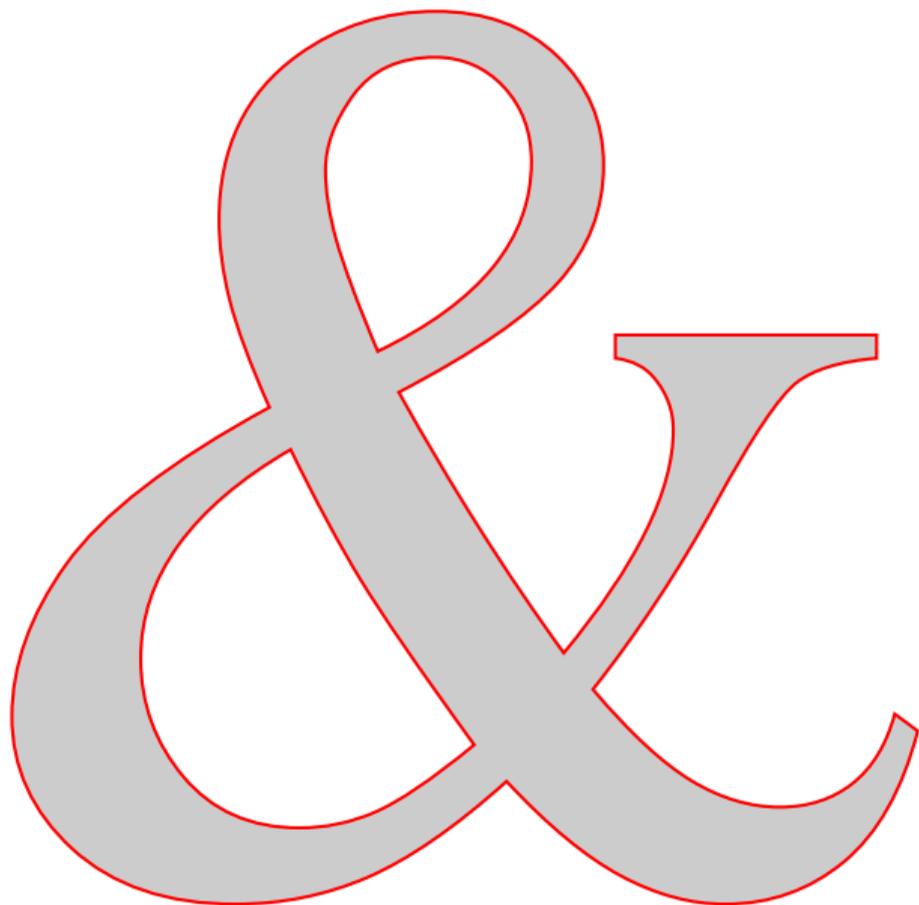


# Representação por fronteira

Jordan (1887)



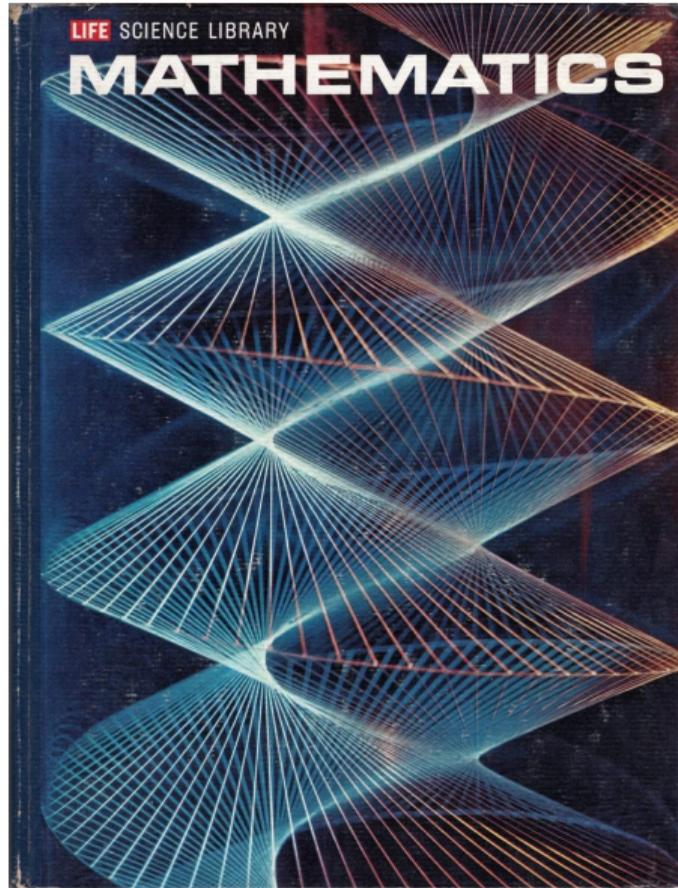
Representação por fronteira



# Representação por fronteira

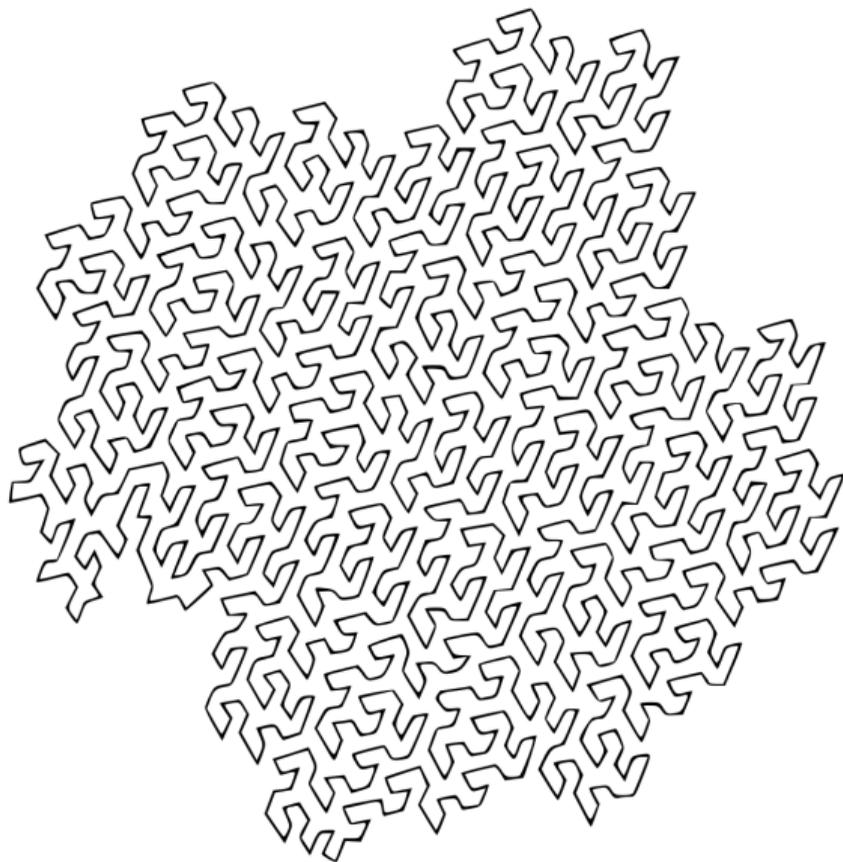


## Representação por fronteira

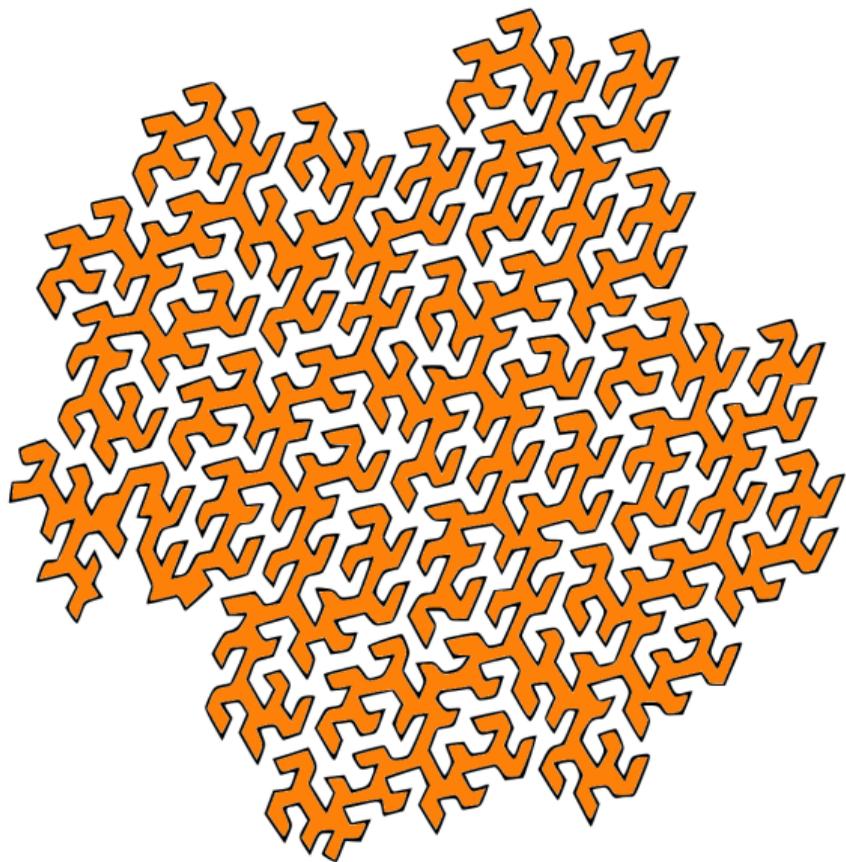


Life Science Library (1963)

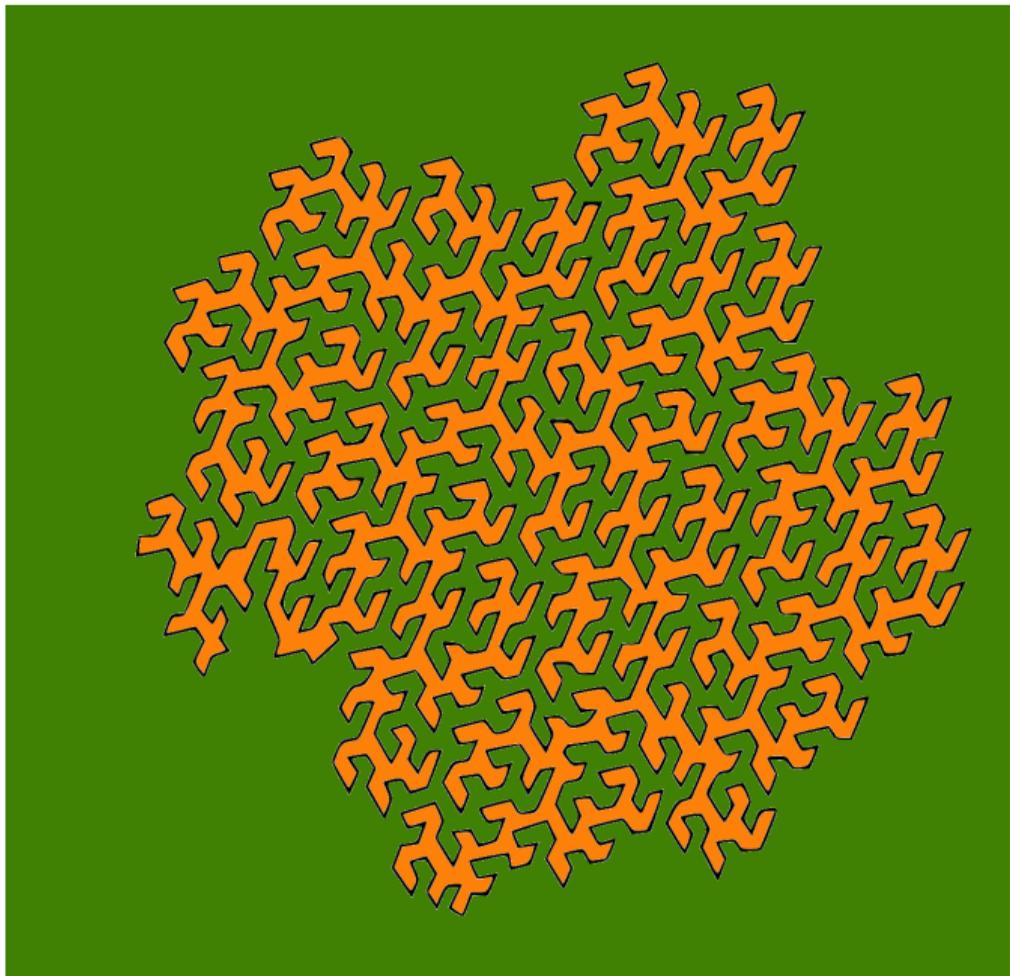
## Representação por fronteira



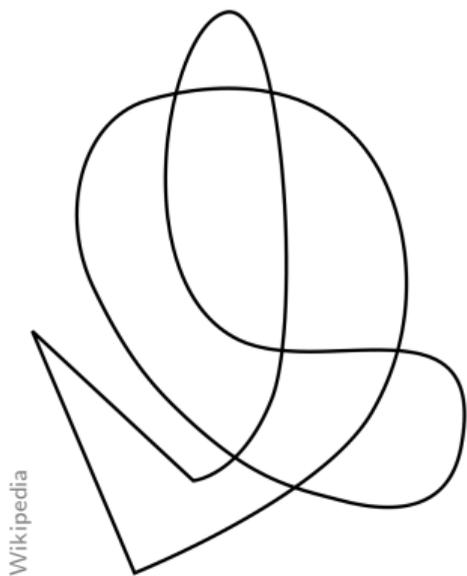
## Representação por fronteira



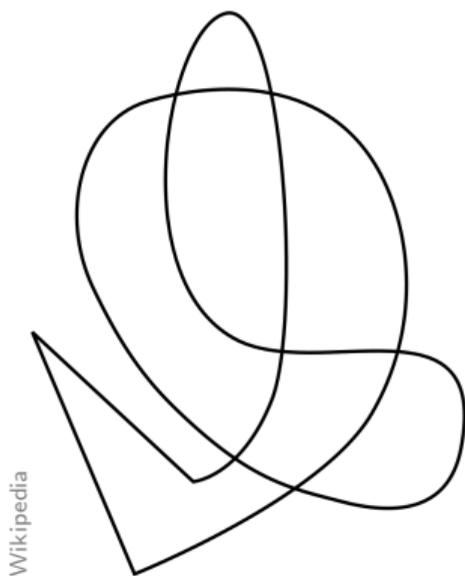
Representação por fronteira



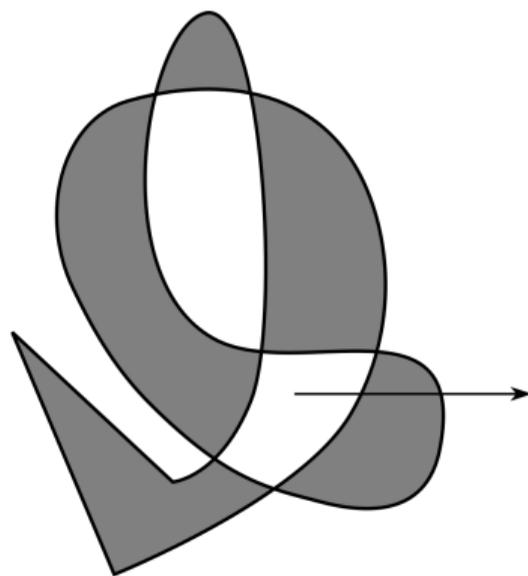
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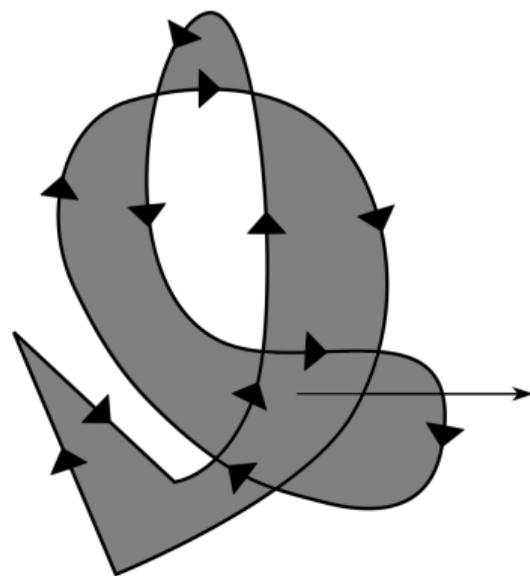
## Representação por fronteira



Wikipedia



paridade

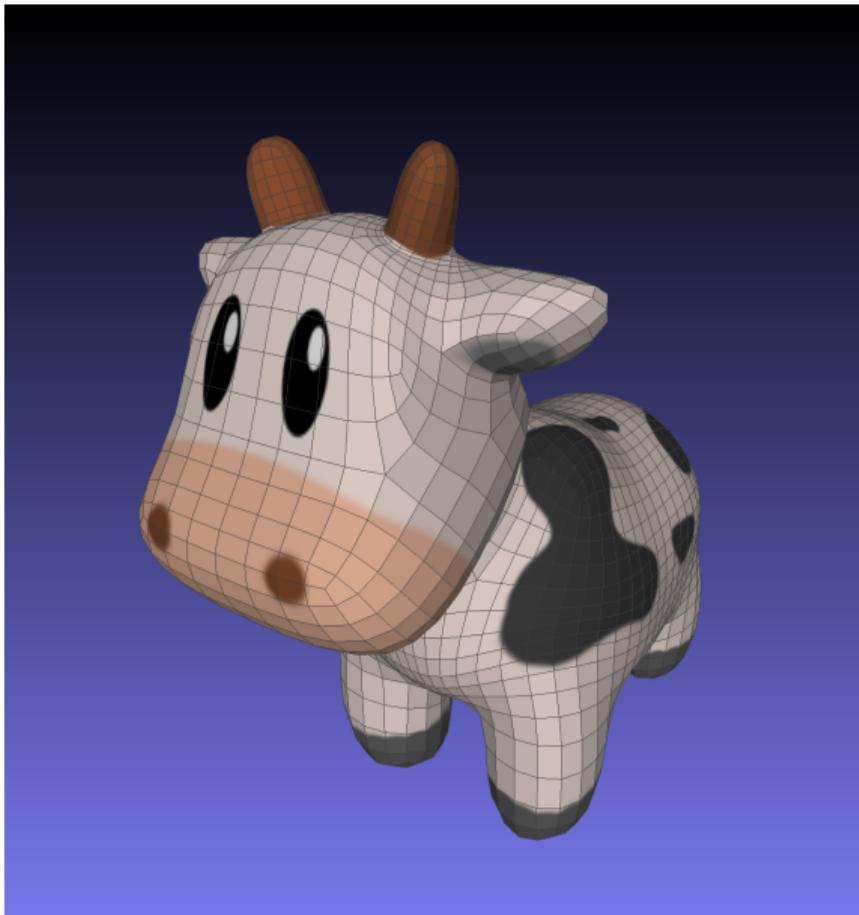


número de rotação

Representação por fronteira

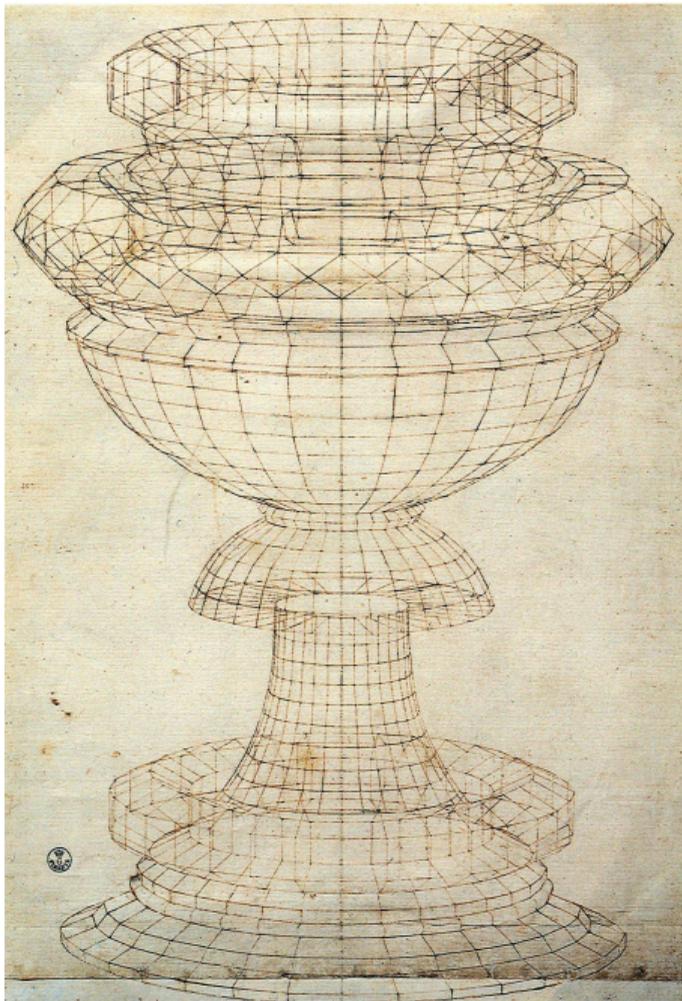






decomposição

Uccello (1450)

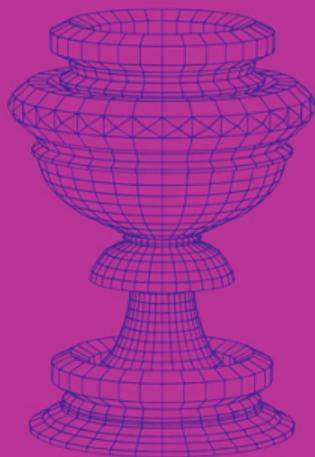




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# COMPUTER AIDED GEOMETRIC DESIGN



Uccello (1450)

## Métodos de decomposição

- decomposição celular
- decomposição intrínseca
- decomposição extrínseca

## Métodos de decomposição

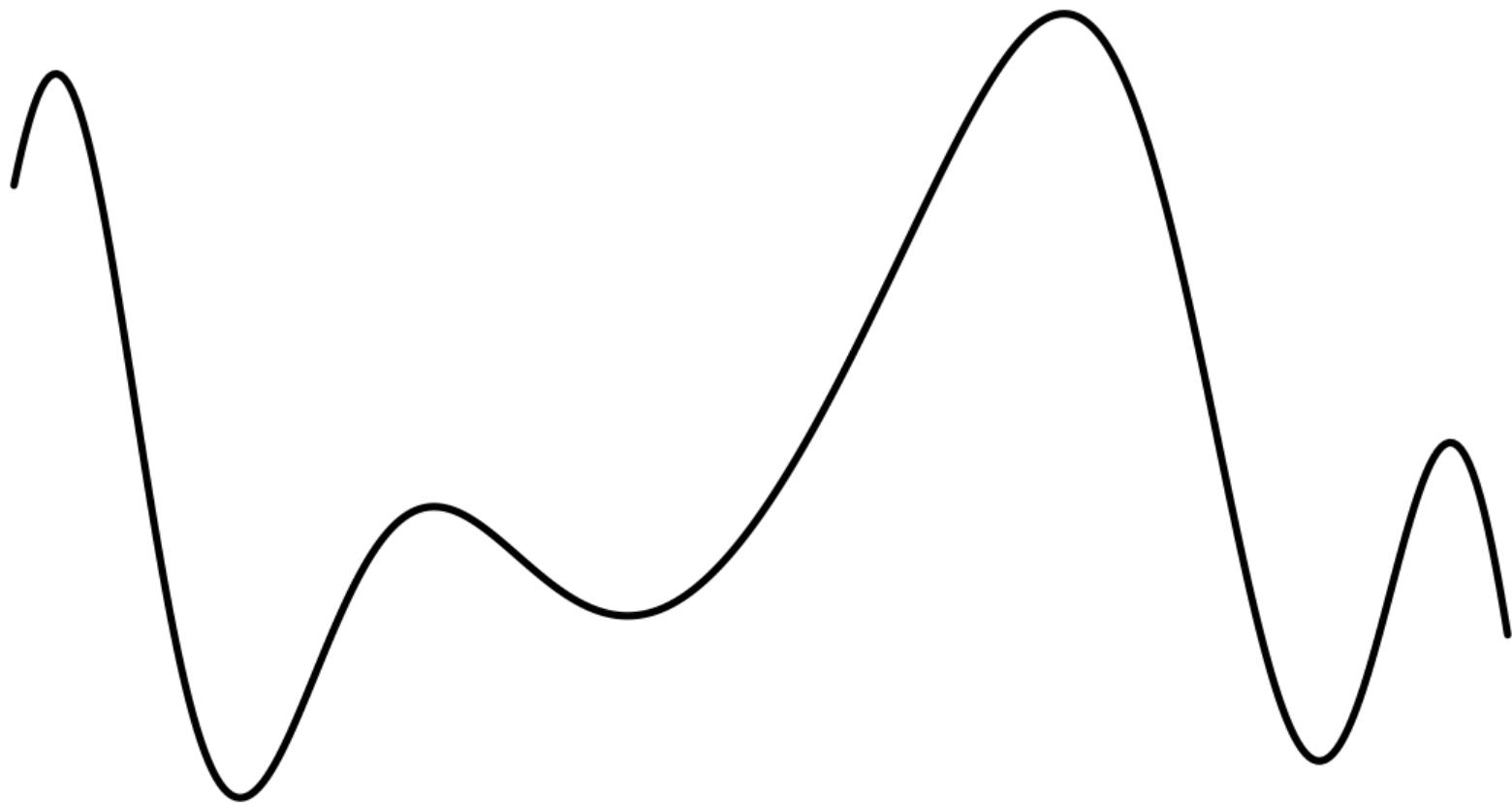
- decomposição celular = discretização  
amostragem do objeto  
estruturação em células
- decomposição intrínseca
- decomposição extrínseca

## Métodos de decomposição

- decomposição celular = discretização  
amostragem do objeto  
estruturação em células
- decomposição intrínseca  
geometria do objeto  
poligonização
- decomposição extrínseca  
localização do objeto  
enumeração espacial

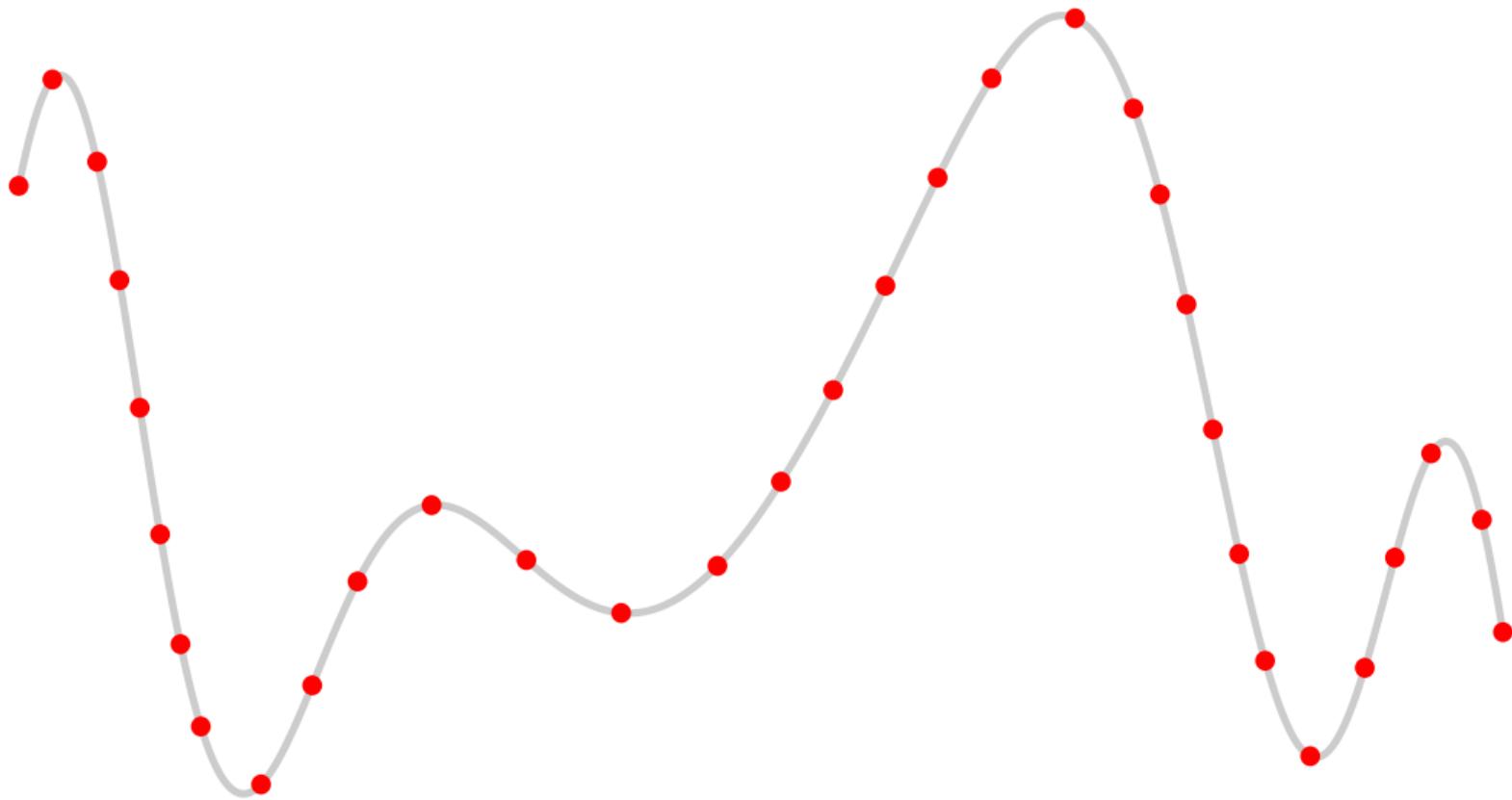
decomposição intríntrica  
poligonização

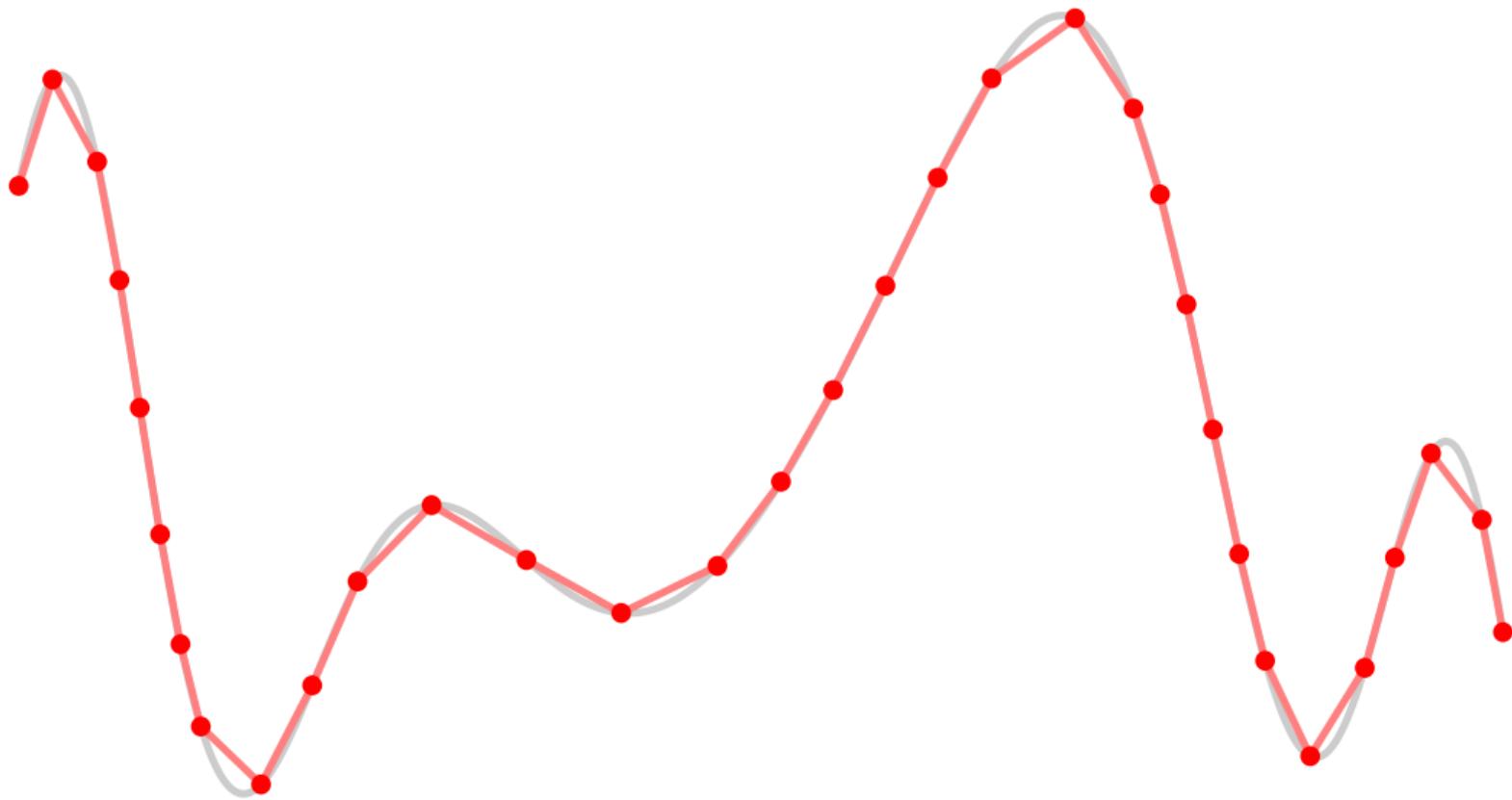
decomposição intríntrica  
amostragem + estruturação



# Decomposição celular intrínseca

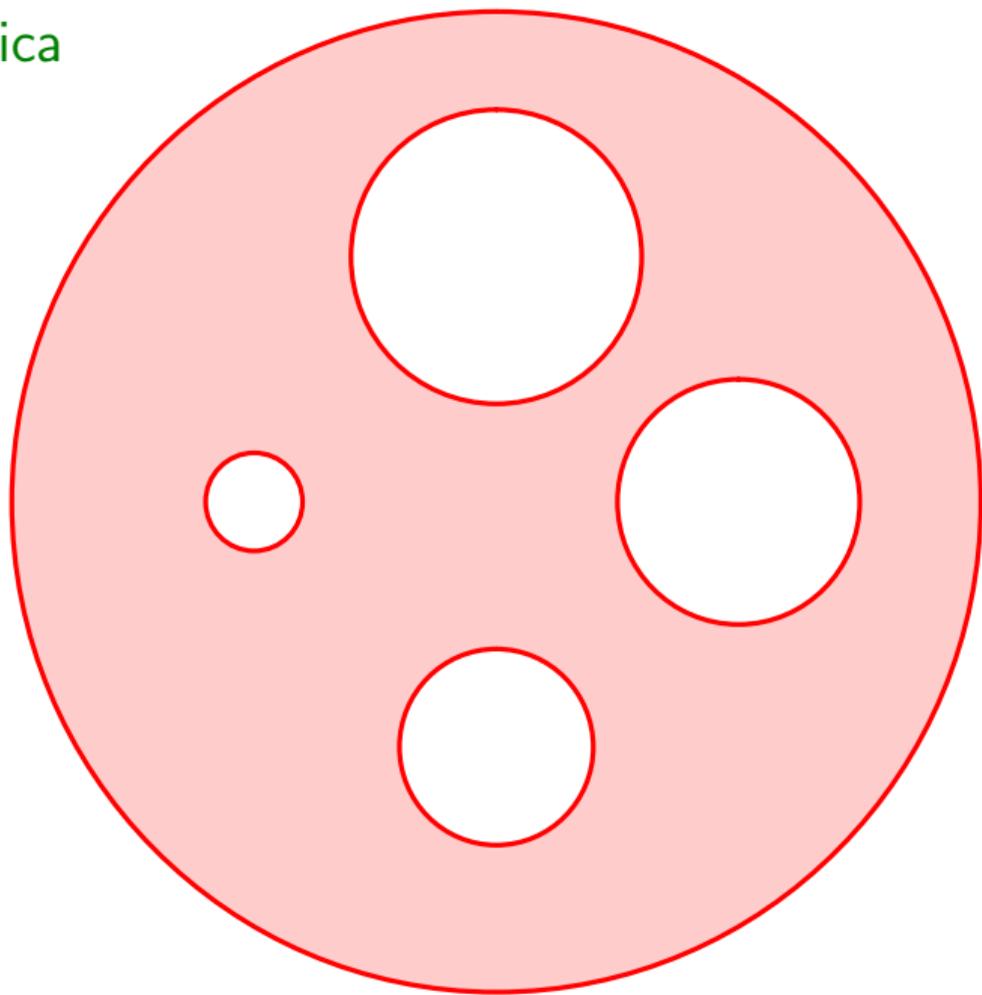
1D





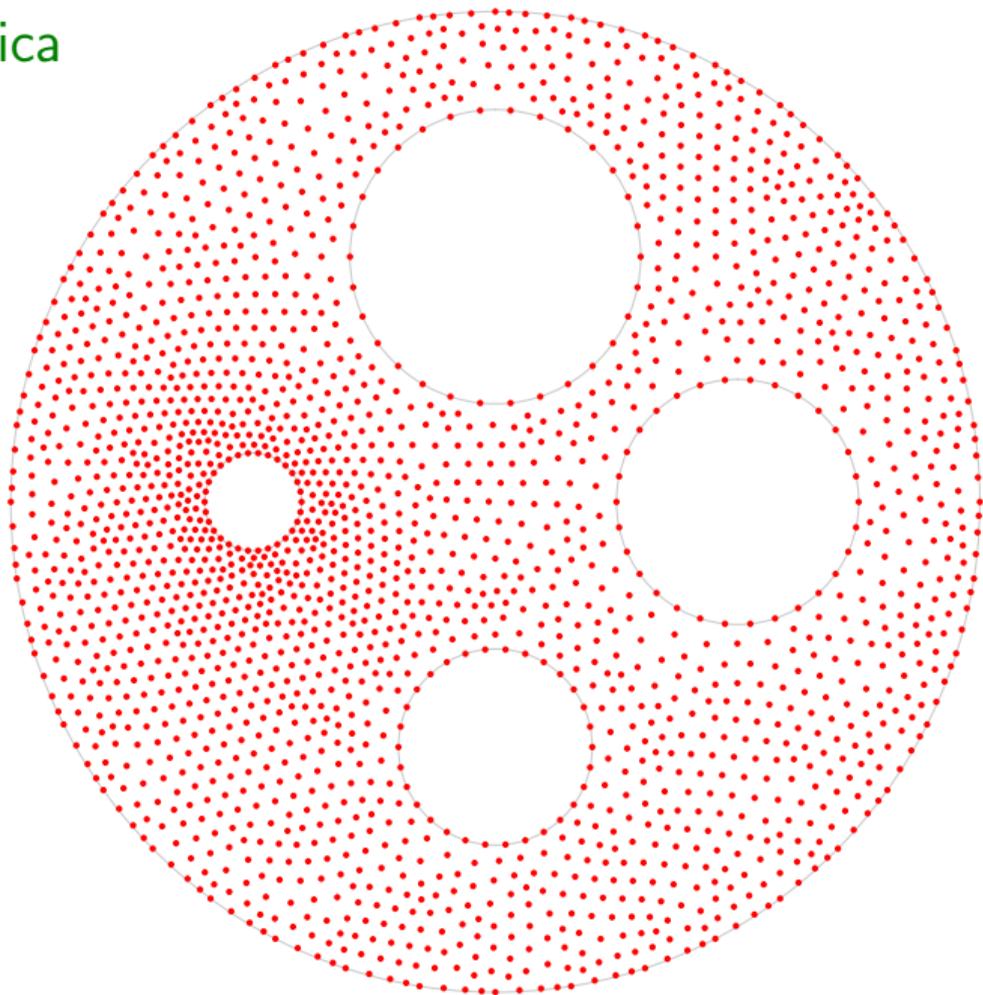
## Decomposição celular intrínseca

2D



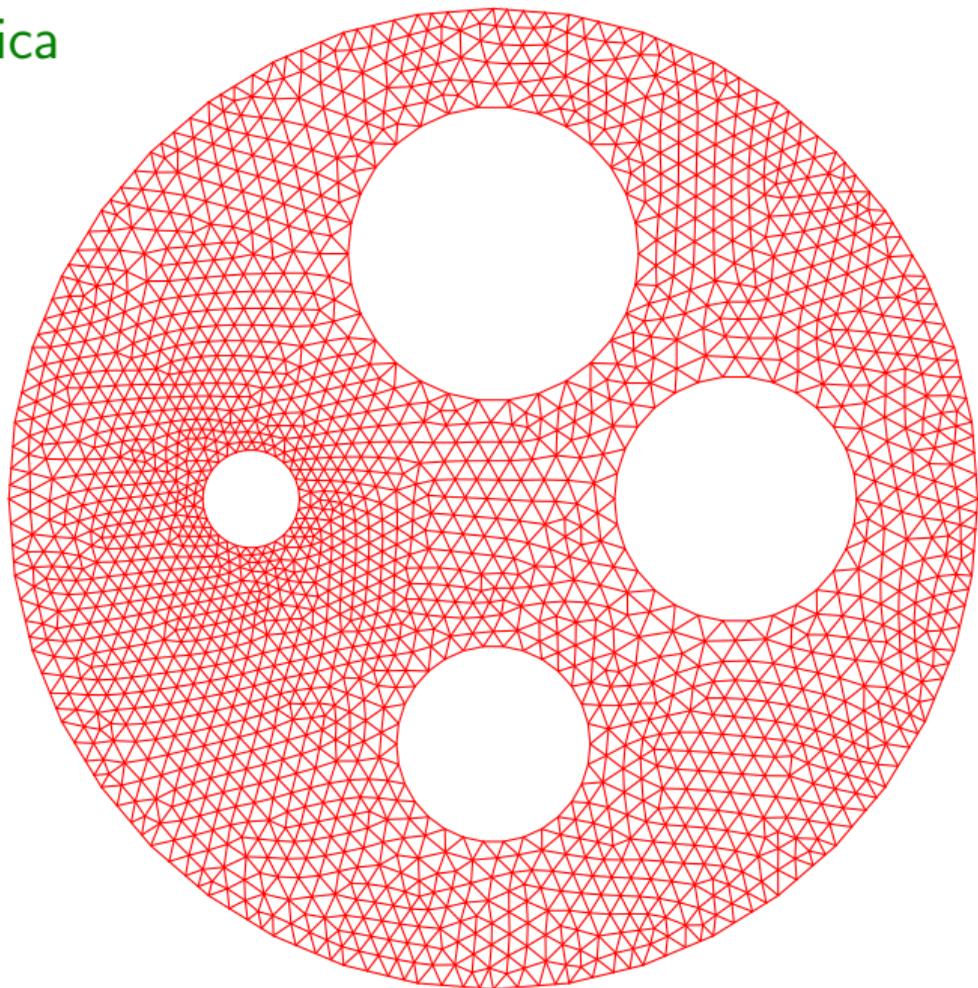
# Decomposição celular intríntrica

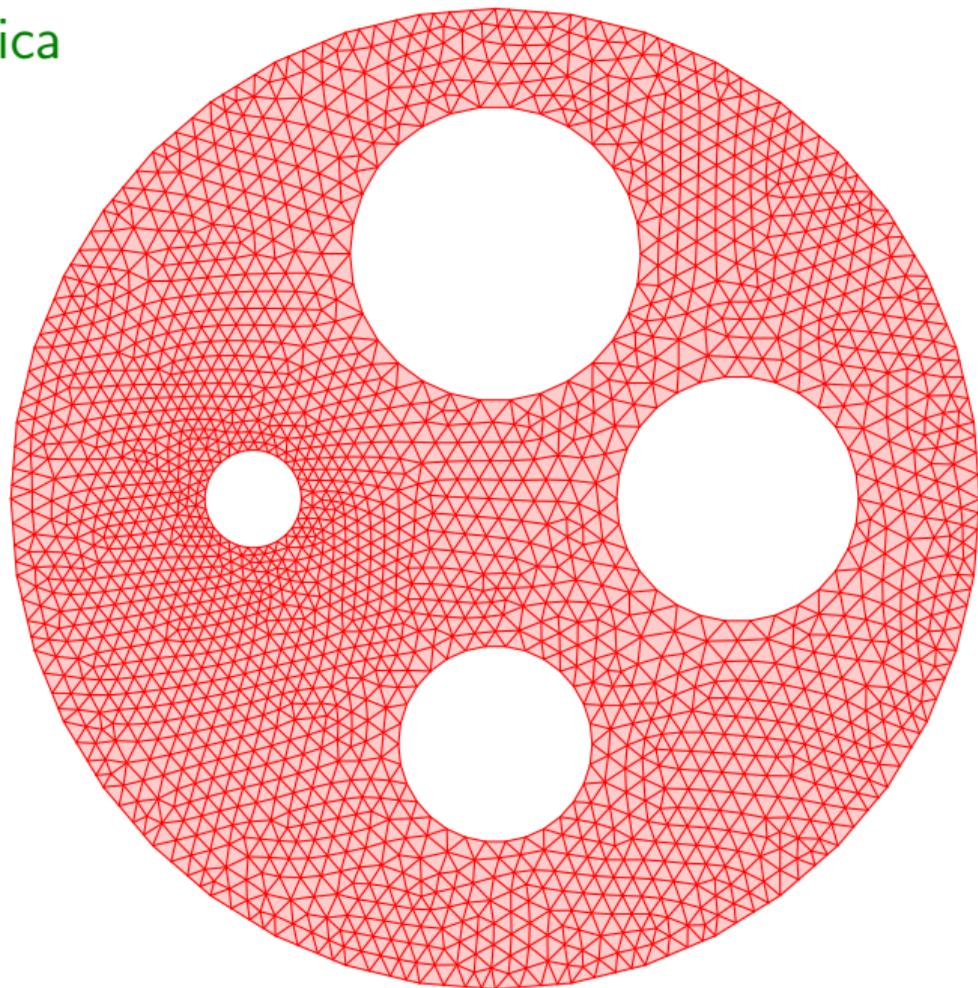
2D

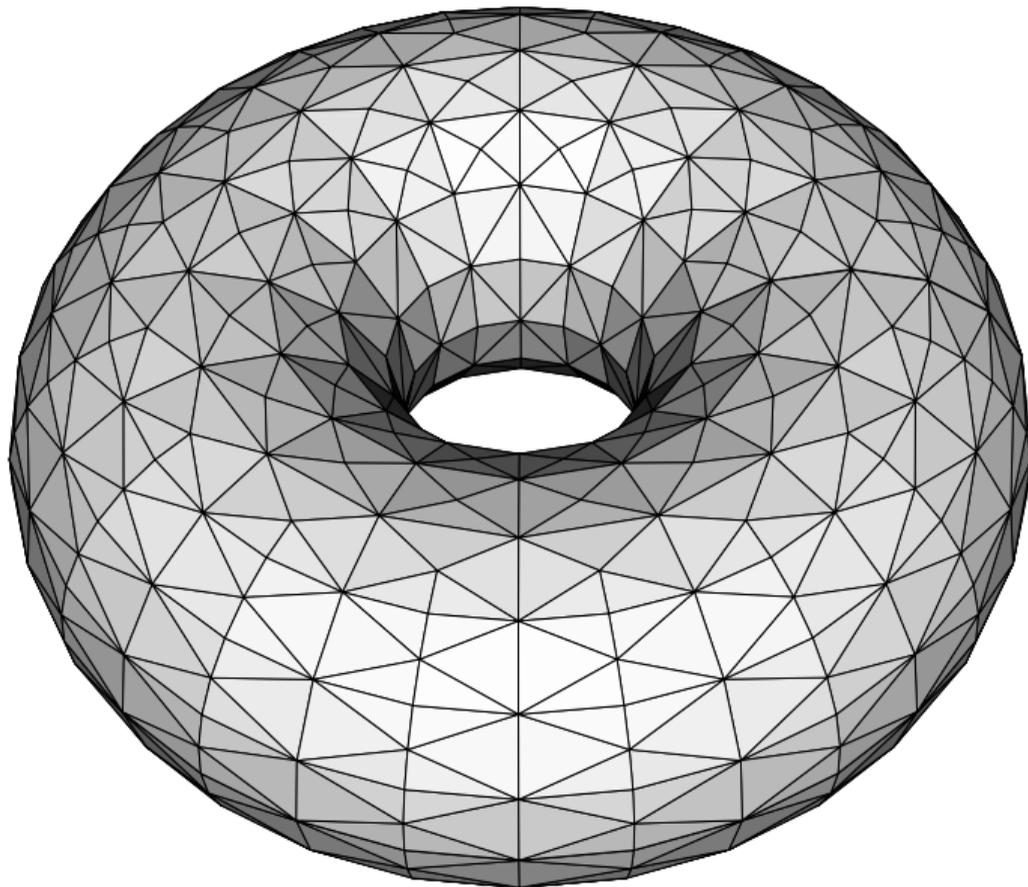


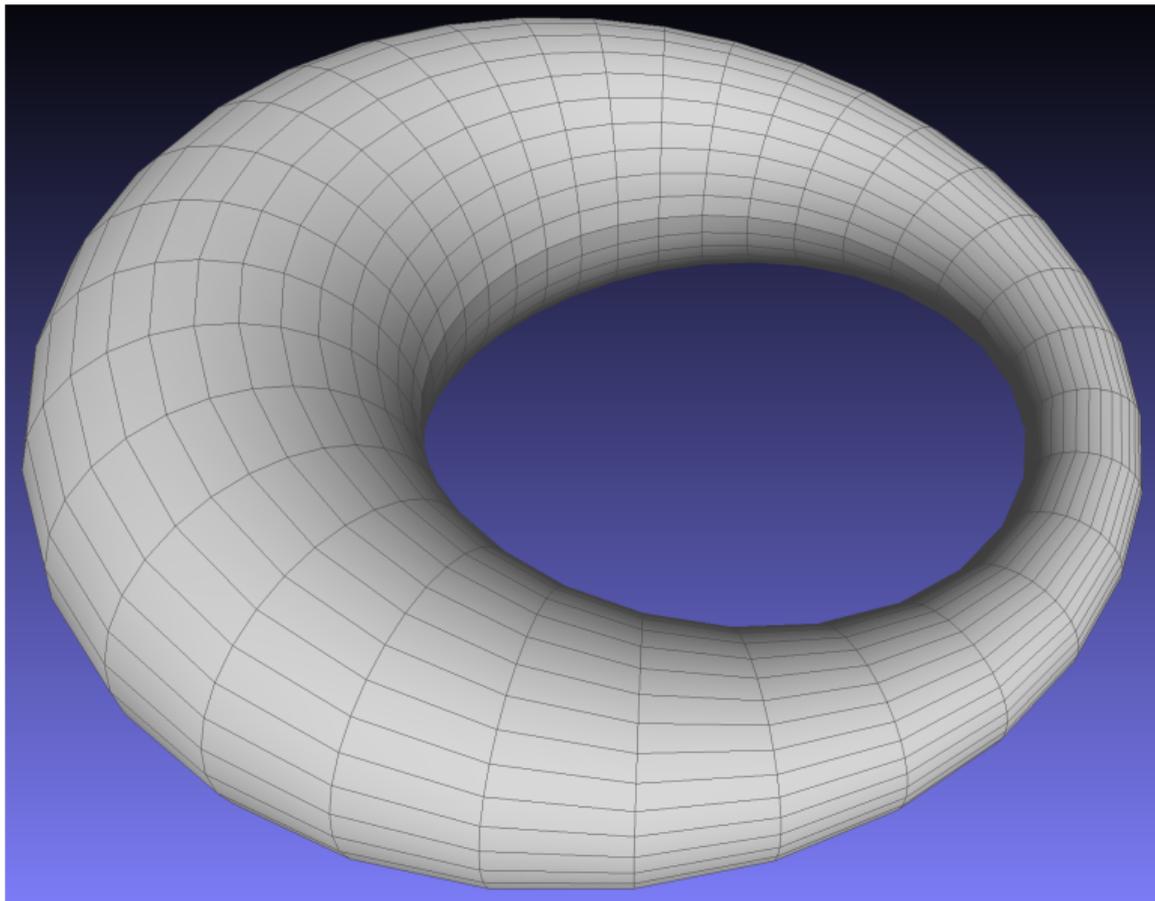
# Decomposição celular intrínseca

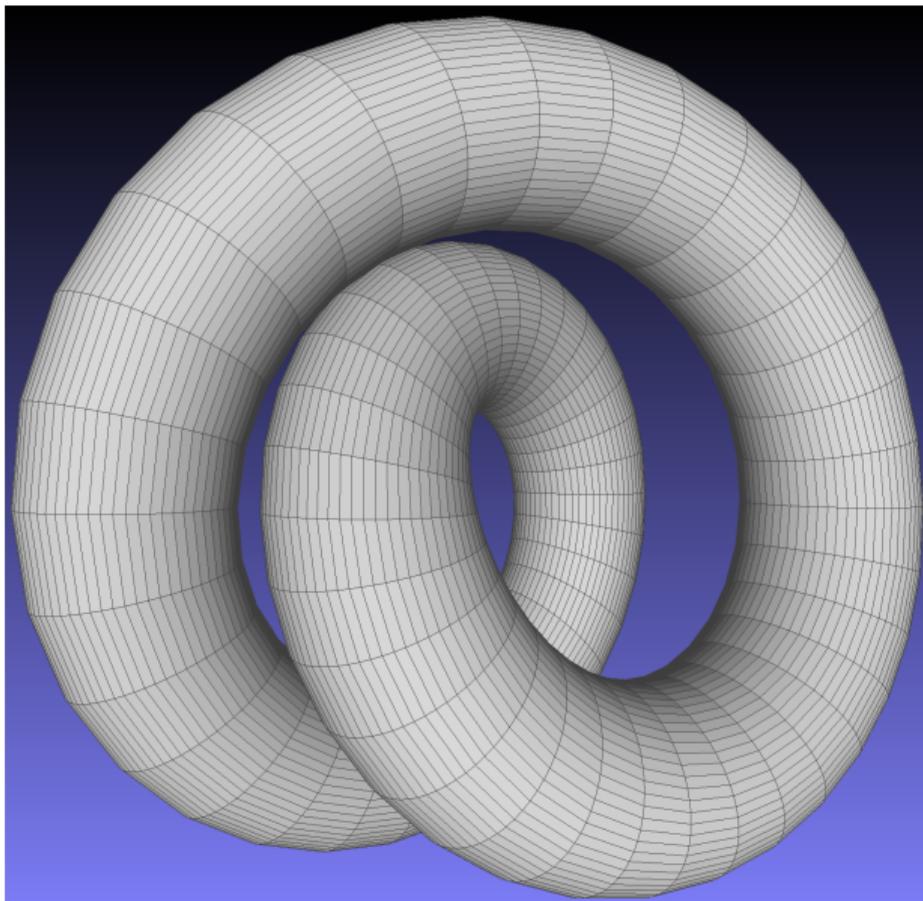
2D











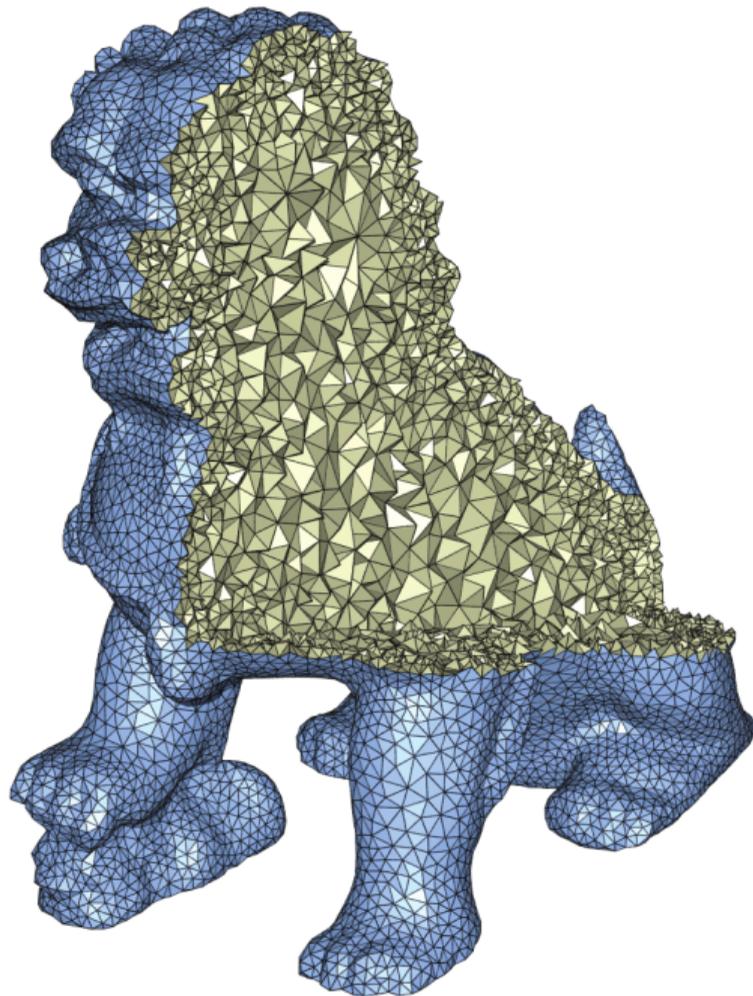
# Decomposição celular intrínseca

3D



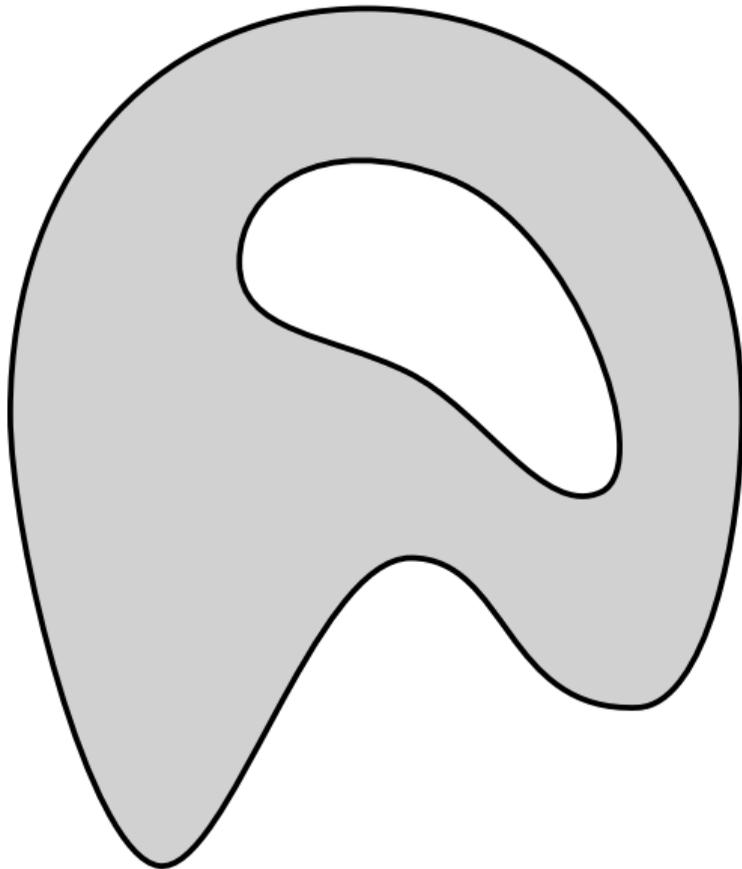
# Decomposição celular intrínseca

3D

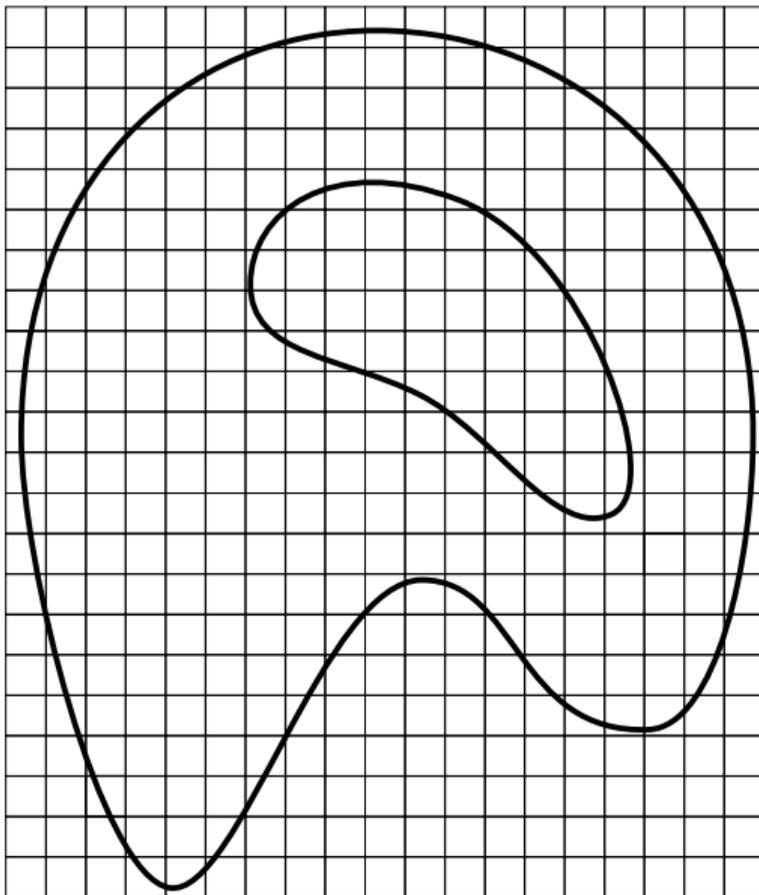


decomposição extrínseca  
enumeração espacial

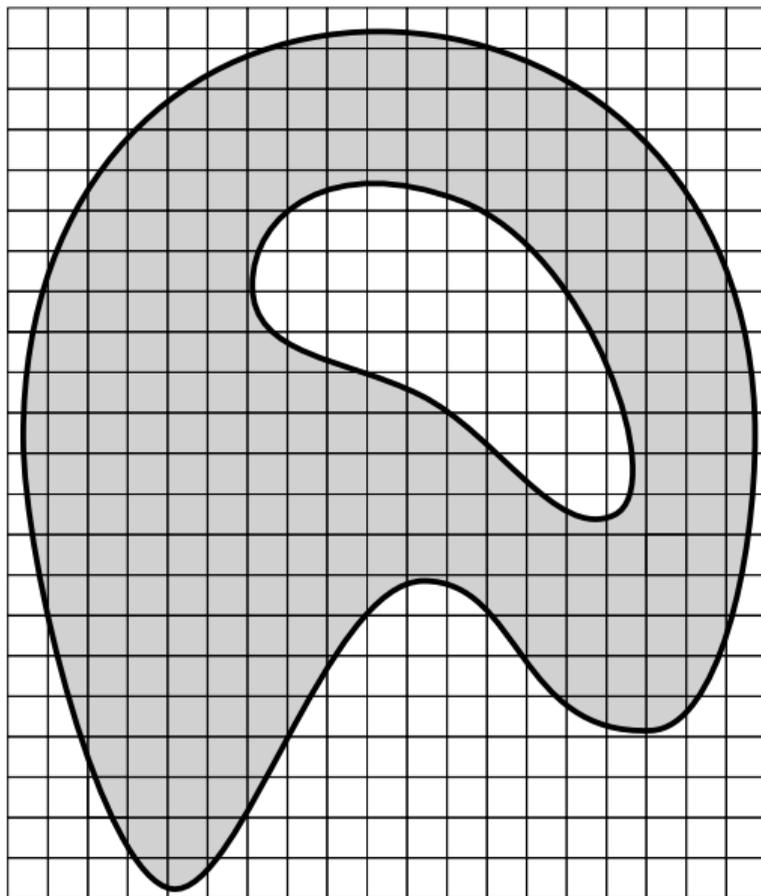
## Decomposição celular extríntrica



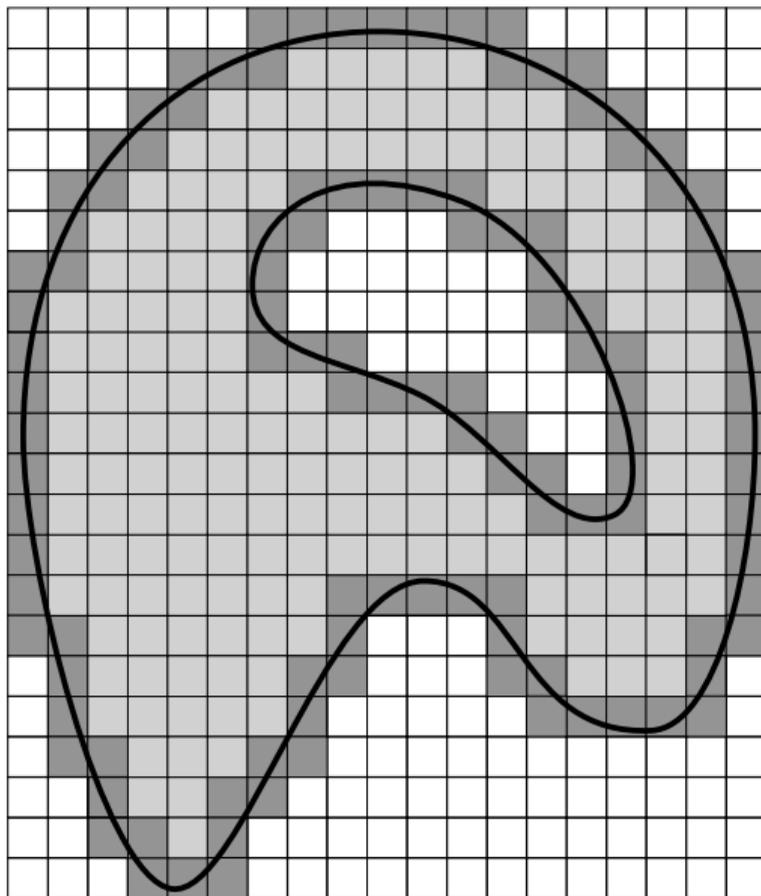
## Decomposição celular extrínseca



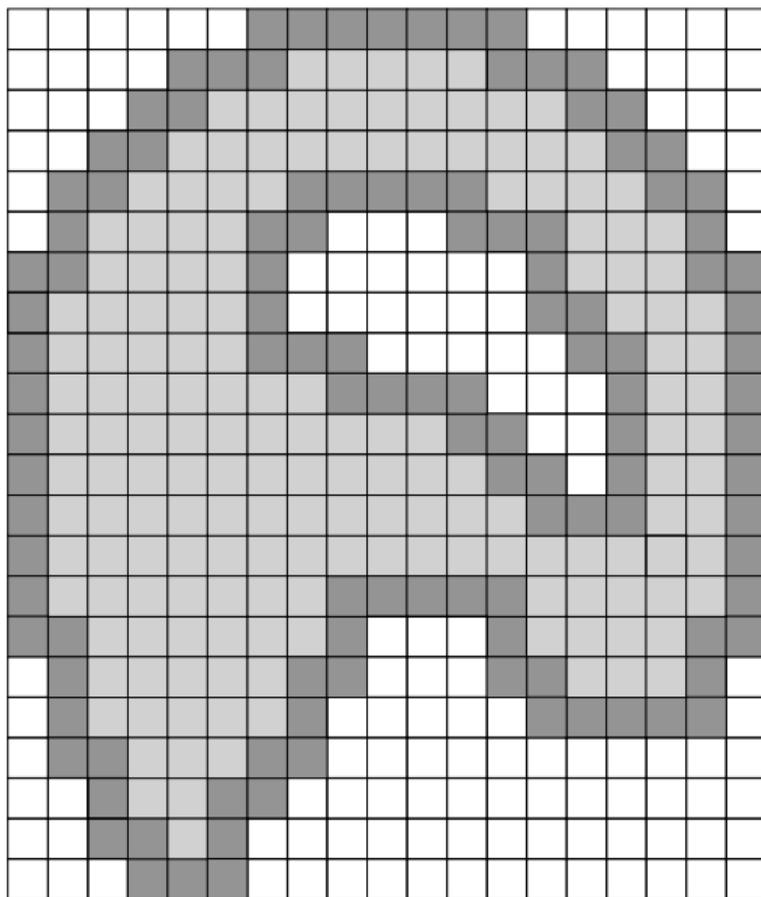
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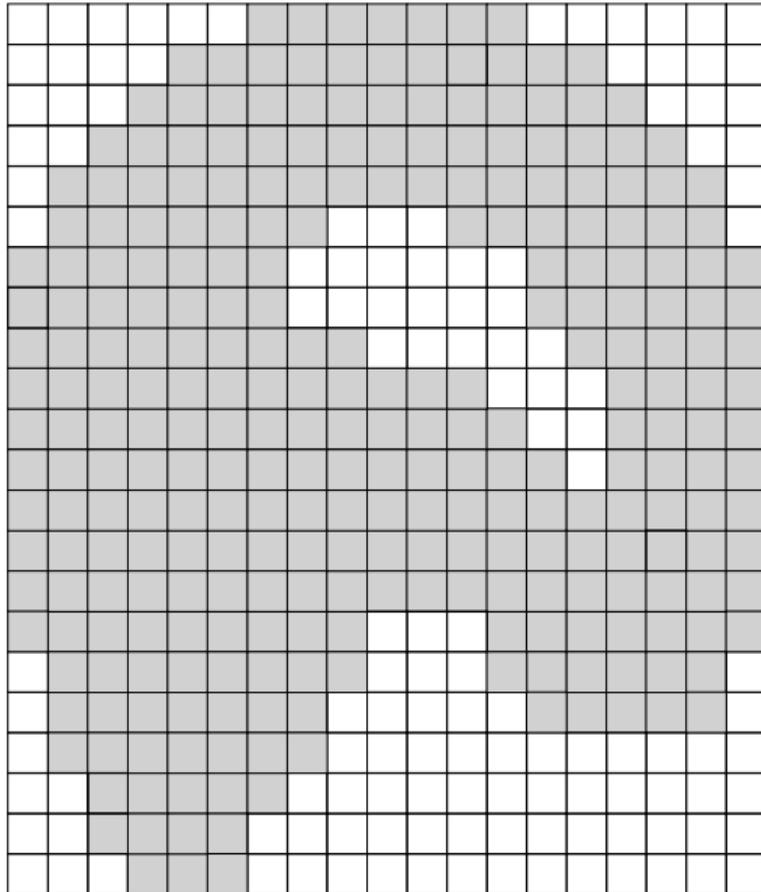
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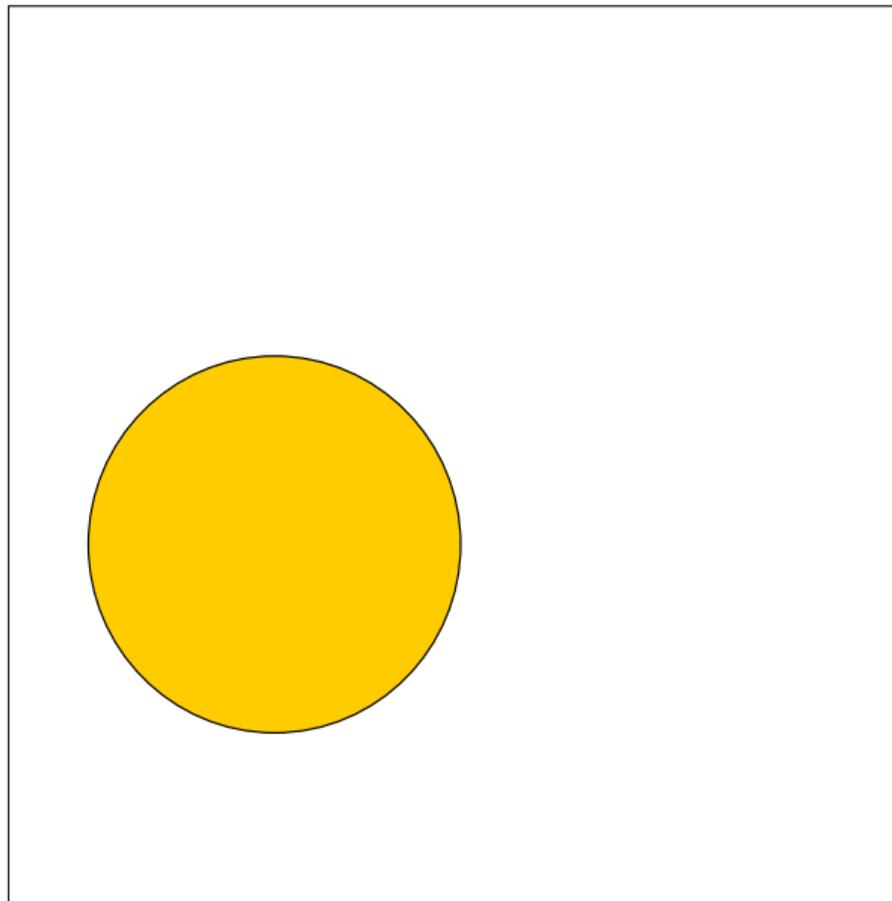
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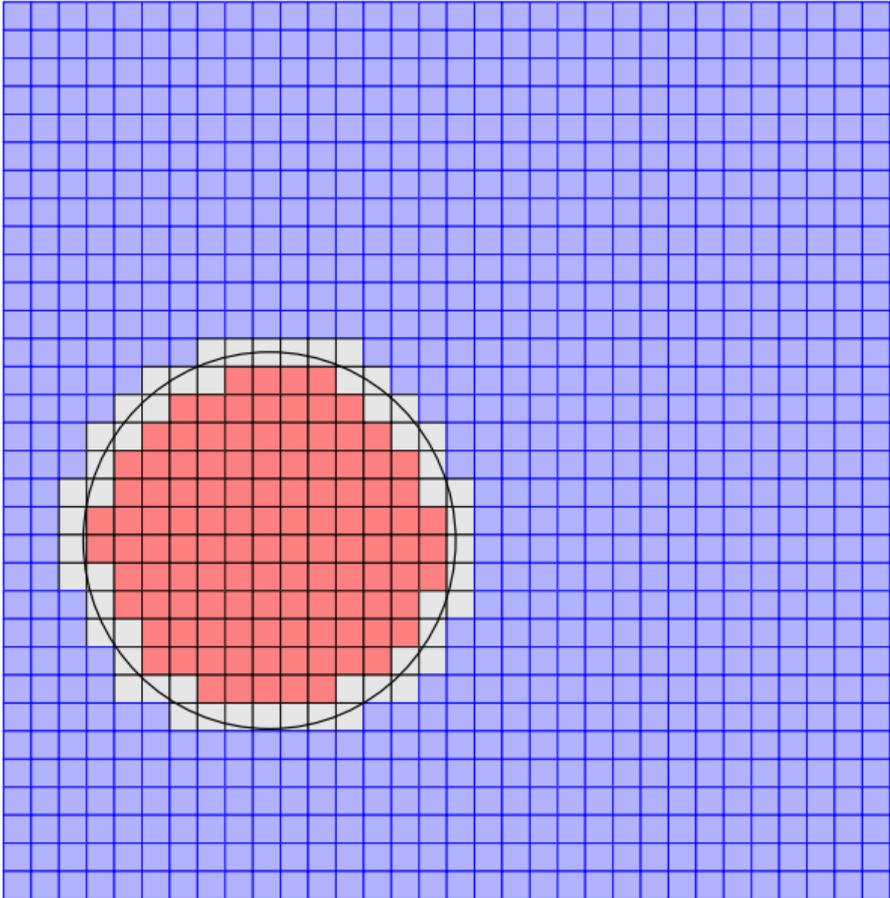
# Decomposição celular extrínseca



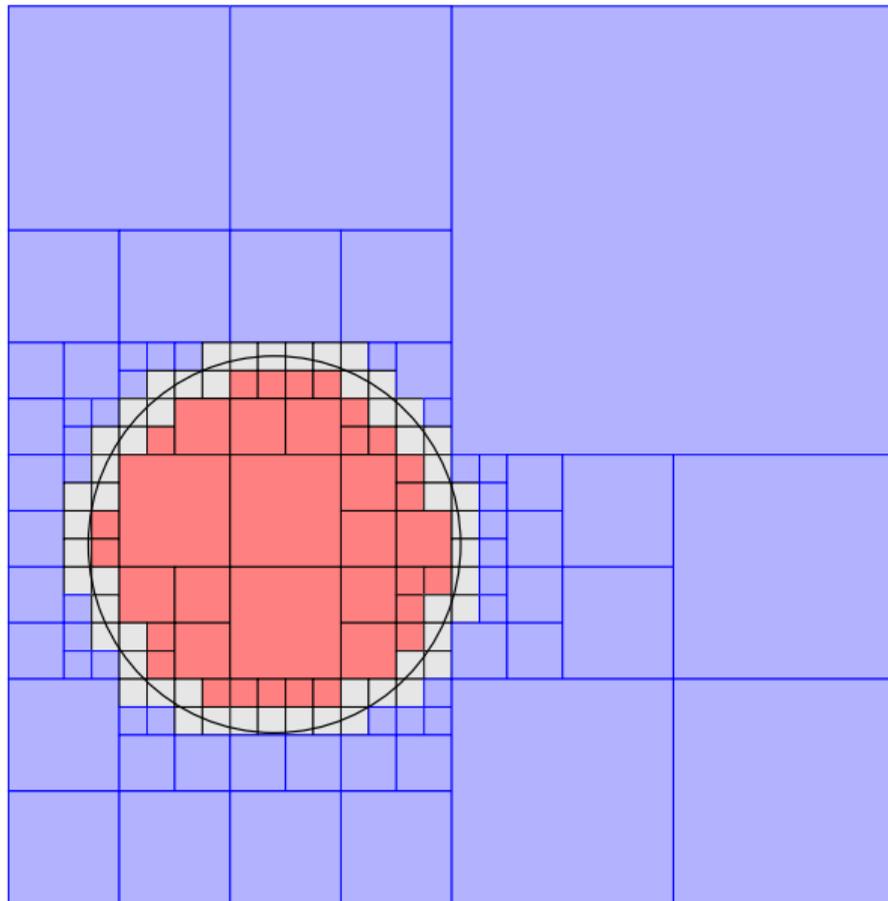
## Enumeração espacial



# Enumeração espacial uniforme



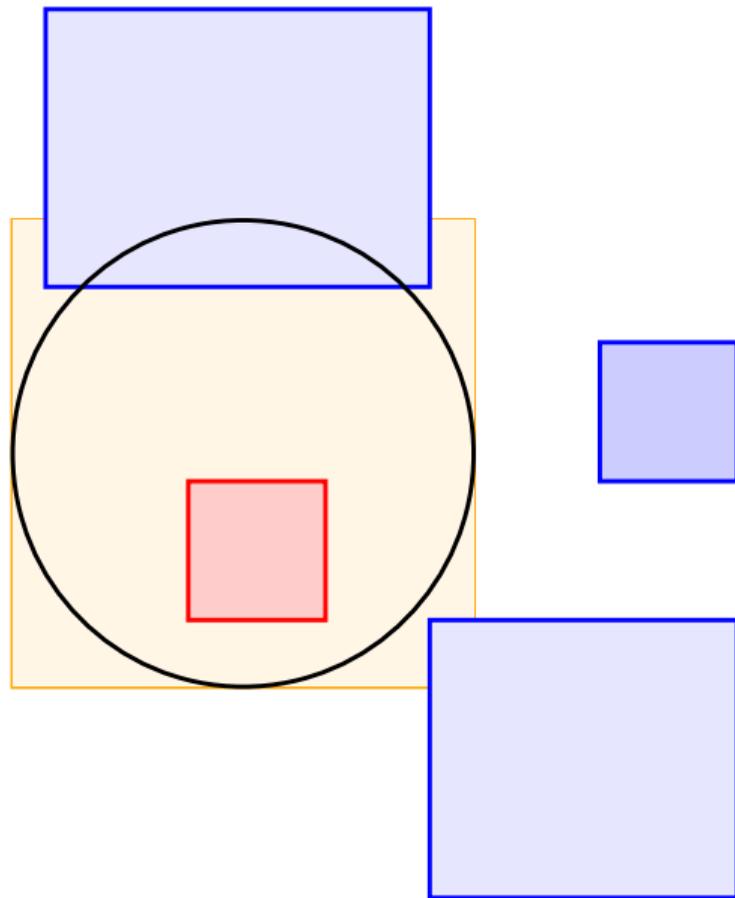
# Enumeração espacial adaptativa



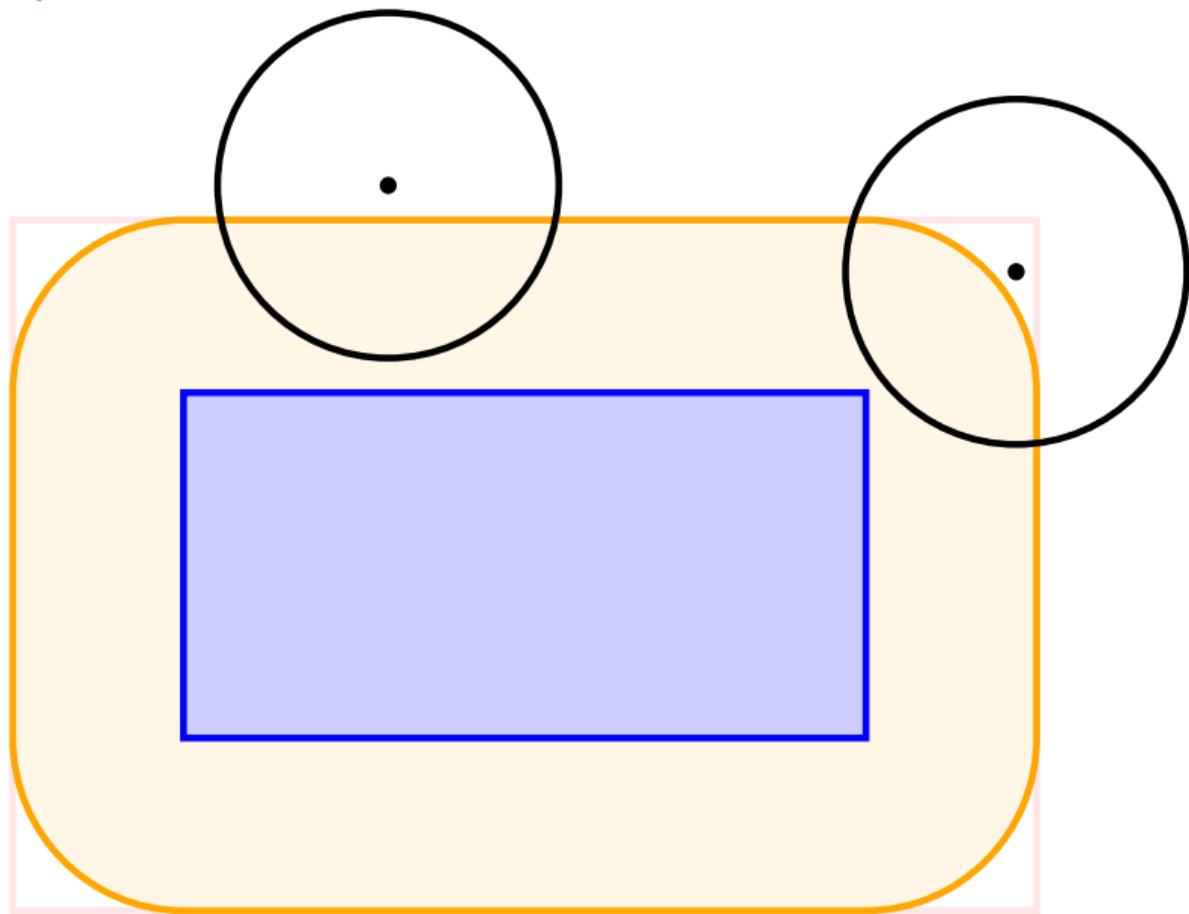
## Enumeração espacial adaptativa

```
function explore(xmin,xmax,ymin,ymax,level)
    local c=classify(xmin,xmax,ymin,ymax)
    if c~="none" or level==0 then
        output(xmin,xmax,ymin,ymax,c)
    else
        local xmid=(xmin+xmax)/2
        local ymid=(ymin+ymax)/2
        explore(xmin,xmid,ymid,ymax,level-1)
        explore(xmid,xmax,ymid,ymax,level-1)
        explore(xmin,xmid,ymin,ymid,level-1)
        explore(xmid,xmax,ymin,ymid,level-1)
    end
end
```

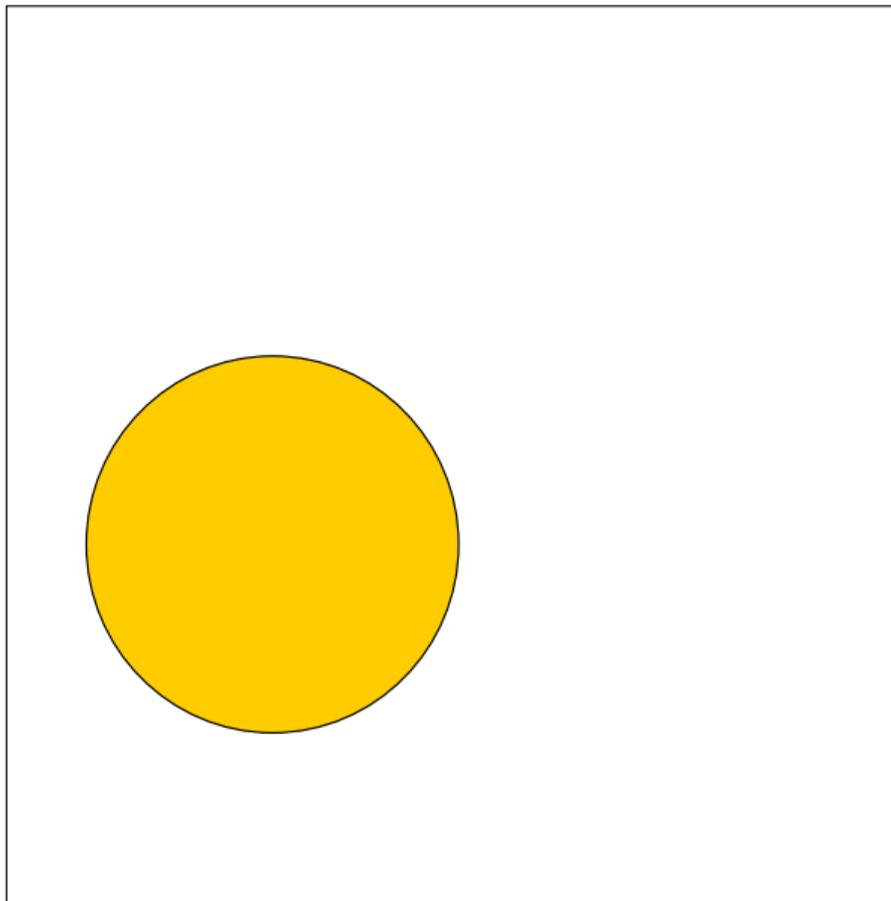
## Enumeração espacial



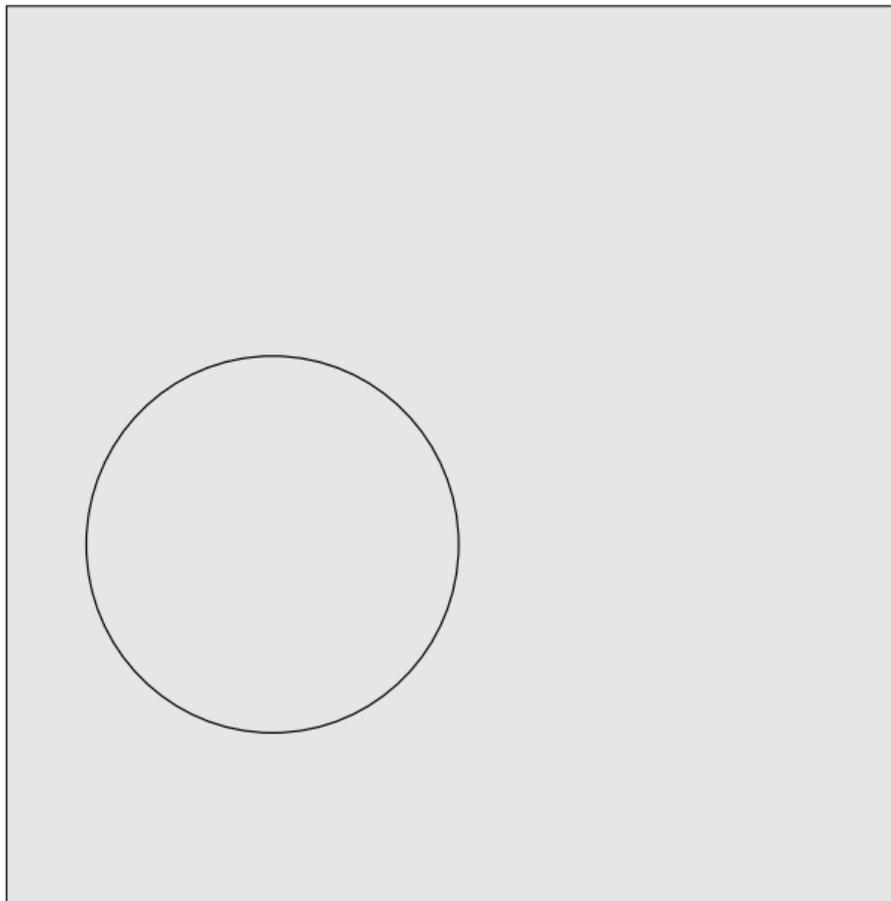
## Enumeração espacial



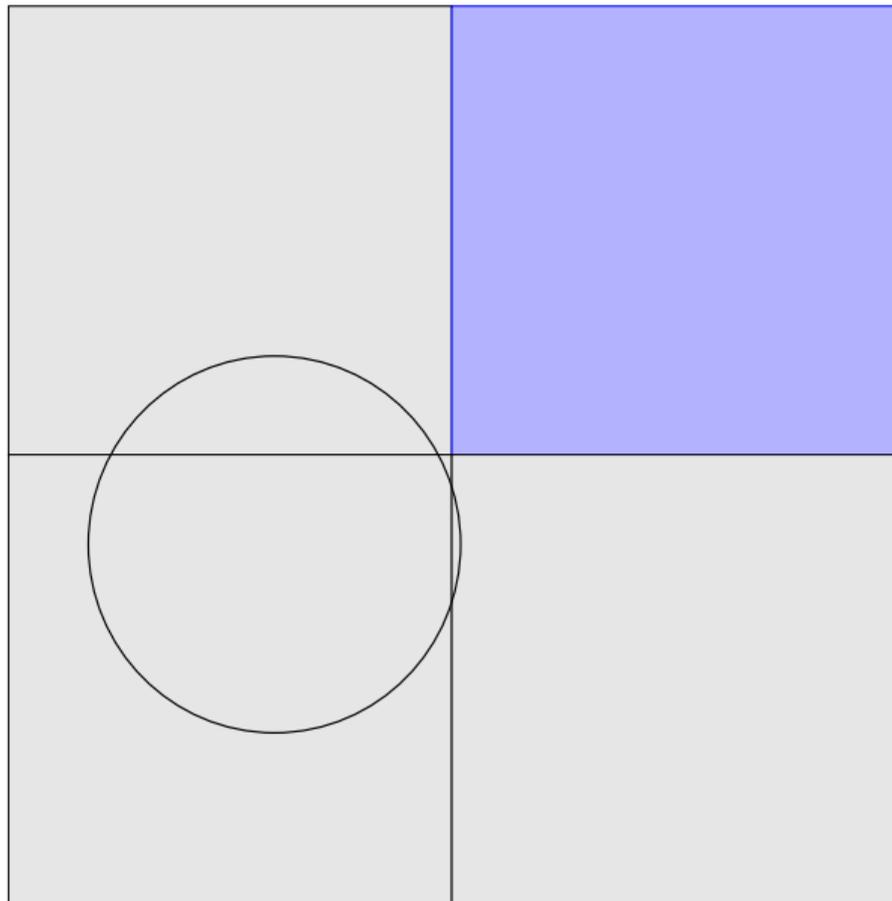
## Enumeração espacial adaptativa



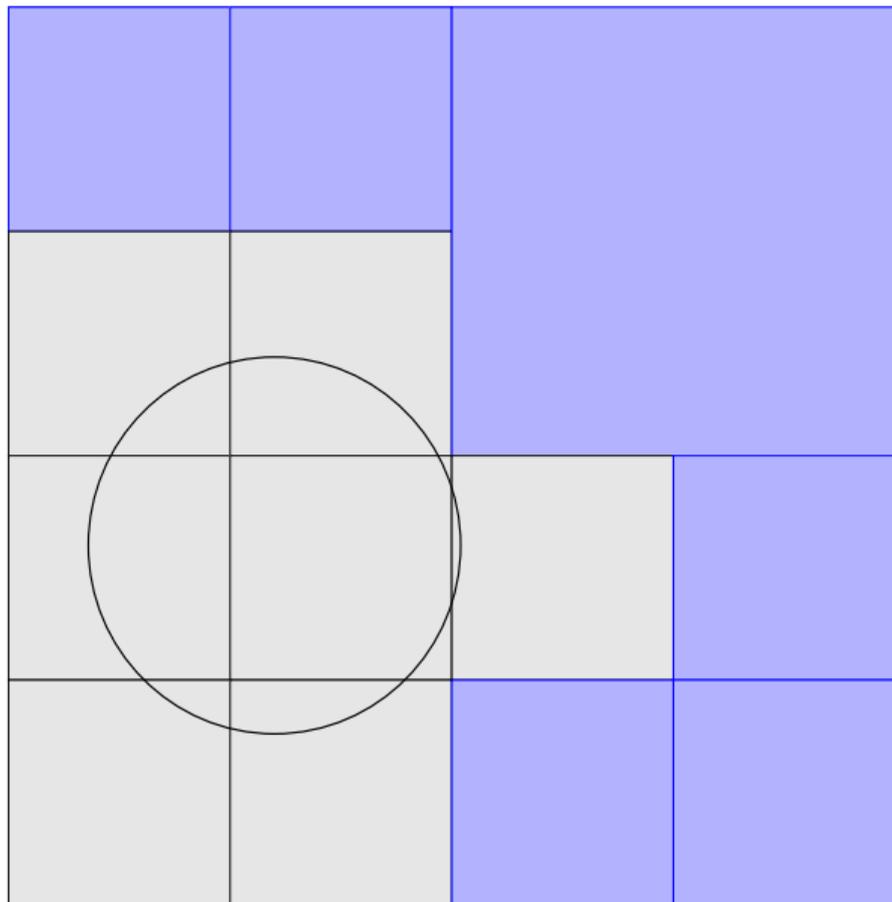
## Enumeração espacial adaptativa



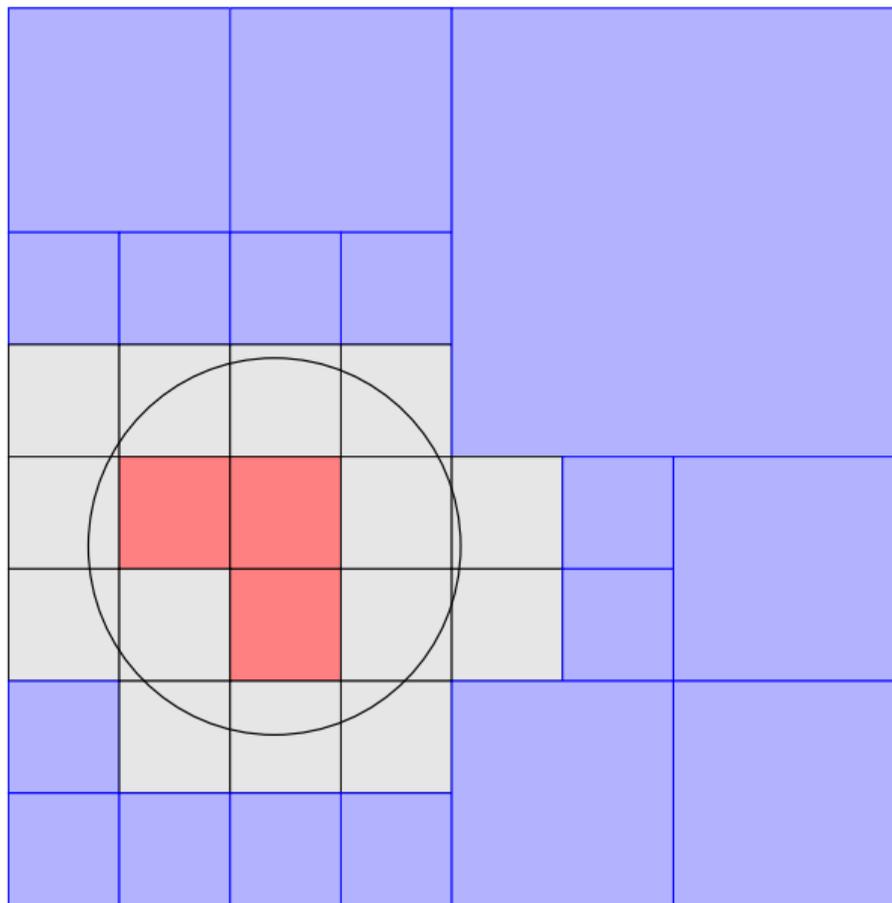
## Enumeração espacial adaptativa



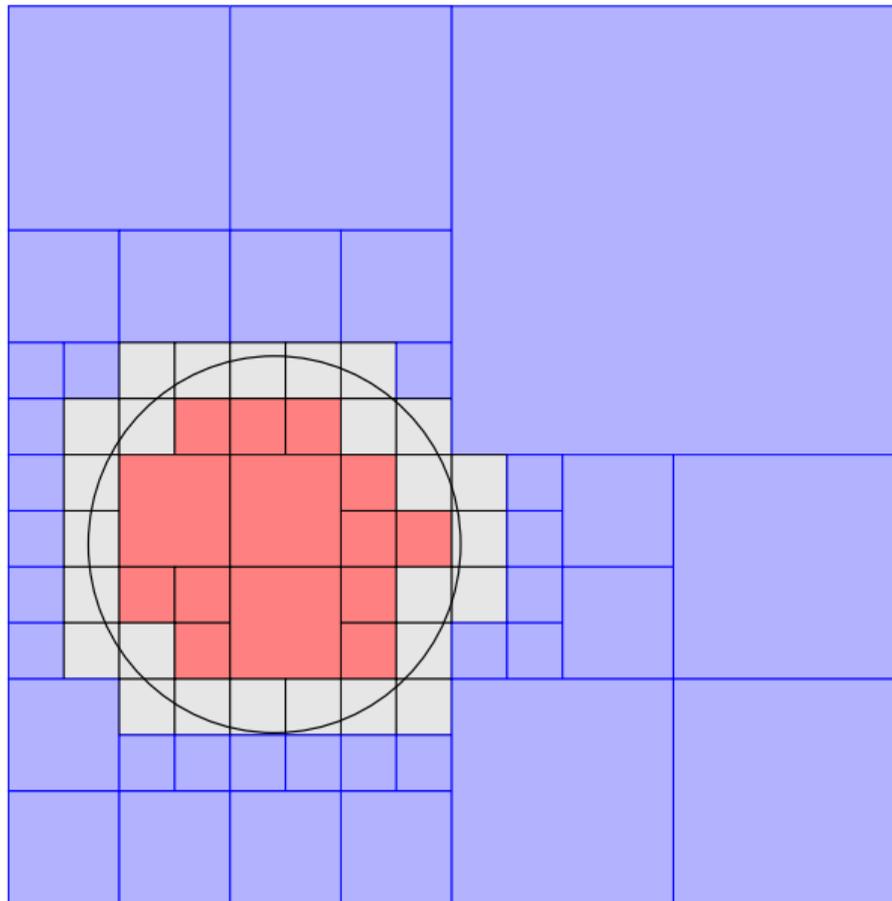
## Enumeração espacial adaptativa



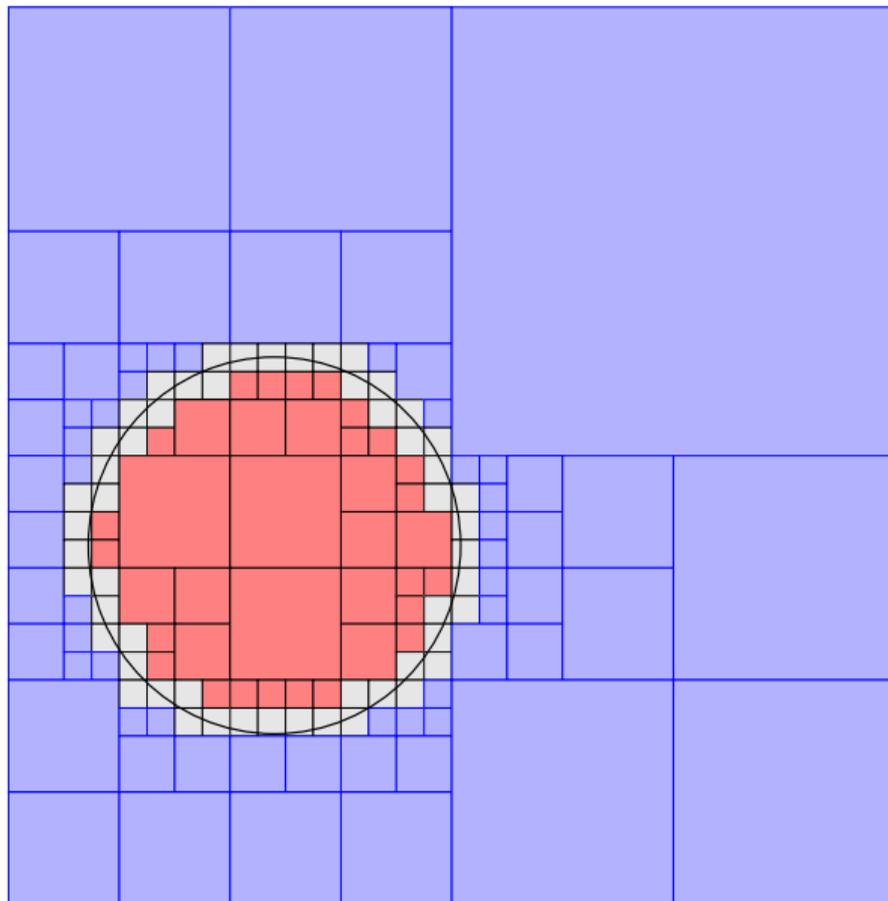
## Enumeração espacial adaptativa



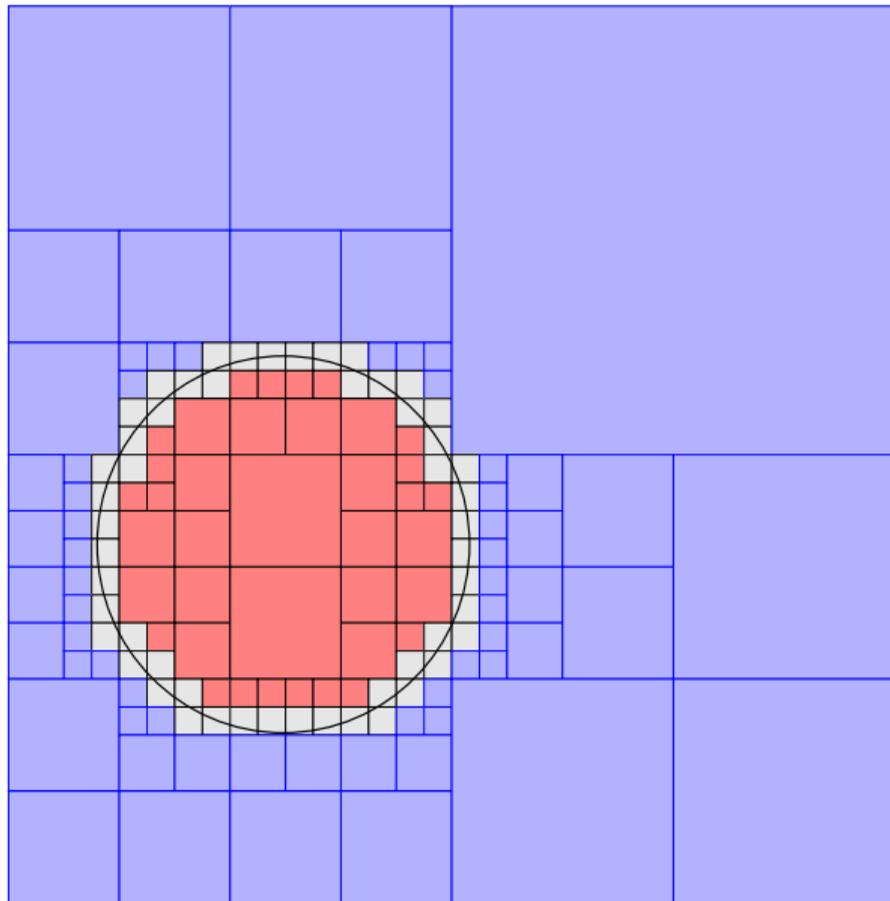
## Enumeração espacial adaptativa



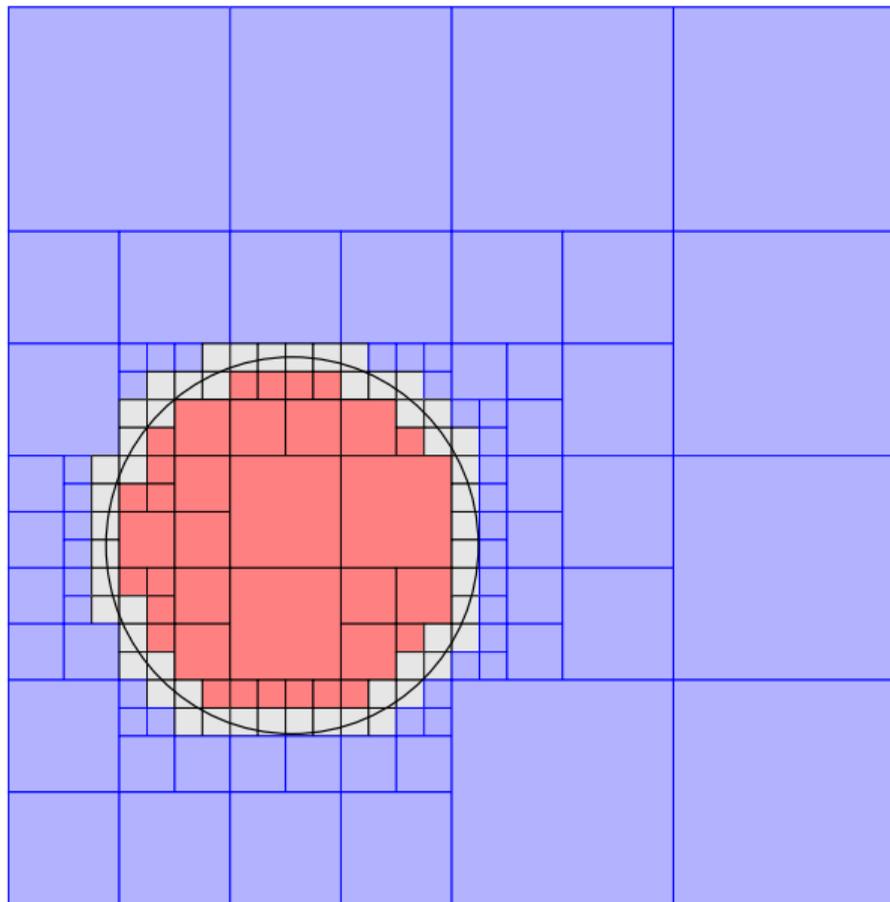
# Enumeração espacial adaptativa



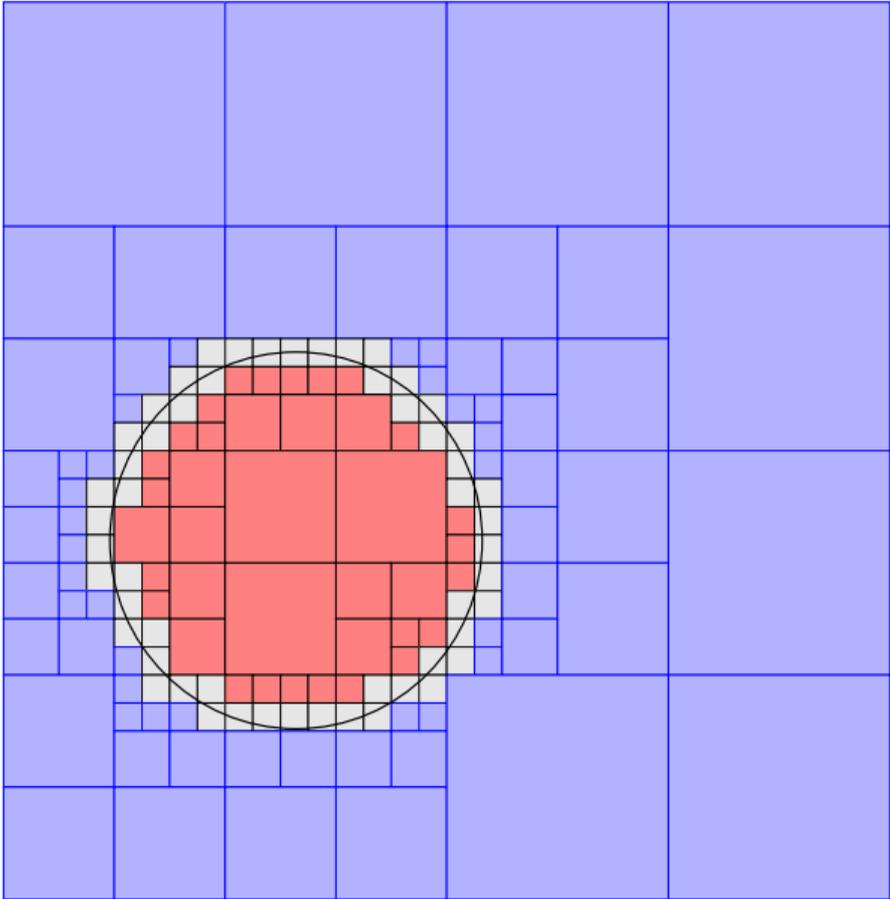
## Enumeração espacial adaptativa



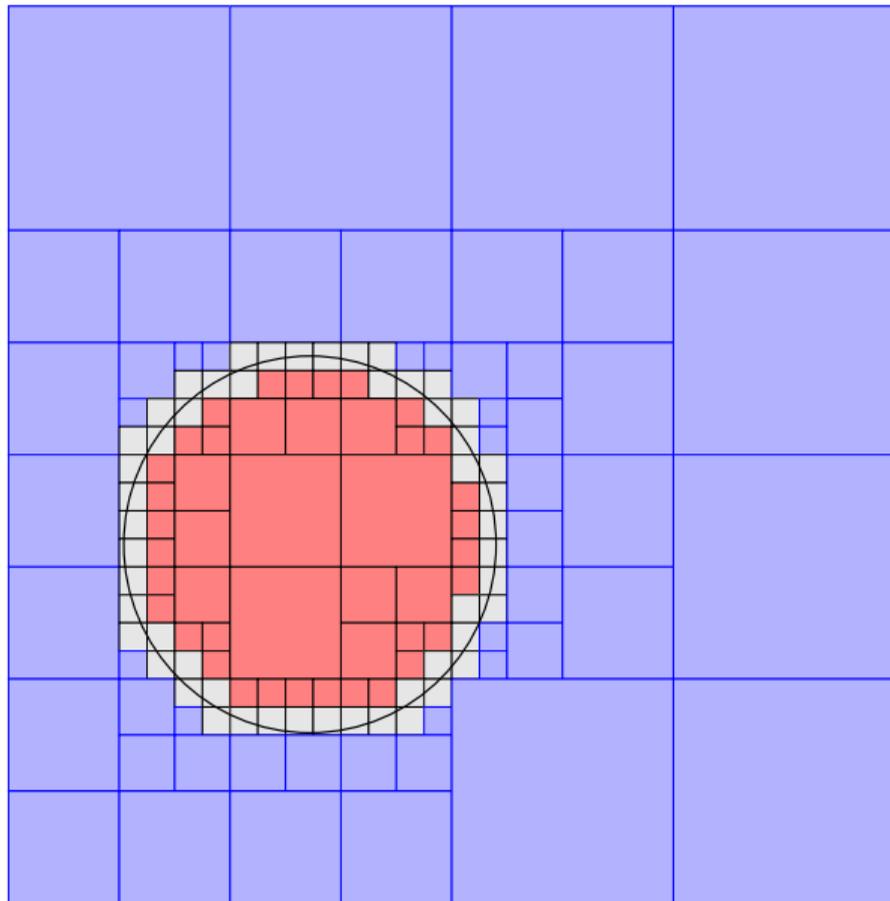
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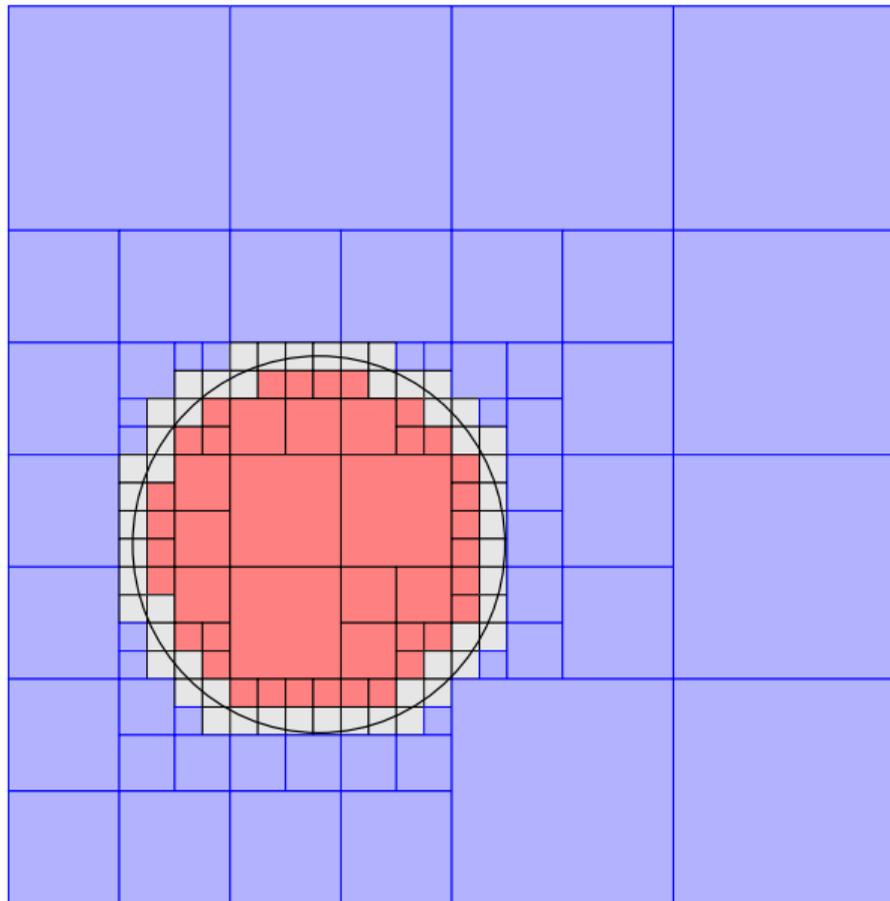
# Enumeração espacial adaptativa



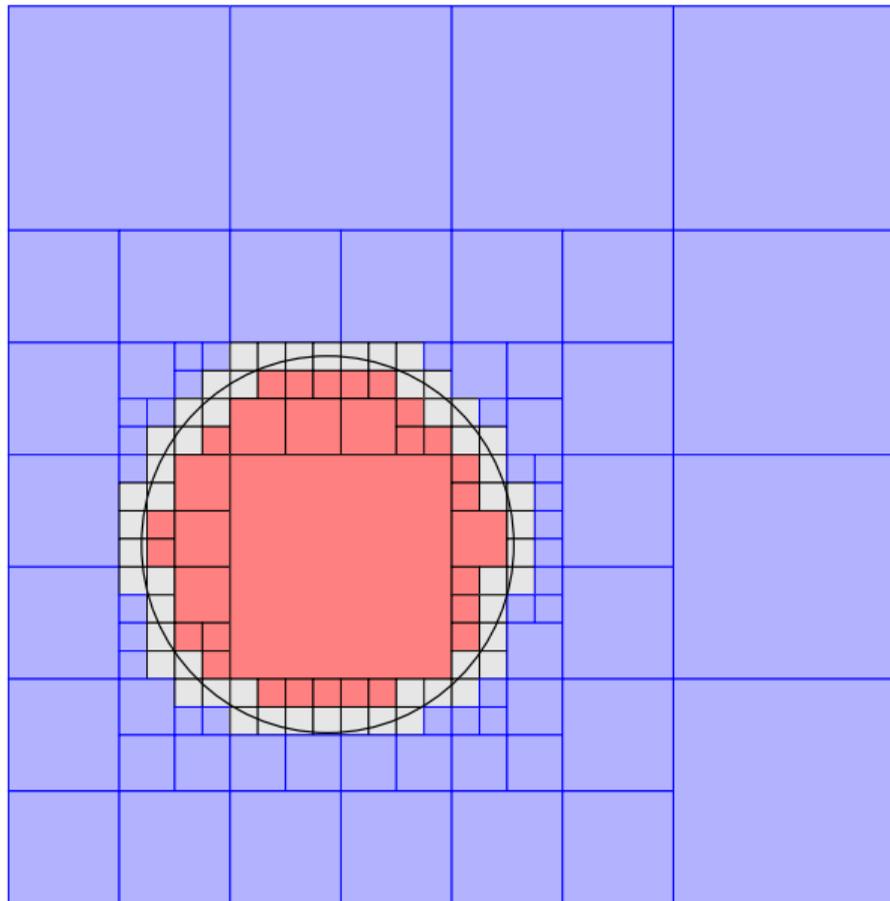
## Enumeração espacial adaptativa



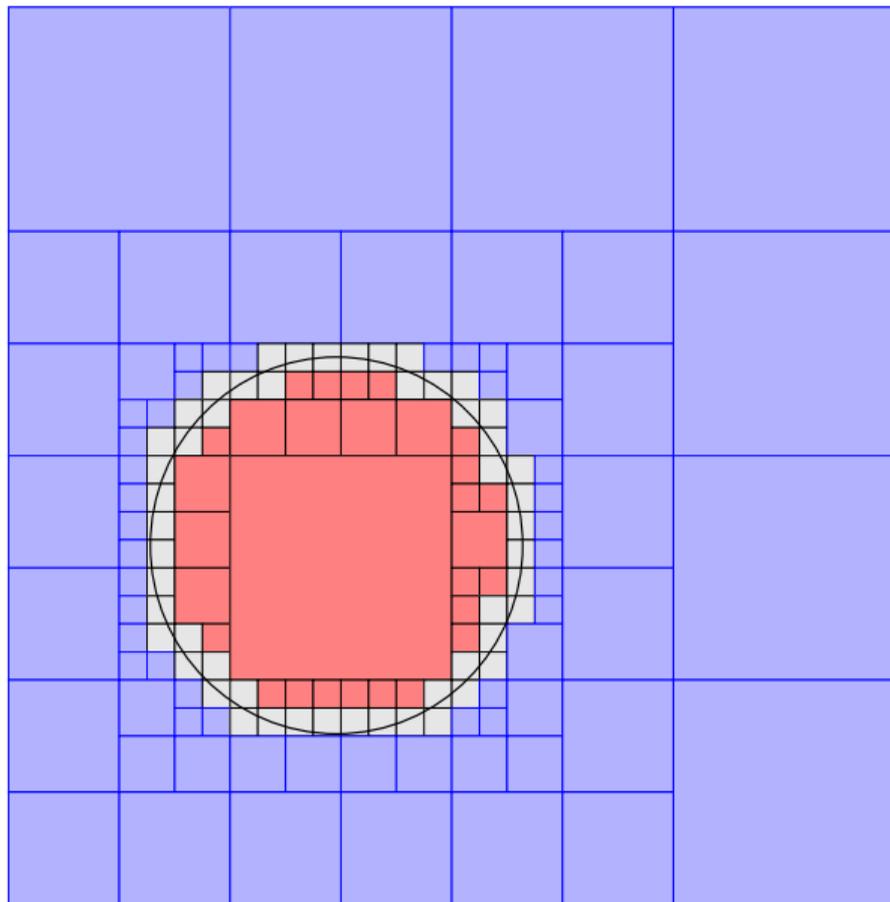
## Enumeração espacial adaptativa



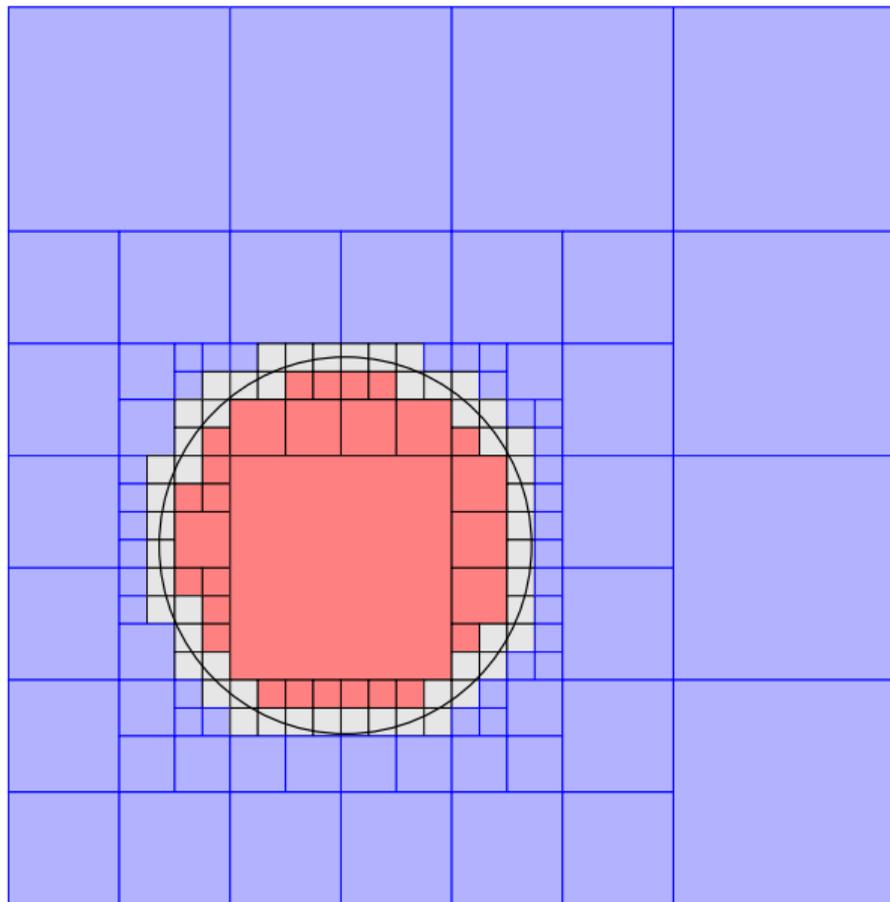
## Enumeração espacial adaptativa



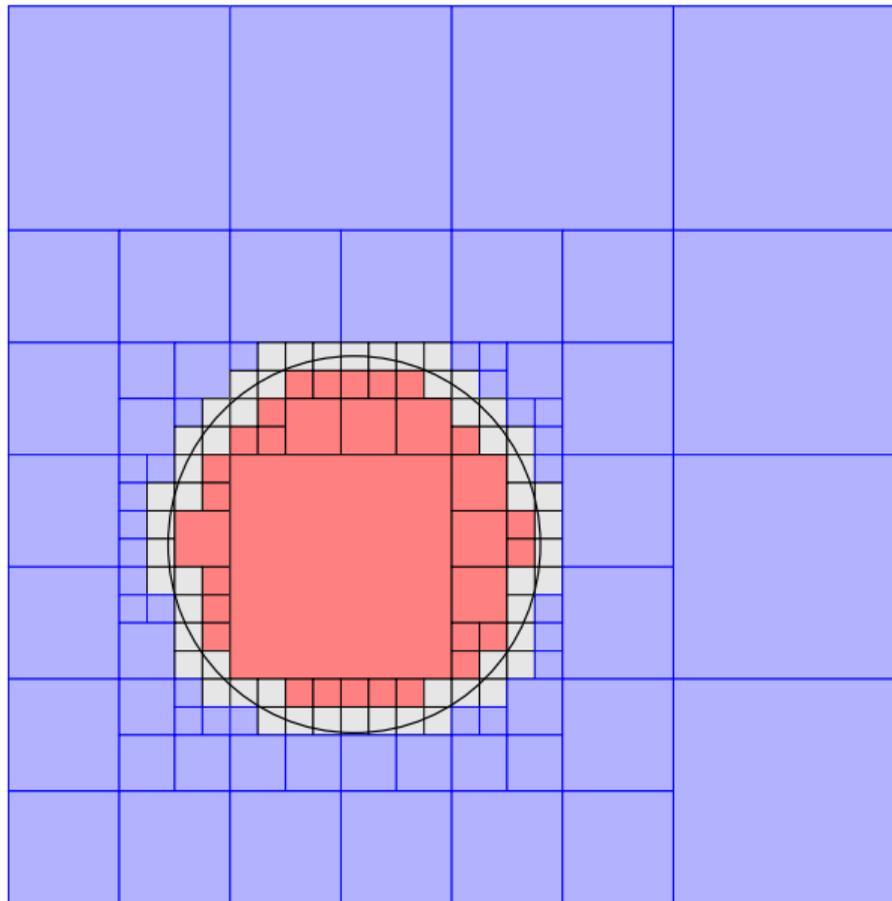
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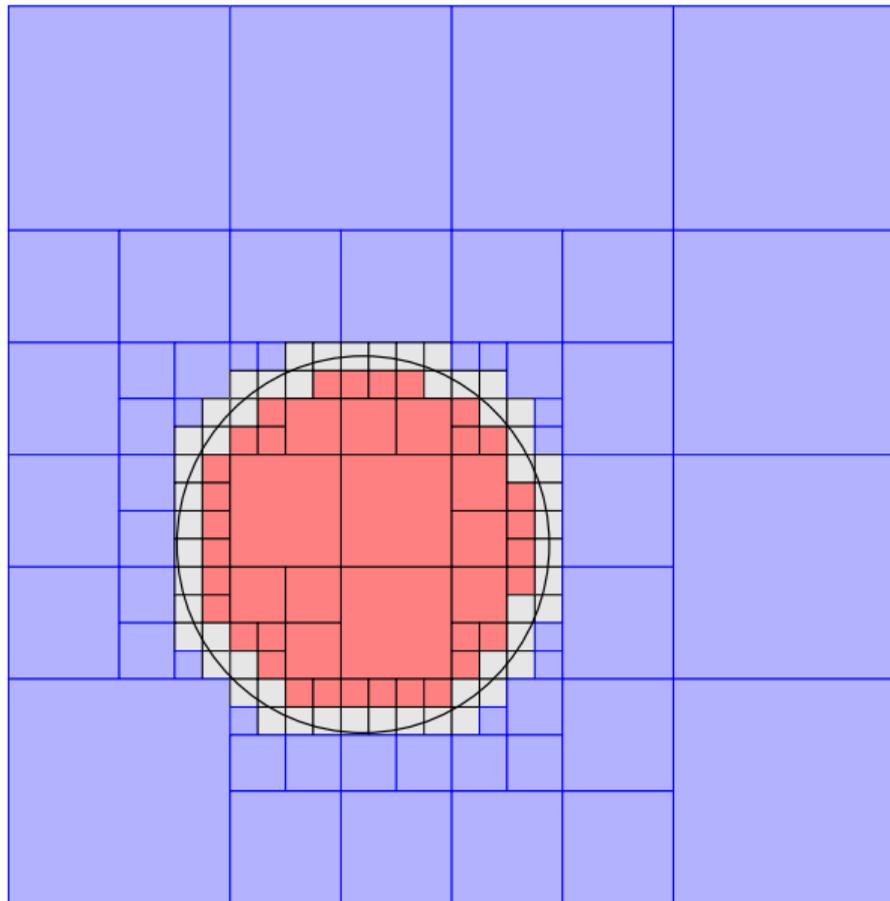
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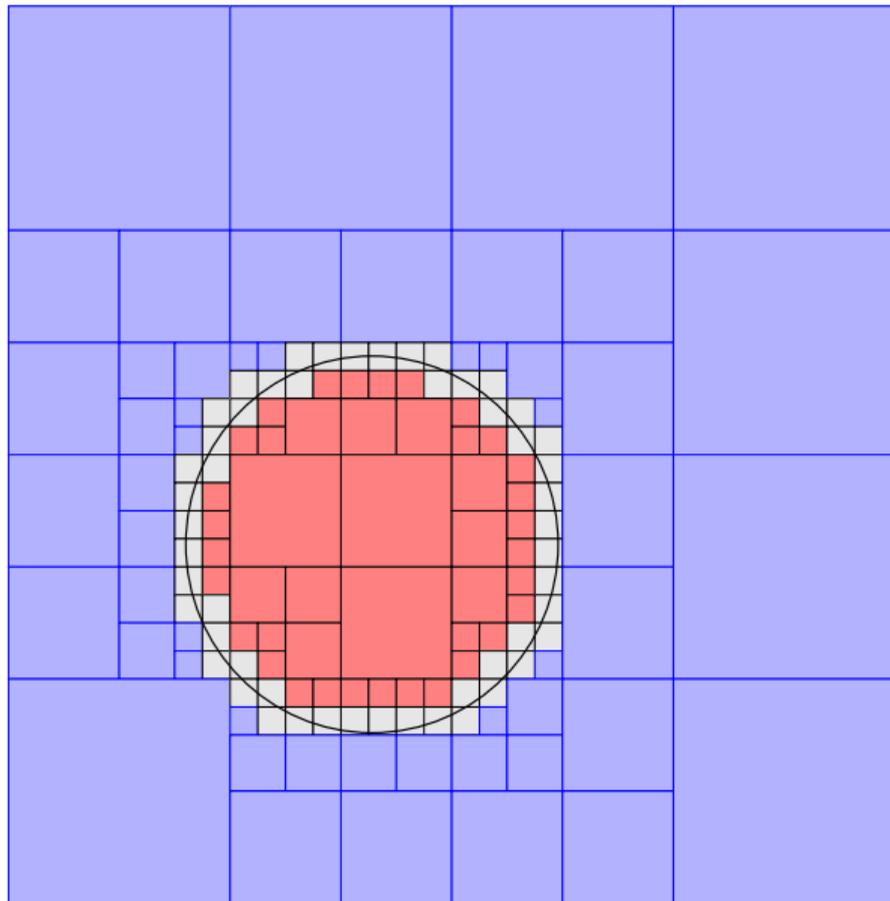
## Enumeração espacial adaptativa



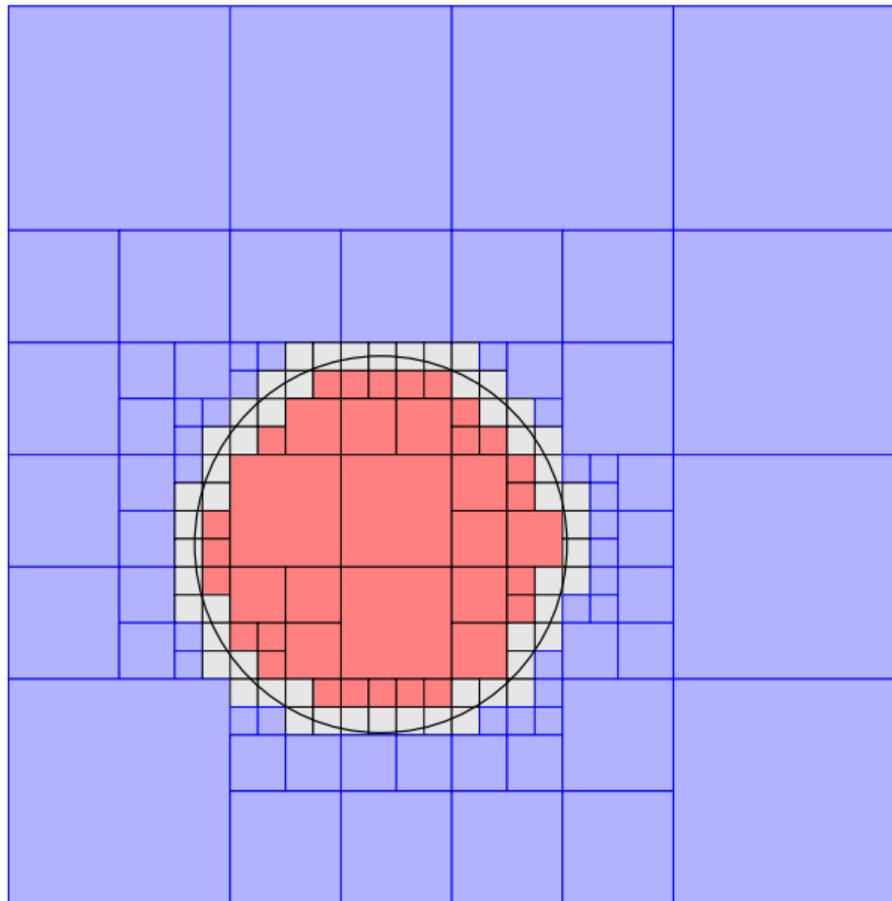
## Enumeração espacial adaptativa



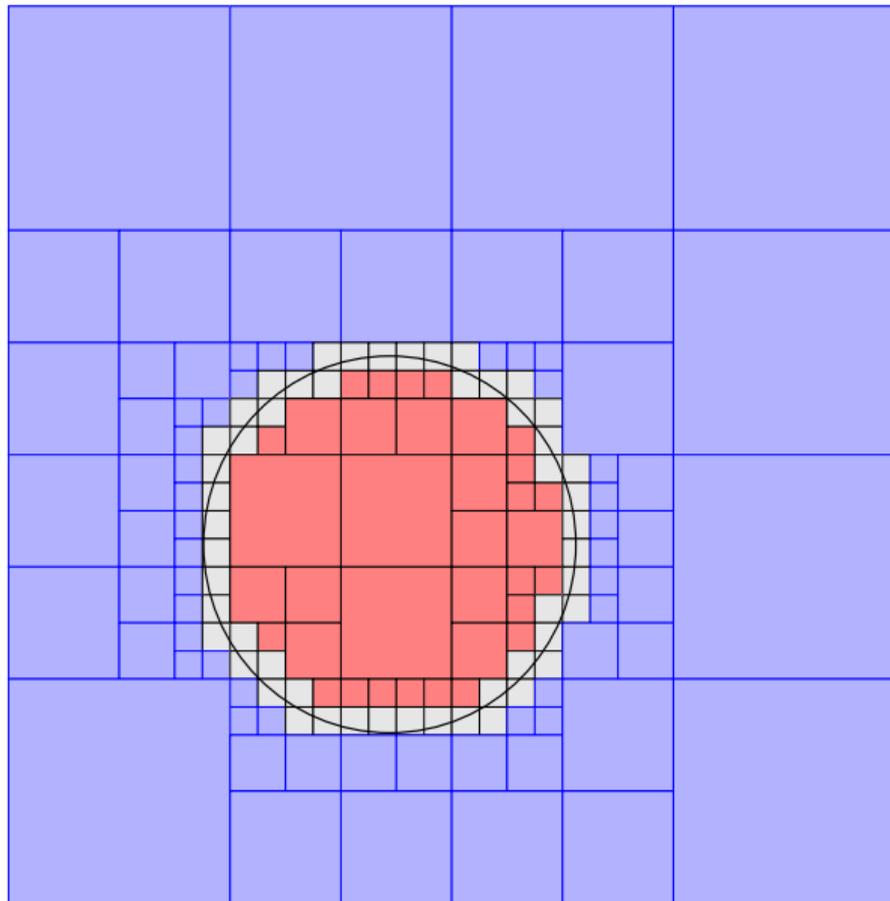
## Enumeração espacial adaptativa



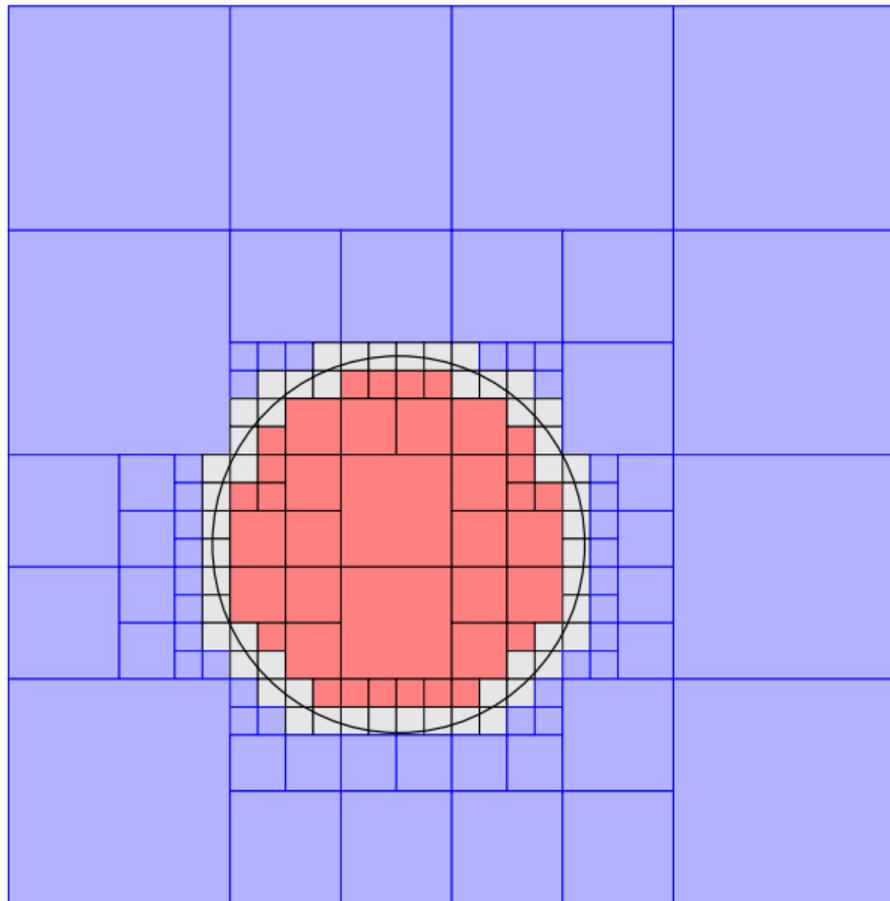
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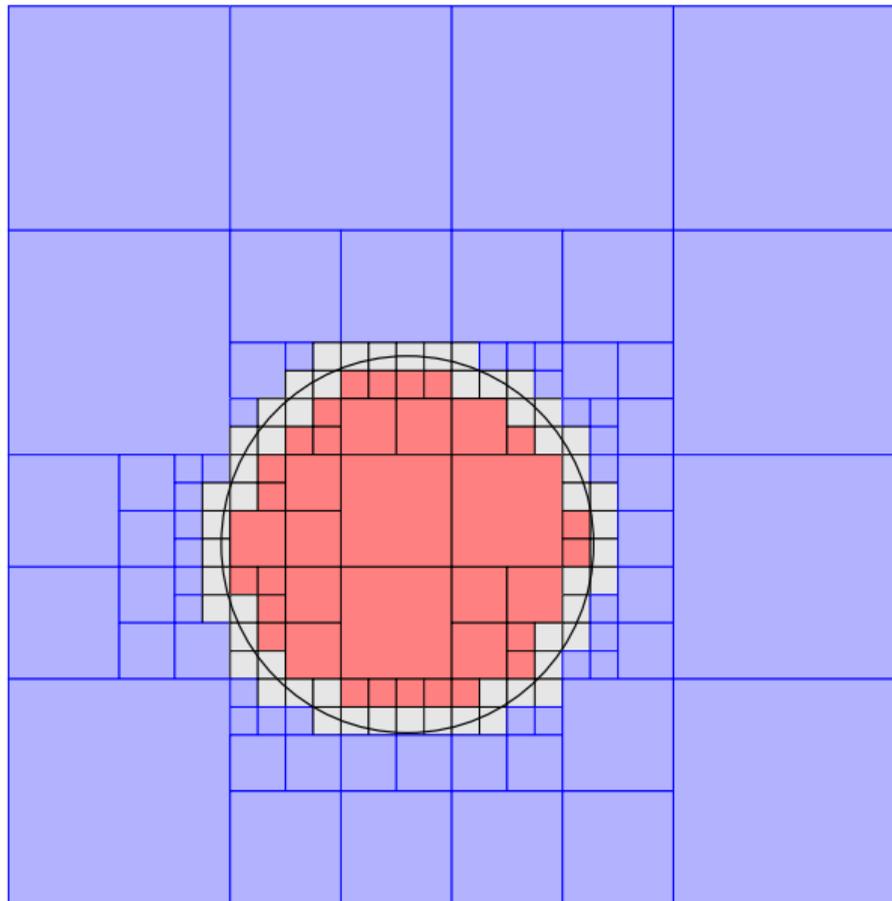
## Enumeração espacial adaptativa



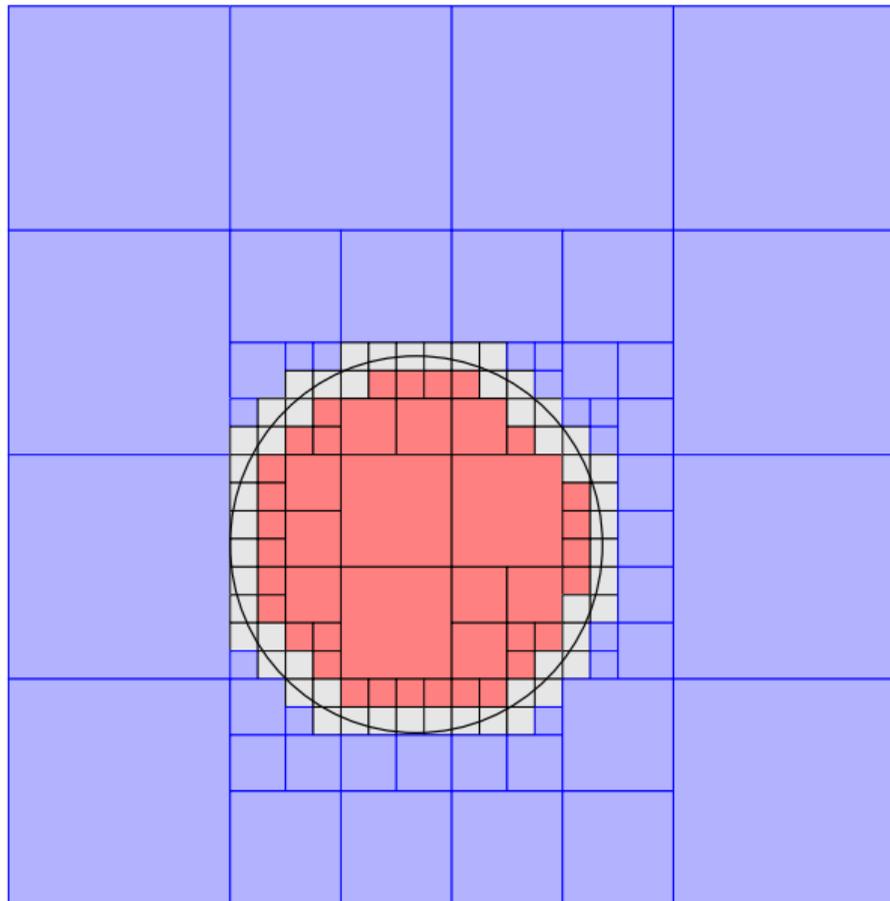
## Enumeração espacial adaptativa



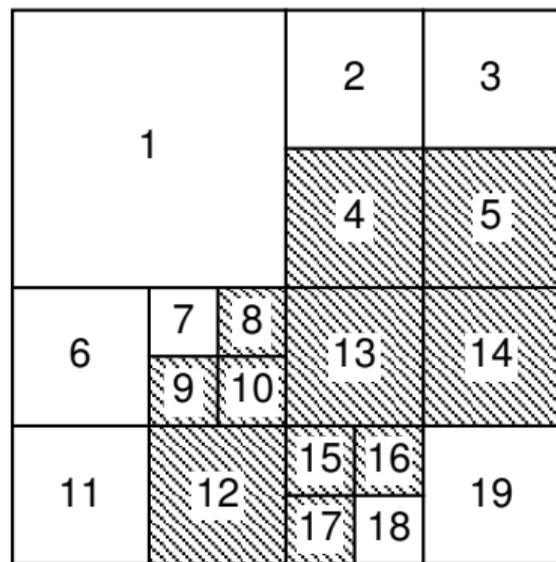
## Enumeração espacial adaptativa



# Enumeração espacial adaptativa

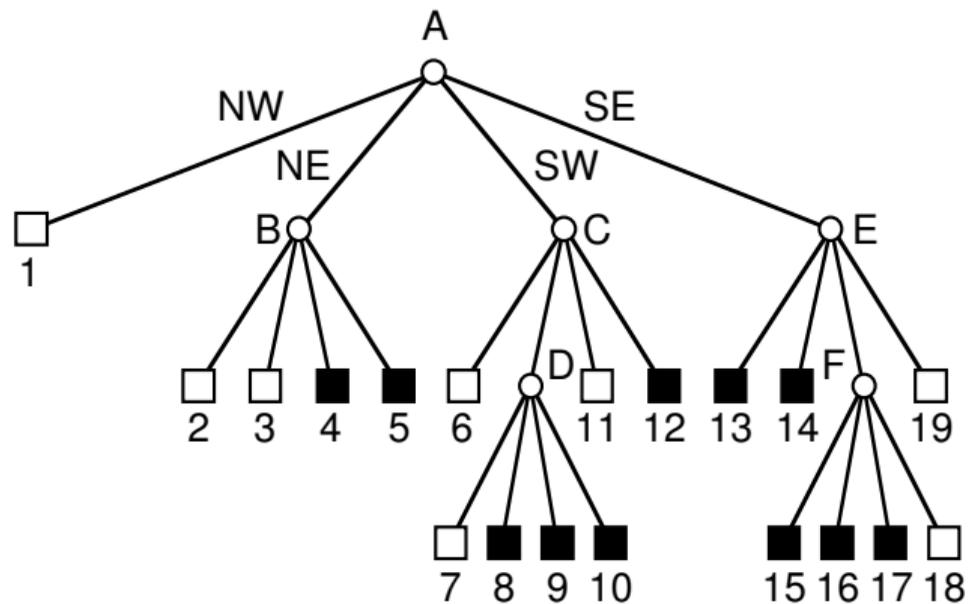


# Quadtree



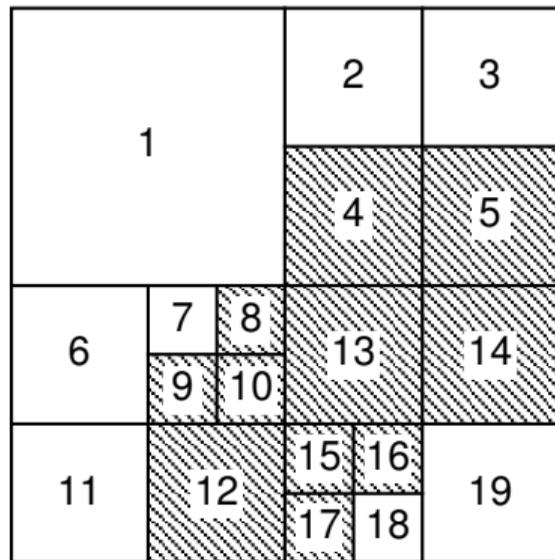
Samet

Finkel-Bentley (1974)

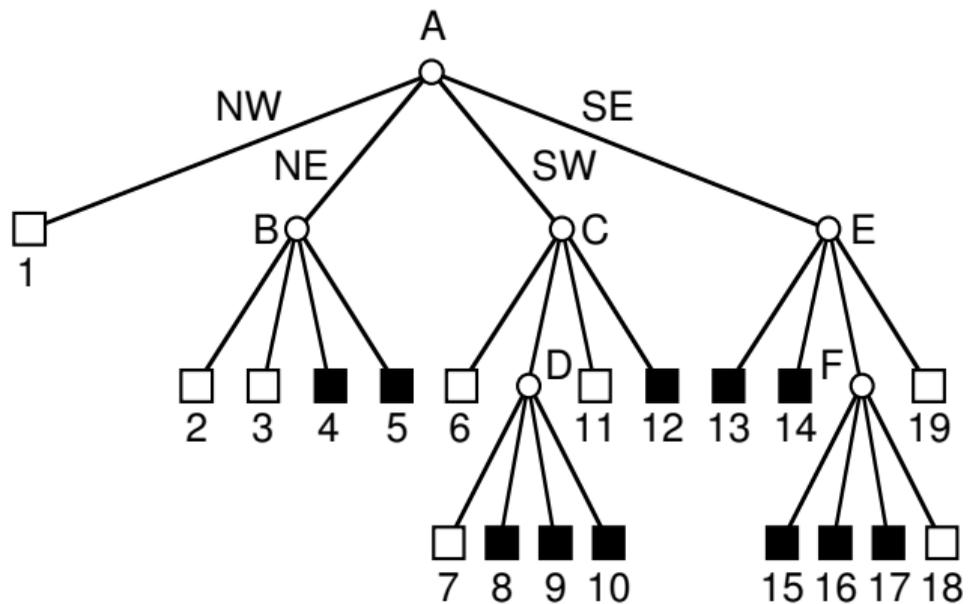


# Quadtree

Finkel-Bentley (1974)



Samet

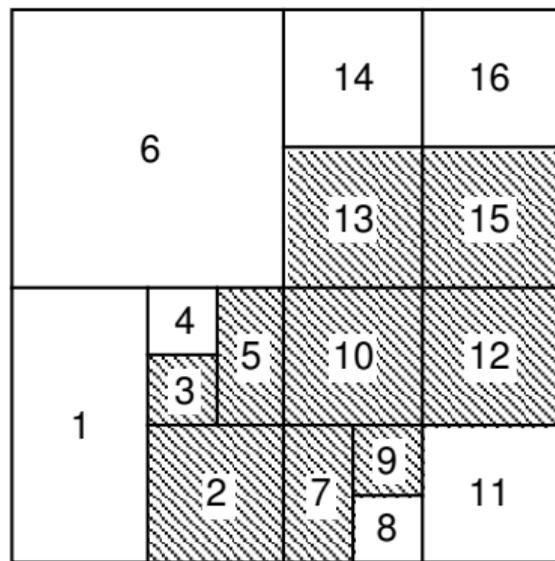


(B(BBPP)(B(BPPP)BP)(PP(PPP)B))

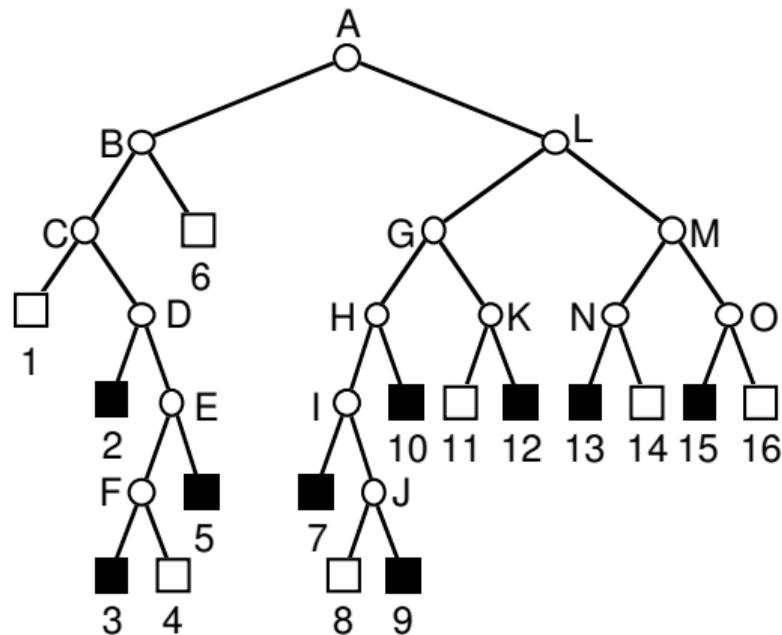


# Bintree

Tamminen et al. (1984)



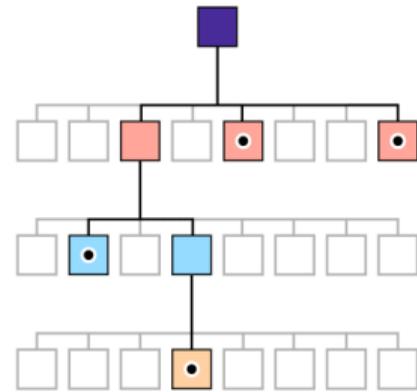
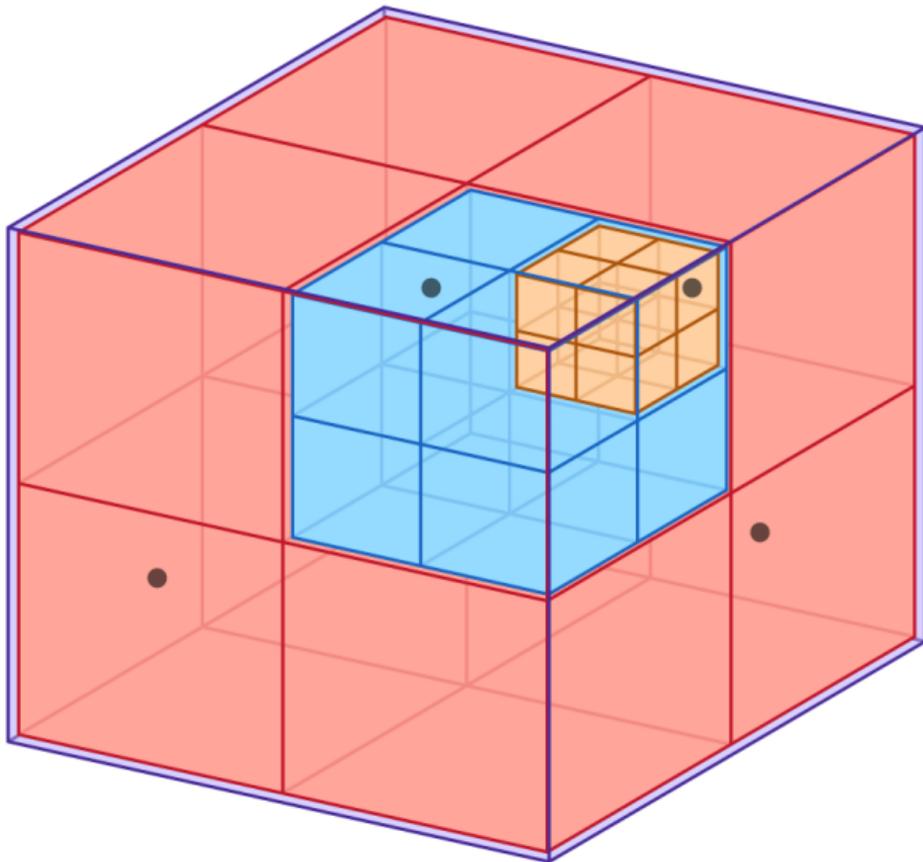
Samet

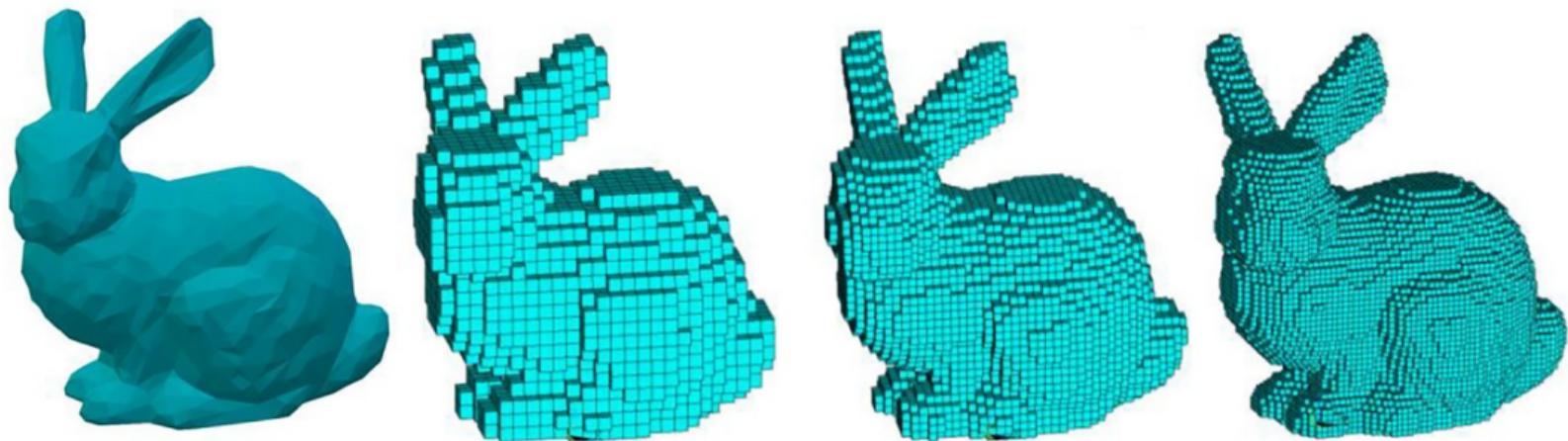


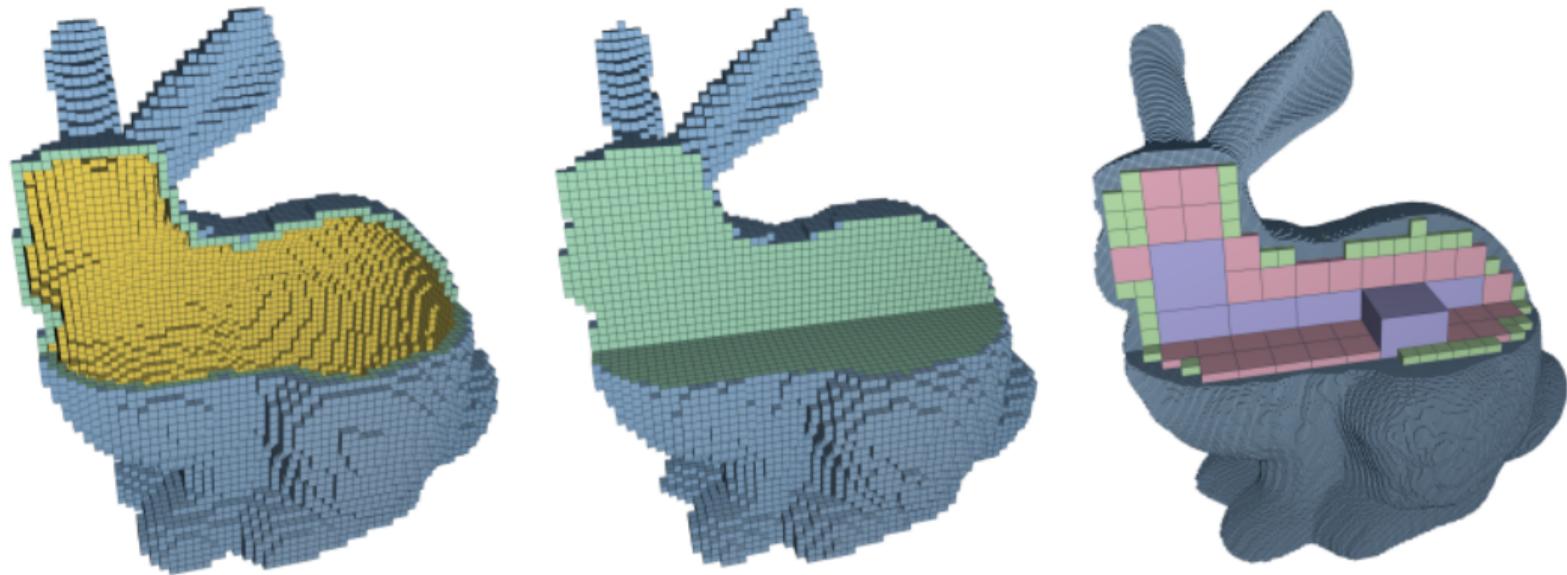
$((((B(P((PB)P)))B)((((P(BP))P)(BP))((PB)(PB))))))$

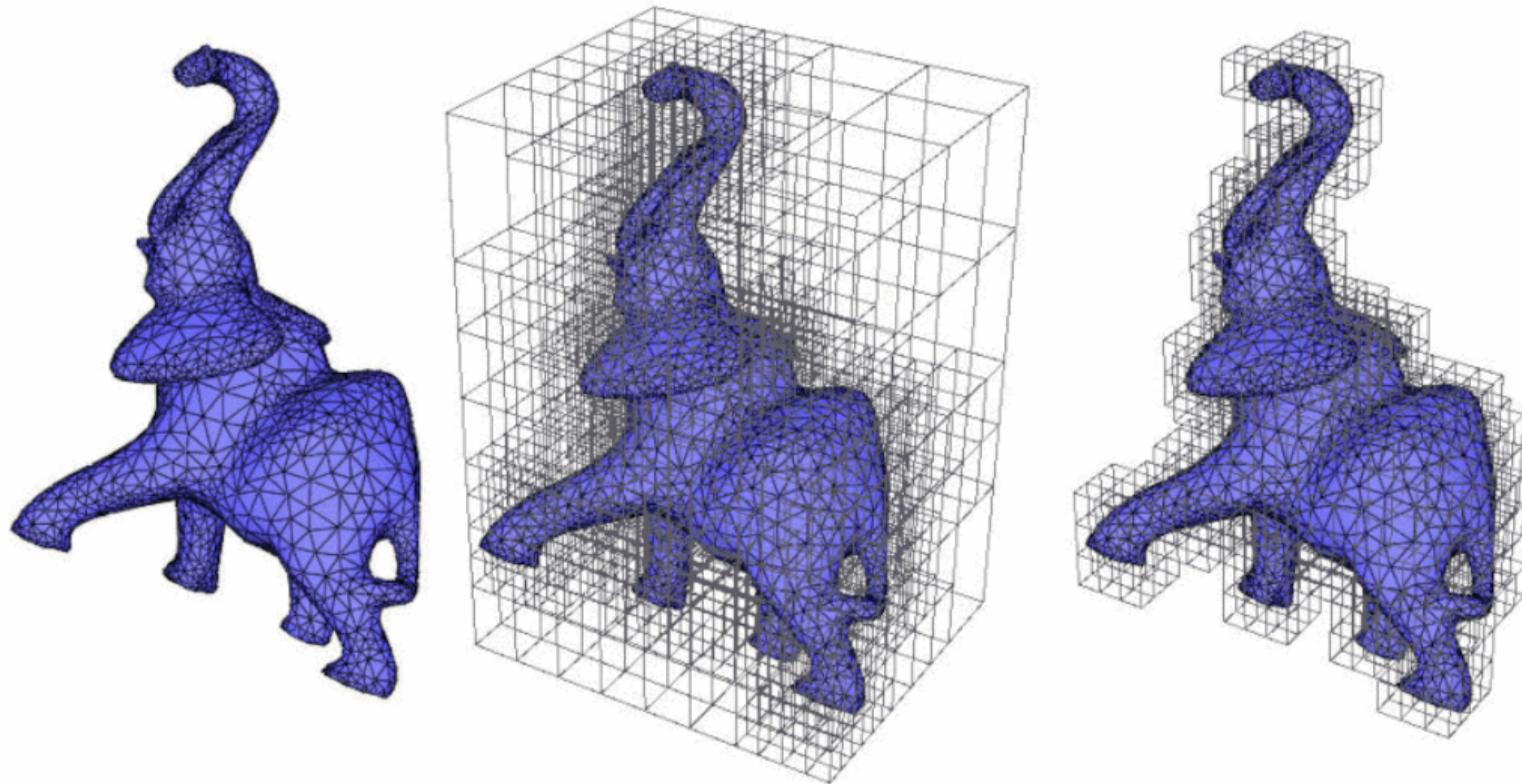
# Octree

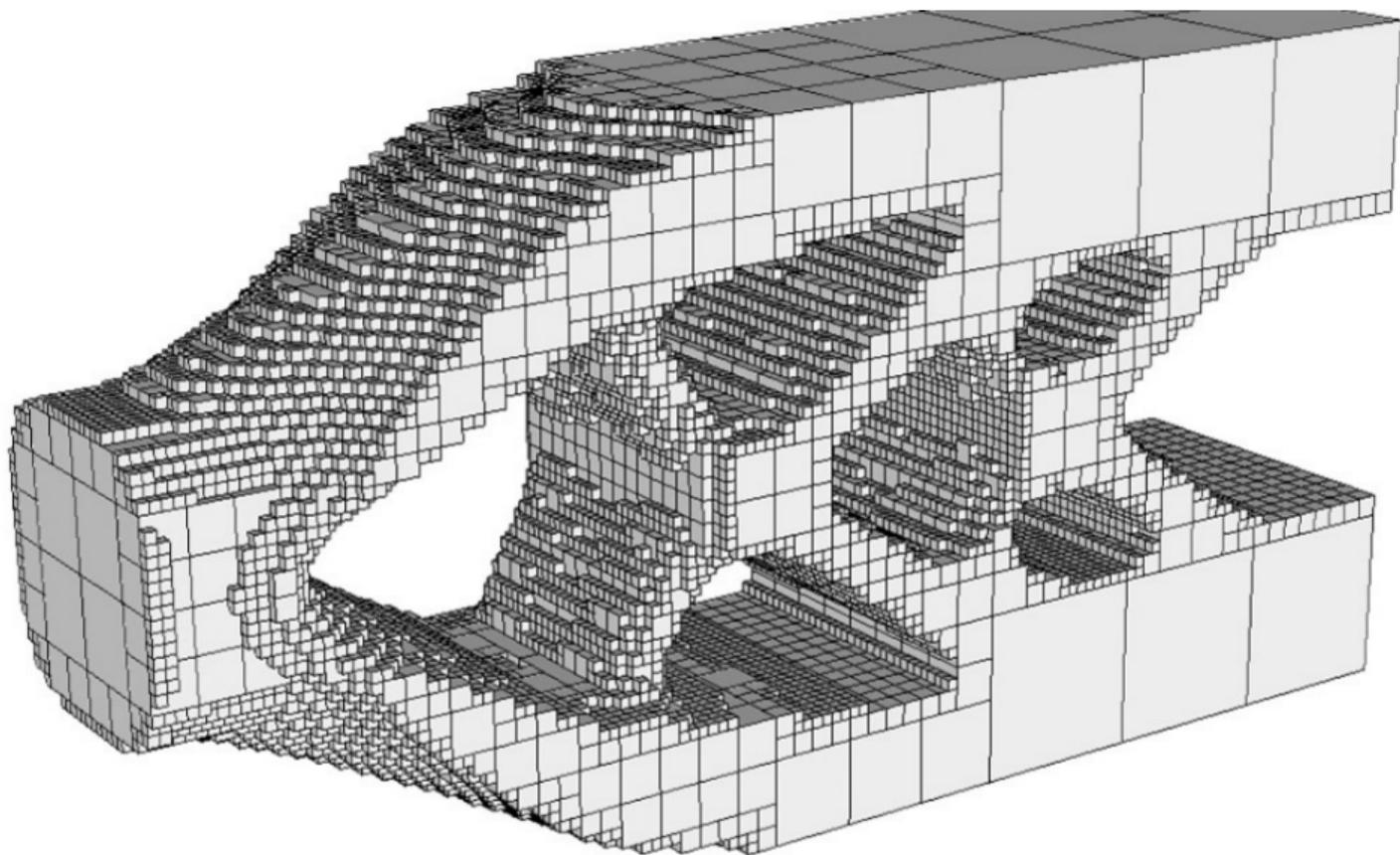
Meagher (1980)

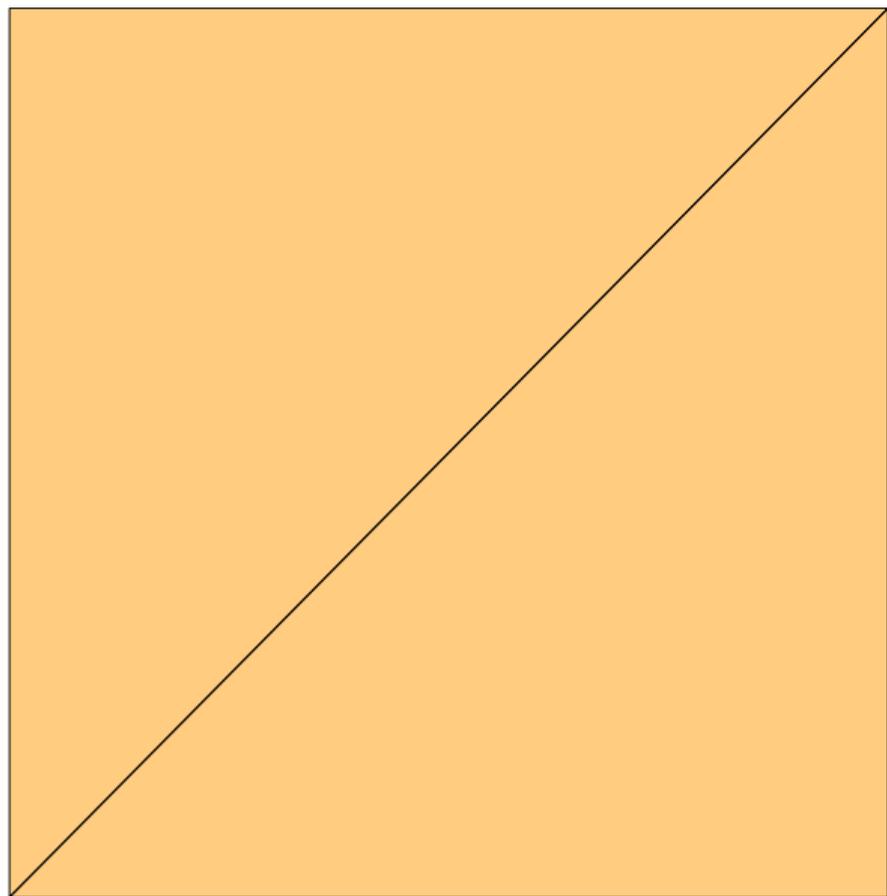


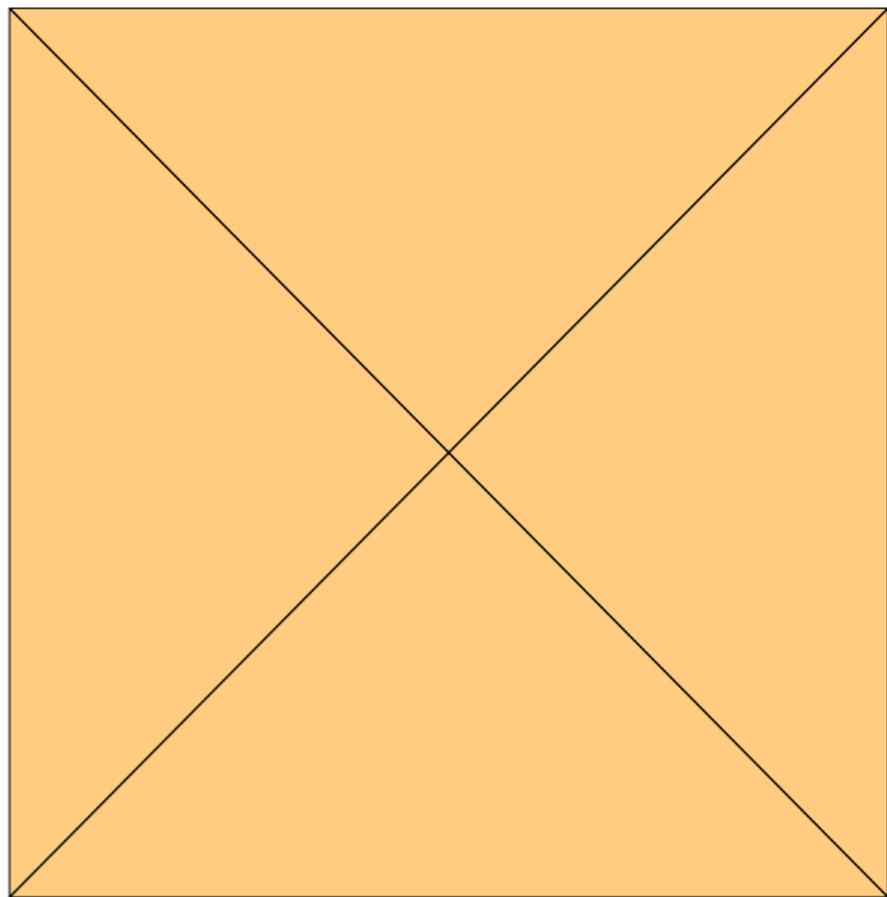


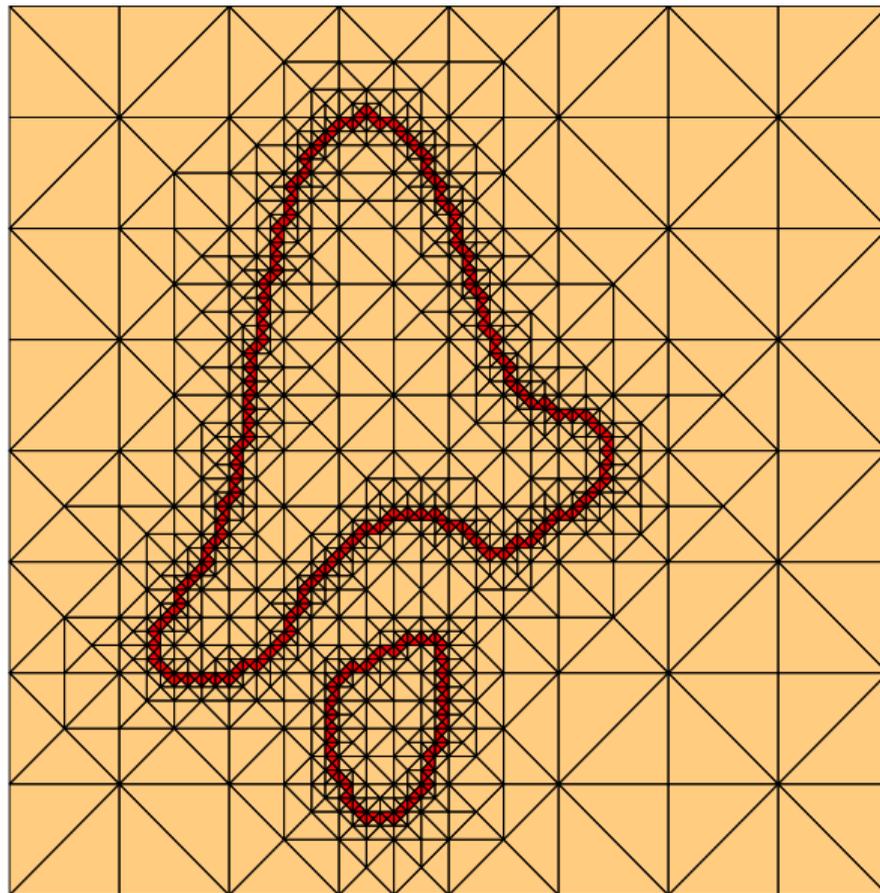






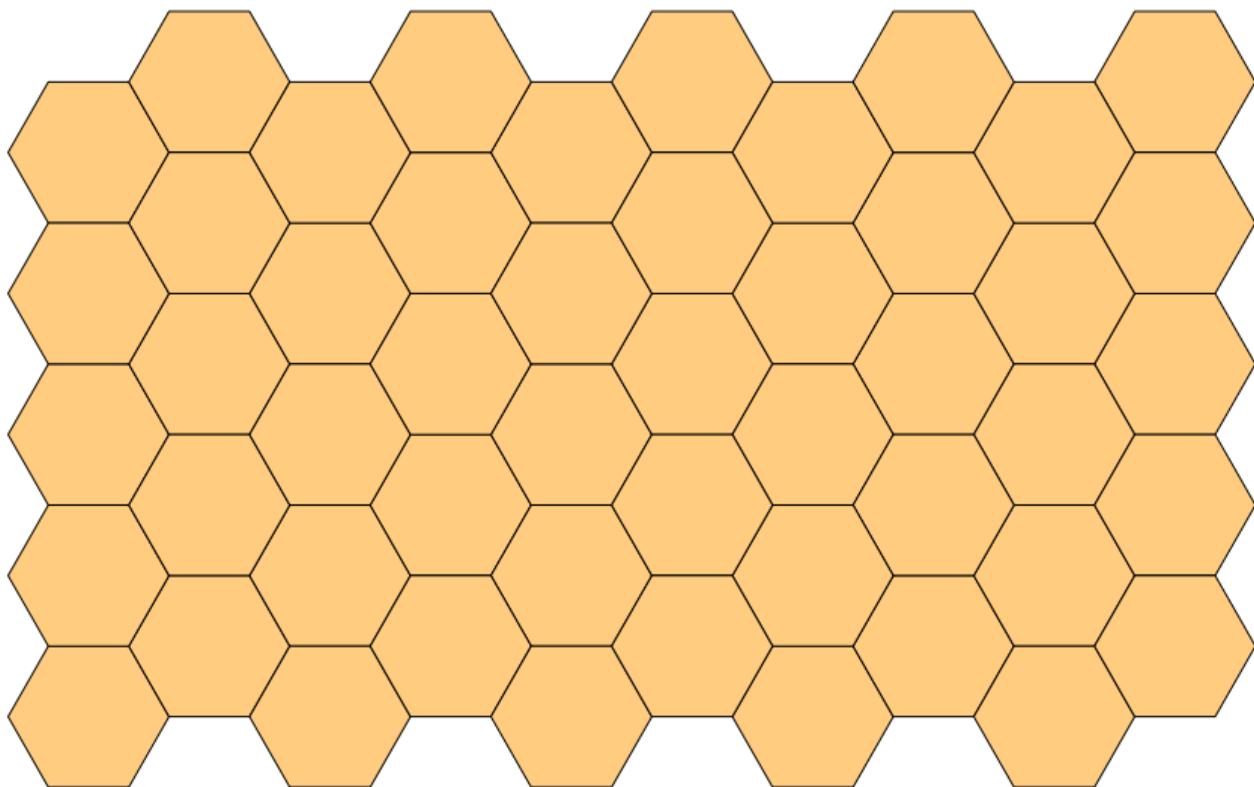






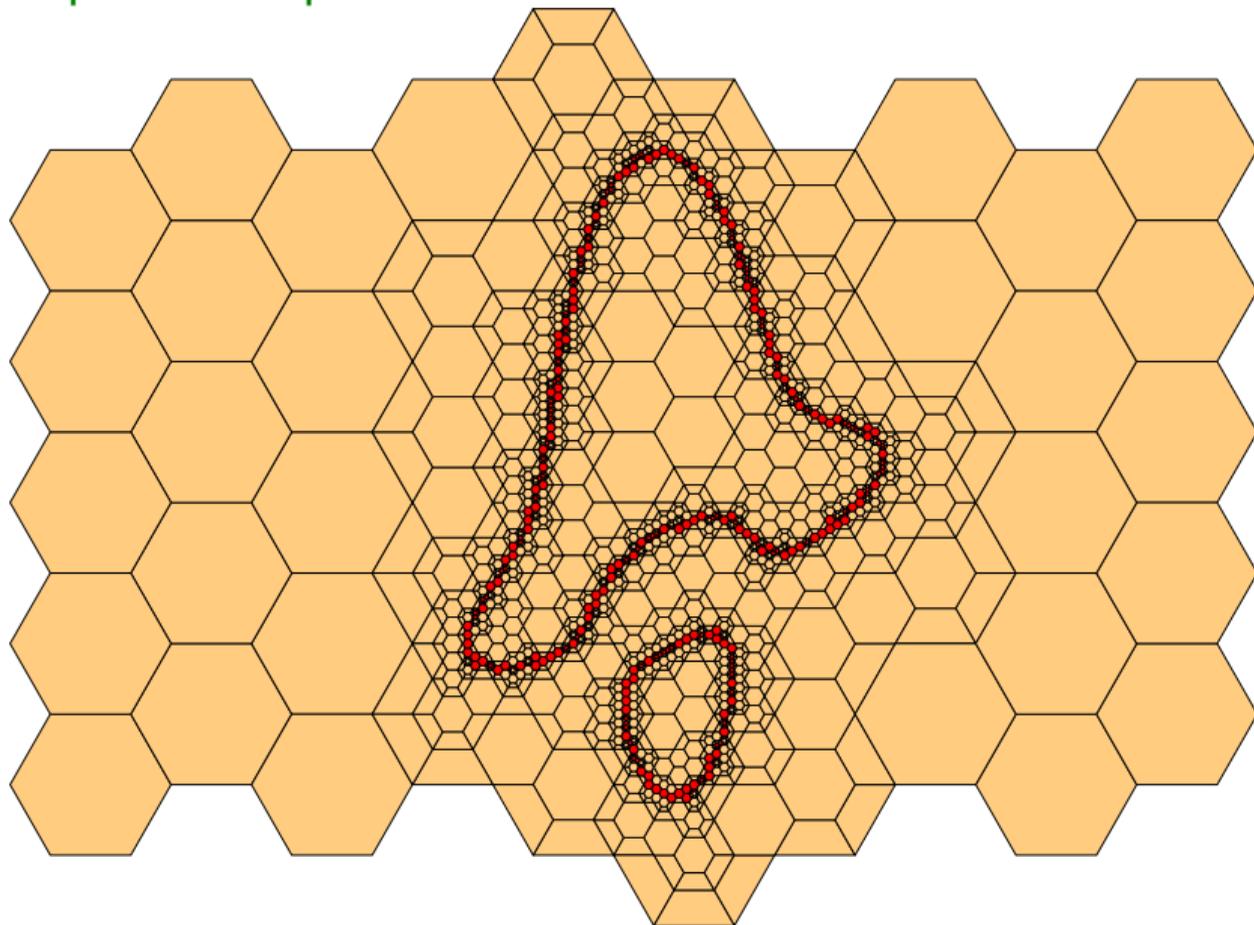
Enumeração espacial adaptativa

hexagonal



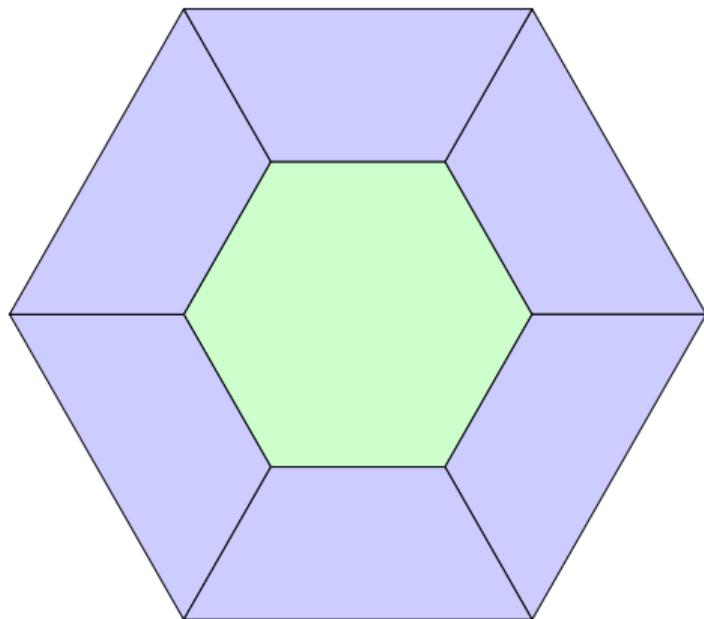
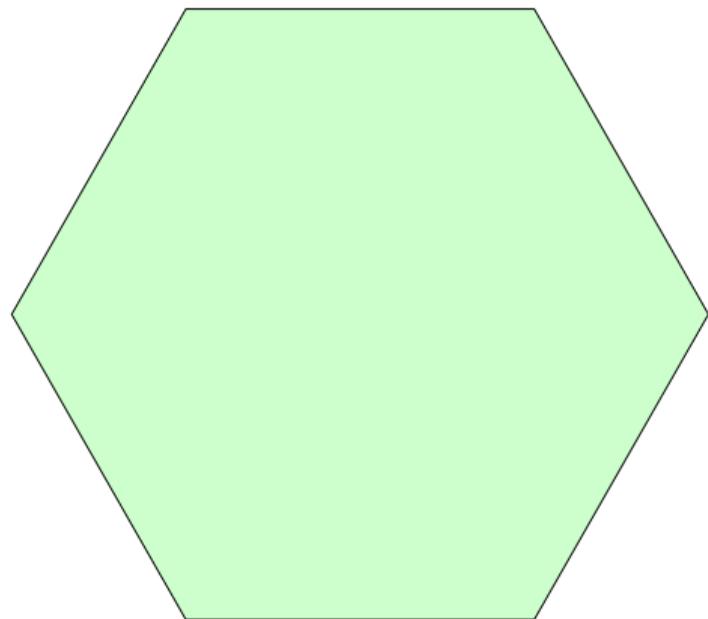
# Enumeração espacial adaptativa

hexagonal



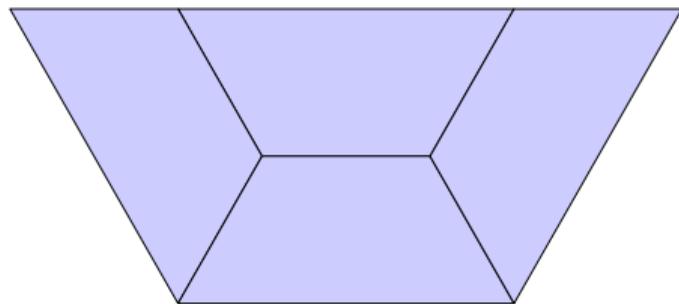
Enumeração espacial adaptativa

hexagonal



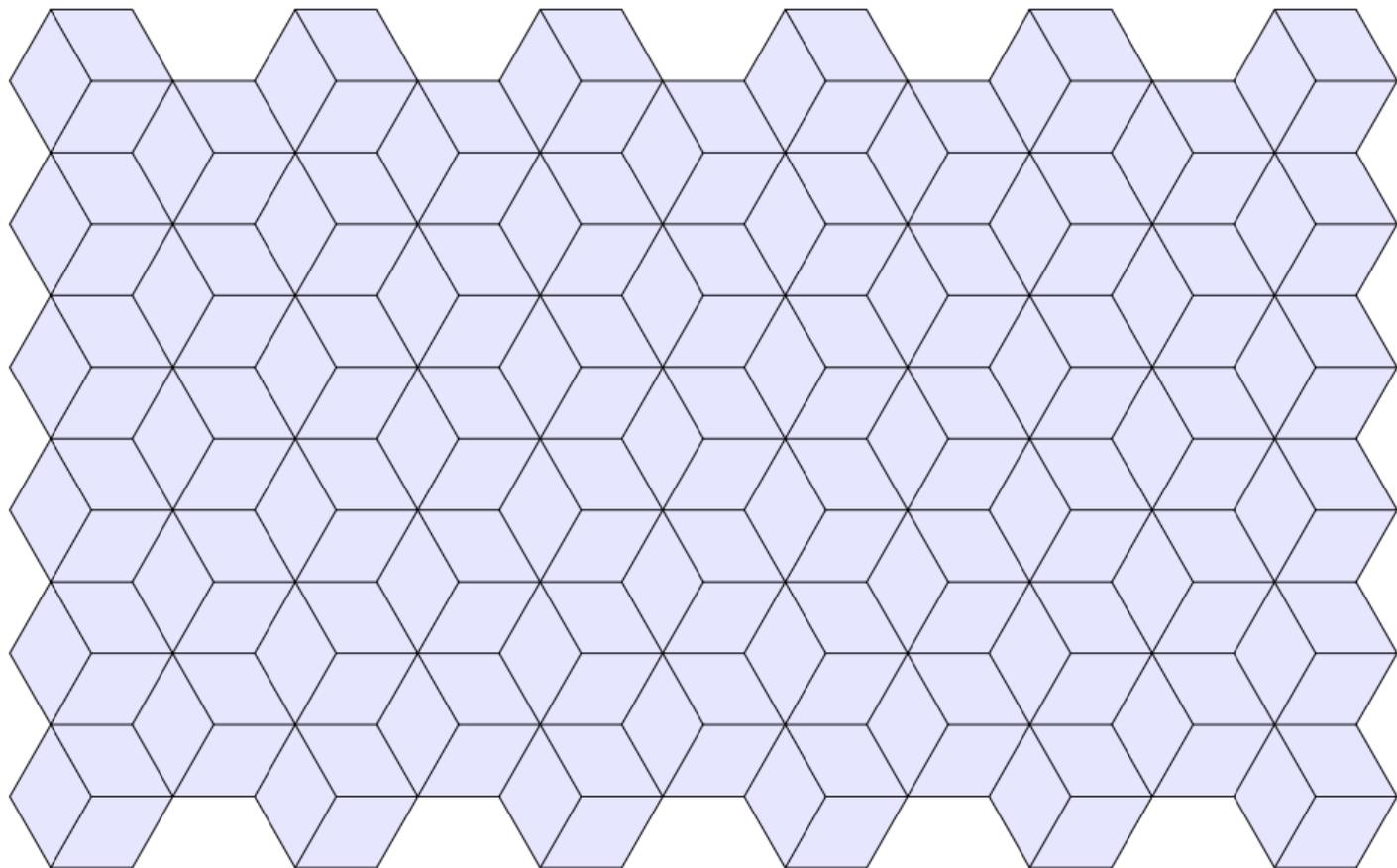
# Enumeração espacial adaptativa

hexagonal



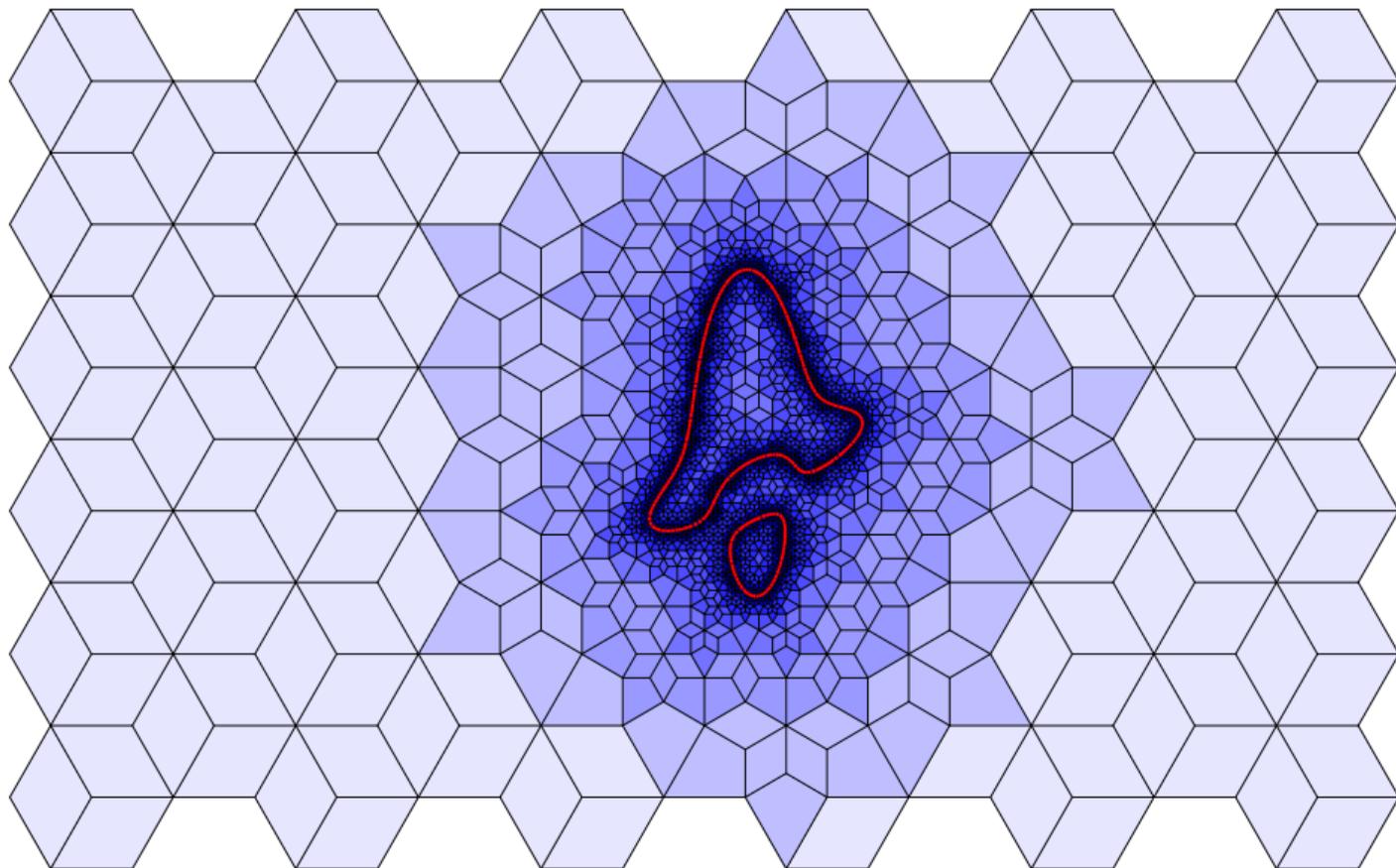
## Enumeração espacial adaptativa

dk



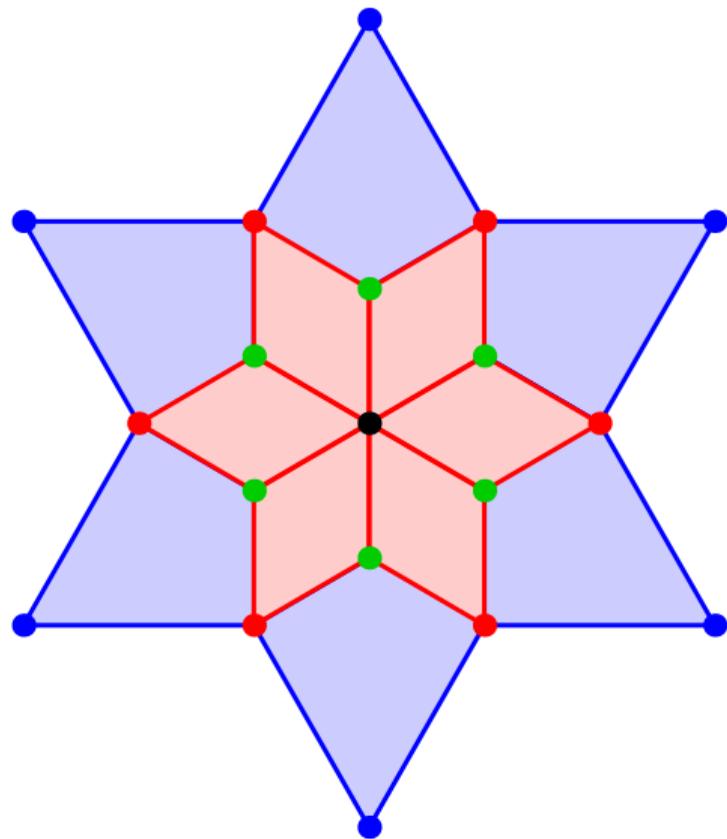
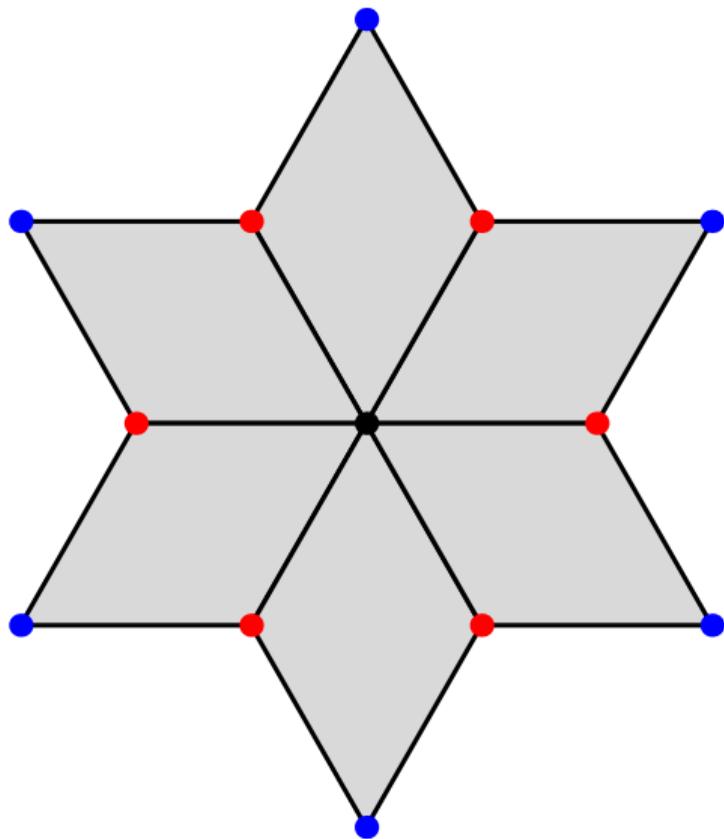
# Enumeração espacial adaptativa

dk



# Enumeração espacial adaptativa

dk



objetos implícitos

## Região implícita

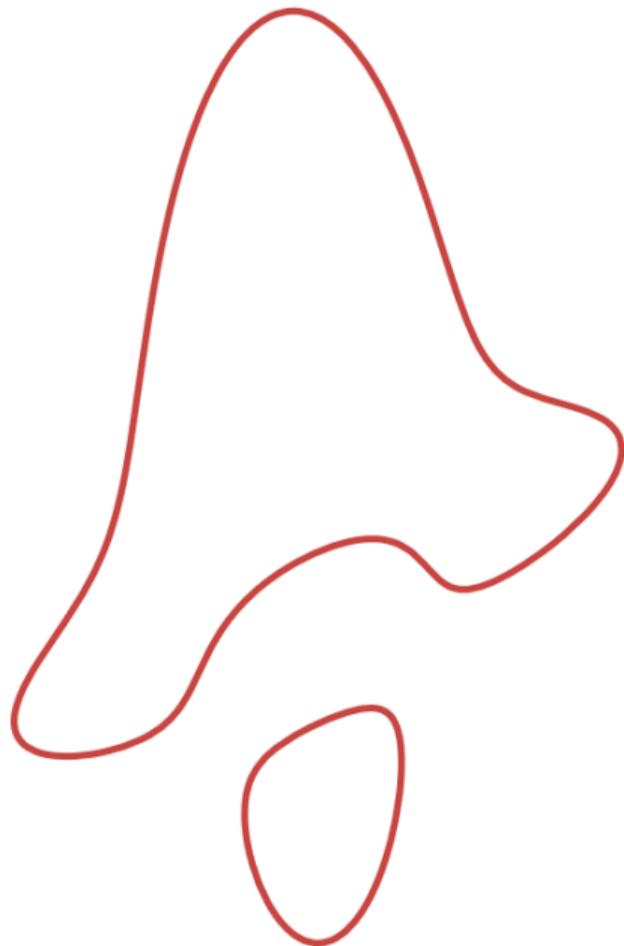
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



## Região implícita

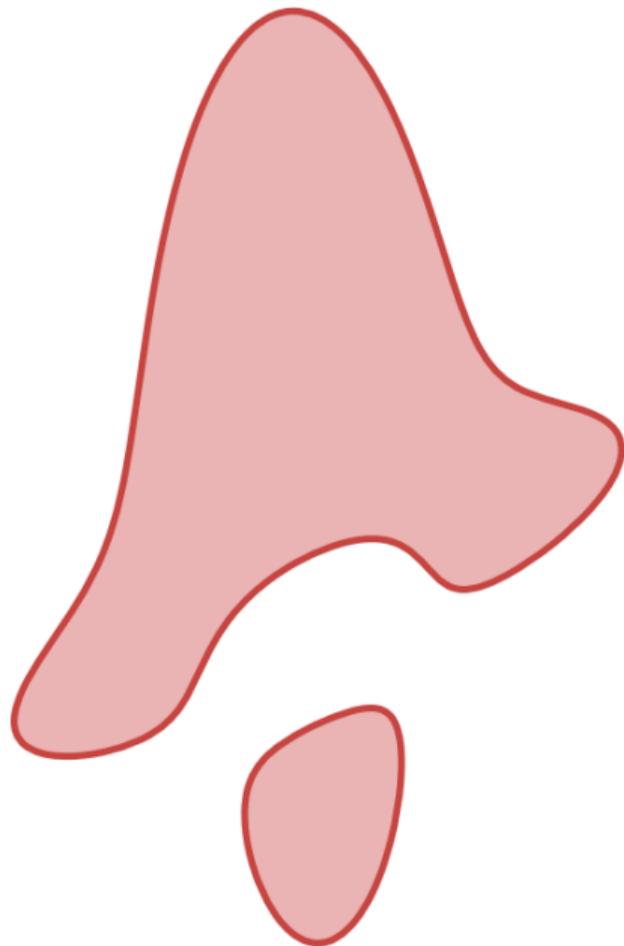
curva = fronteira da região

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região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
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## Região implícita

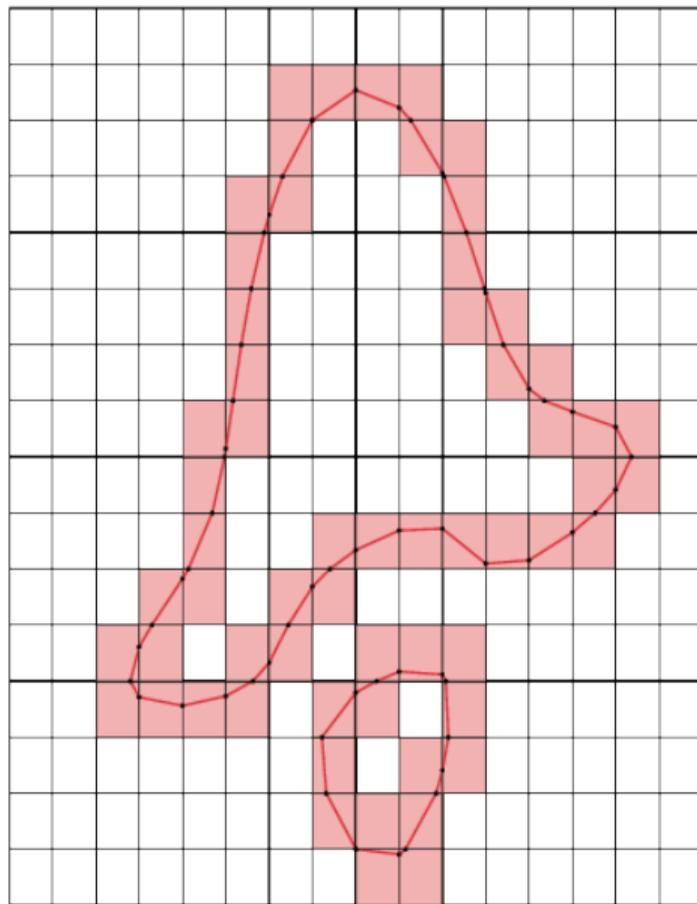
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



## Região implícita

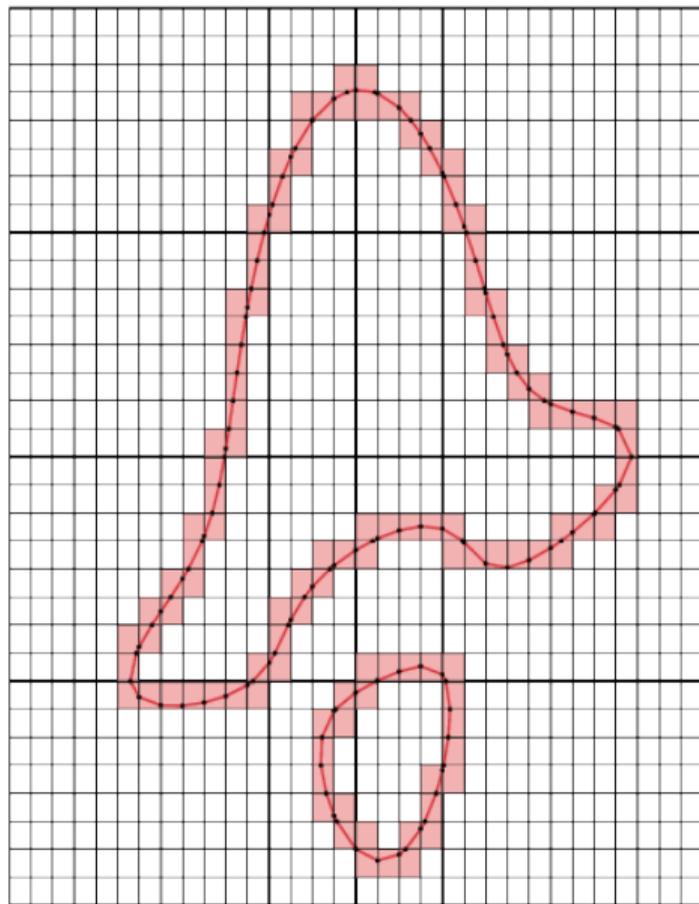
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



## Região implícita

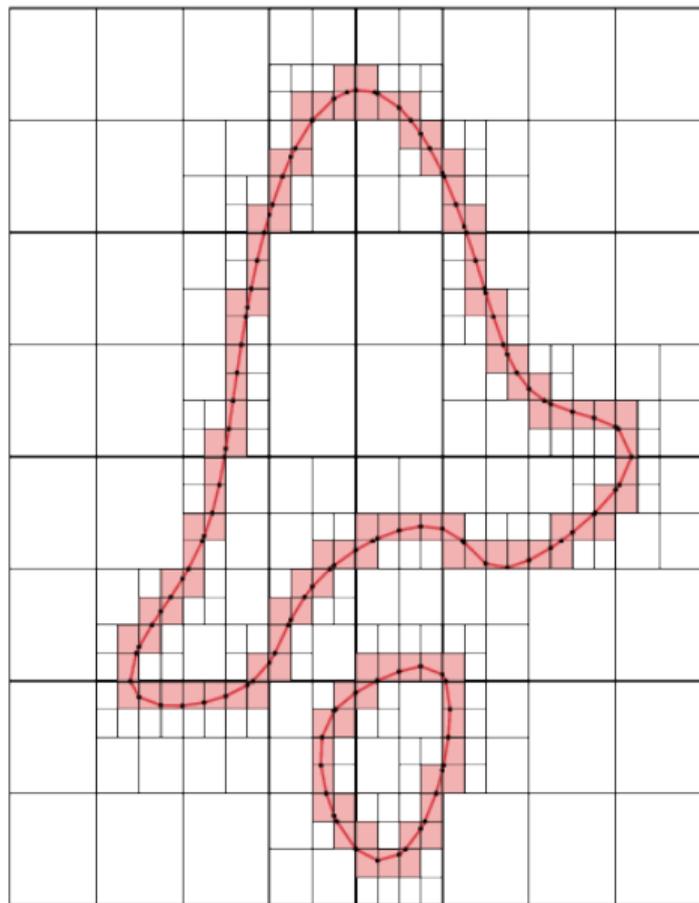
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



■  $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$  contínua

■  $[\alpha, b] \subseteq \Omega$

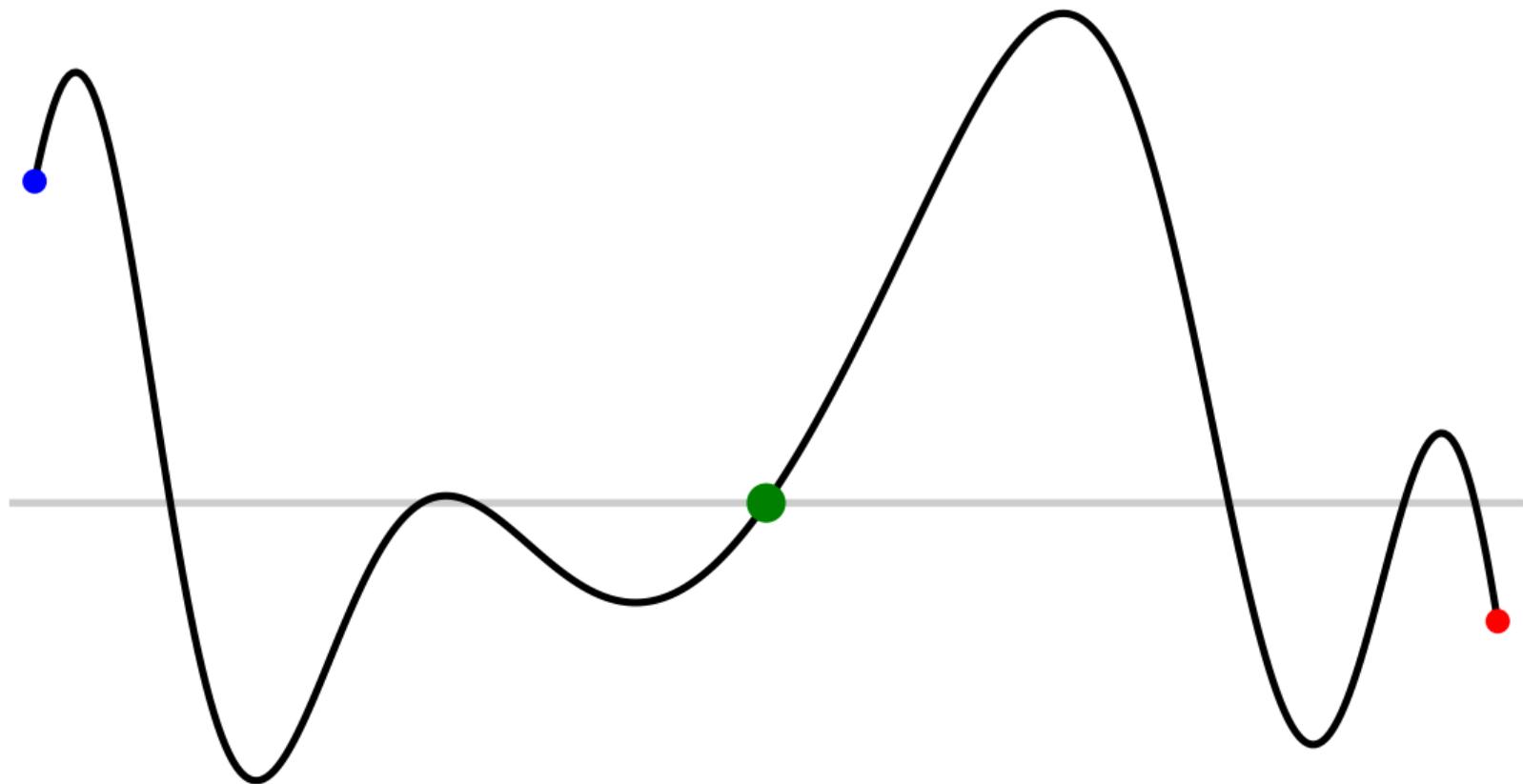
■  $f(\alpha) > 0$

■  $f(b) < 0$

$\implies$  existe  $x^* \in (\alpha, b)$  tal que  $f(x^*) = 0$

# Teorema do valor intermediário

Bolzano (1817)



■  $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$  contínua

■  $[\alpha, b] \subseteq \Omega$

■  $f(\alpha) > 0$

■  $f(b) < 0$

$\implies$  existe  $x^* \in (\alpha, b)$  tal que  $f(x^*) = 0$

■  $f: \Omega = \mathbf{R} \rightarrow \mathbf{R}$  dada por  $f(x) = x^2 - 2$  é contínua

■  $[0, 2] \subseteq \Omega$

■  $f(0) < 0$

■  $f(2) > 0$

$\implies$  existe  $r \in (0, 2)$  tal que  $f(r) = 0$ , isto é,  $r^2 = 2$

$$r = \sqrt{2}$$

■  $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$  contínua

■  $[\alpha, b] \subseteq \Omega$

■  $f(\alpha) > 0$

■  $f(b) < 0$

$\implies$  existe  $x^* \in (\alpha, b)$  tal que  $f(x^*) = 0$

## Teorema do valor intermediário – prova construtiva: método da bisseção

■  $a_0 = a, \quad b_0 = b$

$$f(a) > 0, \quad f(b) < 0$$

## Teorema do valor intermediário – prova construtiva: método da biseção

- $a_0 = a, \quad b_0 = b$

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- $m = \frac{a_n + b_n}{2}$

## Teorema do valor intermediário – prova construtiva: método da biseção

■  $a_0 = a, \quad b_0 = b$

$f(a) > 0, \quad f(b) < 0$

■  $m = \frac{a_n + b_n}{2}$

■ se  $f(m) = 0$ :  $x^* = m$

fim

## Teorema do valor intermediário – prova construtiva: método da biseção

■  $a_0 = a, \quad b_0 = b$

$$f(a) > 0, \quad f(b) < 0$$

■  $m = \frac{a_n + b_n}{2}$

■ se  $f(m) = 0$ :  $x^* = m$

fim

■ se  $f(m) > 0$ :  $a_{n+1} = m, \quad b_{n+1} = b_n$

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$$

## Teorema do valor intermediário – prova construtiva: método da biseção

■  $a_0 = a, \quad b_0 = b$

$$f(a) > 0, \quad f(b) < 0$$

■  $m = \frac{a_n + b_n}{2}$

■ se  $f(m) = 0$ :  $x^* = m$

fim

■ se  $f(m) > 0$ :  $a_{n+1} = m, \quad b_{n+1} = b_n$

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$$

■ se  $f(m) < 0$ :  $a_{n+1} = a_n, \quad b_{n+1} = m$

$$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$$

## Teorema do valor intermediário – prova construtiva: método da bisseção

■  $a_0 = a, \quad b_0 = b$

$$f(a) > 0, \quad f(b) < 0$$

■  $m = \frac{a_n + b_n}{2}$

■ se  $f(m) = 0$ :  $x^* = m$

fim

■ se  $f(m) > 0$ :  $a_{n+1} = m, \quad b_{n+1} = b_n$

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$$

■ se  $f(m) < 0$ :  $a_{n+1} = a_n, \quad b_{n+1} = m$

$$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$$

■  $f(a_n) > 0, \quad f(b_n) < 0$

invariante

## Teorema do valor intermediário – prova construtiva: método da bisseção

■  $a_0 = a, \quad b_0 = b$   $f(a) > 0, \quad f(b) < 0$

■  $m = \frac{a_n + b_n}{2}$

■ se  $f(m) = 0$ :  $x^* = m$  fim

■ se  $f(m) > 0$ :  $a_{n+1} = m, \quad b_{n+1} = b_n$   $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$

■ se  $f(m) < 0$ :  $a_{n+1} = a_n, \quad b_{n+1} = m$   $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

■  $f(a_n) > 0, \quad f(b_n) < 0$  invariante

■  $a = a_0 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq \dots \leq b_n \leq \dots \leq b_2 \leq b_1 \leq b_0 = b$

## Teorema do valor intermediário – prova construtiva: método da biseção

■  $a_0 = a, \quad b_0 = b$   $f(a) > 0, \quad f(b) < 0$

■  $m = \frac{a_n + b_n}{2}$

■ se  $f(m) = 0$ :  $x^* = m$  fim

■ se  $f(m) > 0$ :  $a_{n+1} = m, \quad b_{n+1} = b_n$   $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$

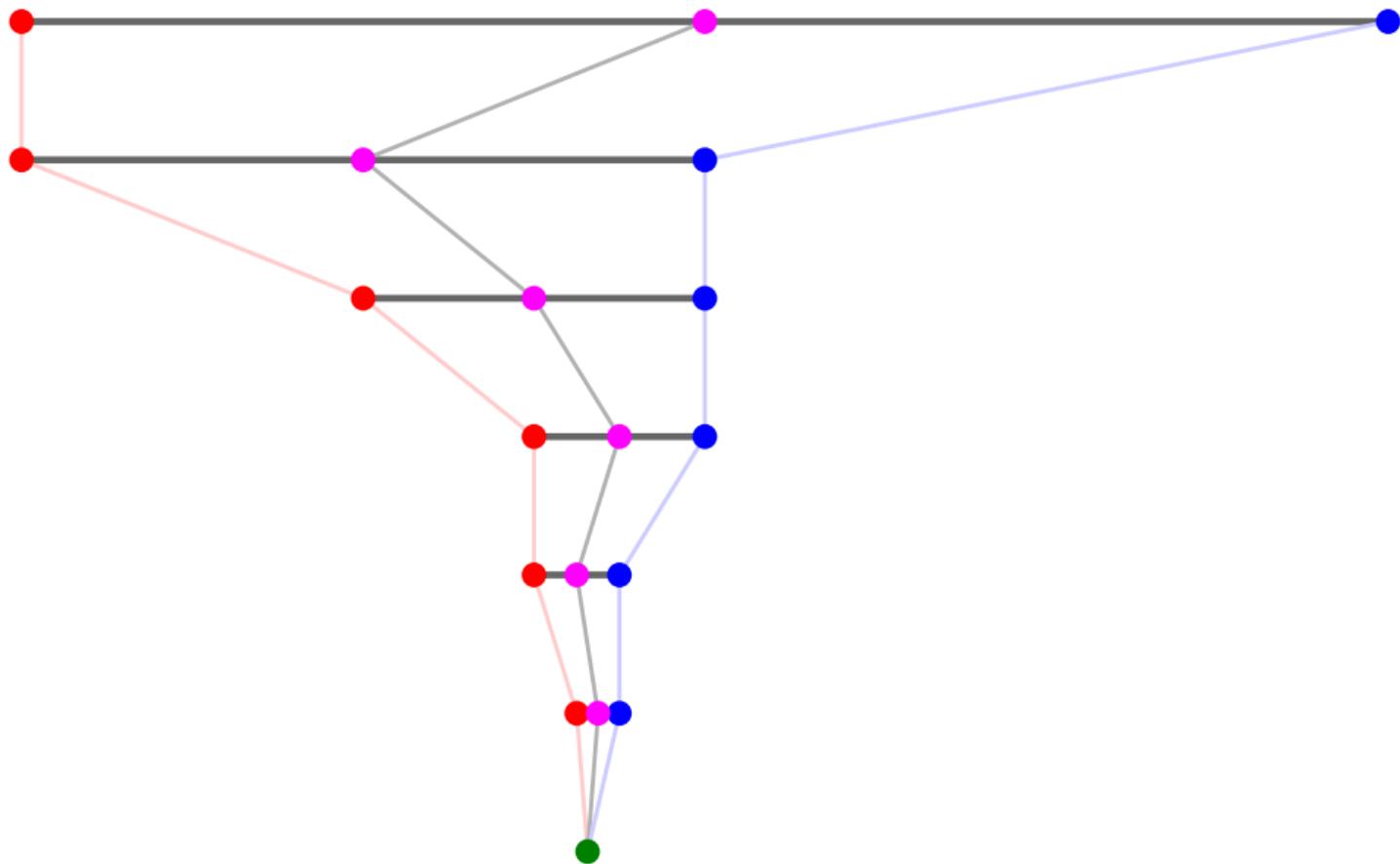
■ se  $f(m) < 0$ :  $a_{n+1} = a_n, \quad b_{n+1} = m$   $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

■  $f(a_n) > 0, \quad f(b_n) < 0$  invariante

■  $a = a_0 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq \dots \leq b_n \leq \dots \leq b_2 \leq b_1 \leq b_0 = b$

$\implies x^* = \lim a_n = \lim b_n, \quad f(x^*) = 0$   $b_n - a_n = \frac{b - a}{2^n} \rightarrow 0$

# Teorema do valor intermediário – método da biseção para $\sqrt{2}$



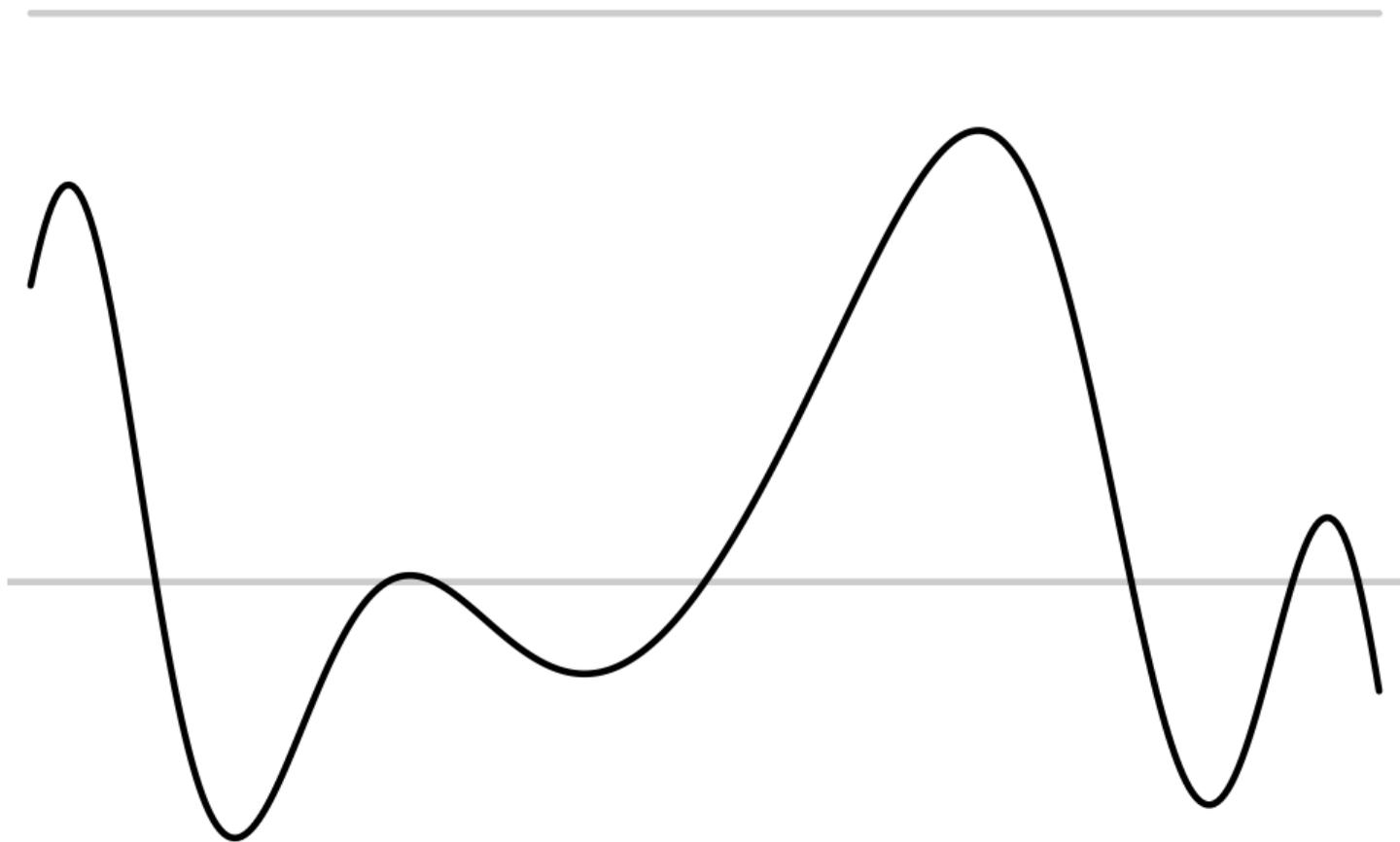
## Método da bisseção

```
function bisect(f,a,b)
    local c=(a+b)/2
    if abs(a-b)<delta or c==a or c==b then
        return c
    end
    if f(c)>0 then
        return bisect(f,a,c)
    else
        return bisect(f,c,b)
    end
end
```

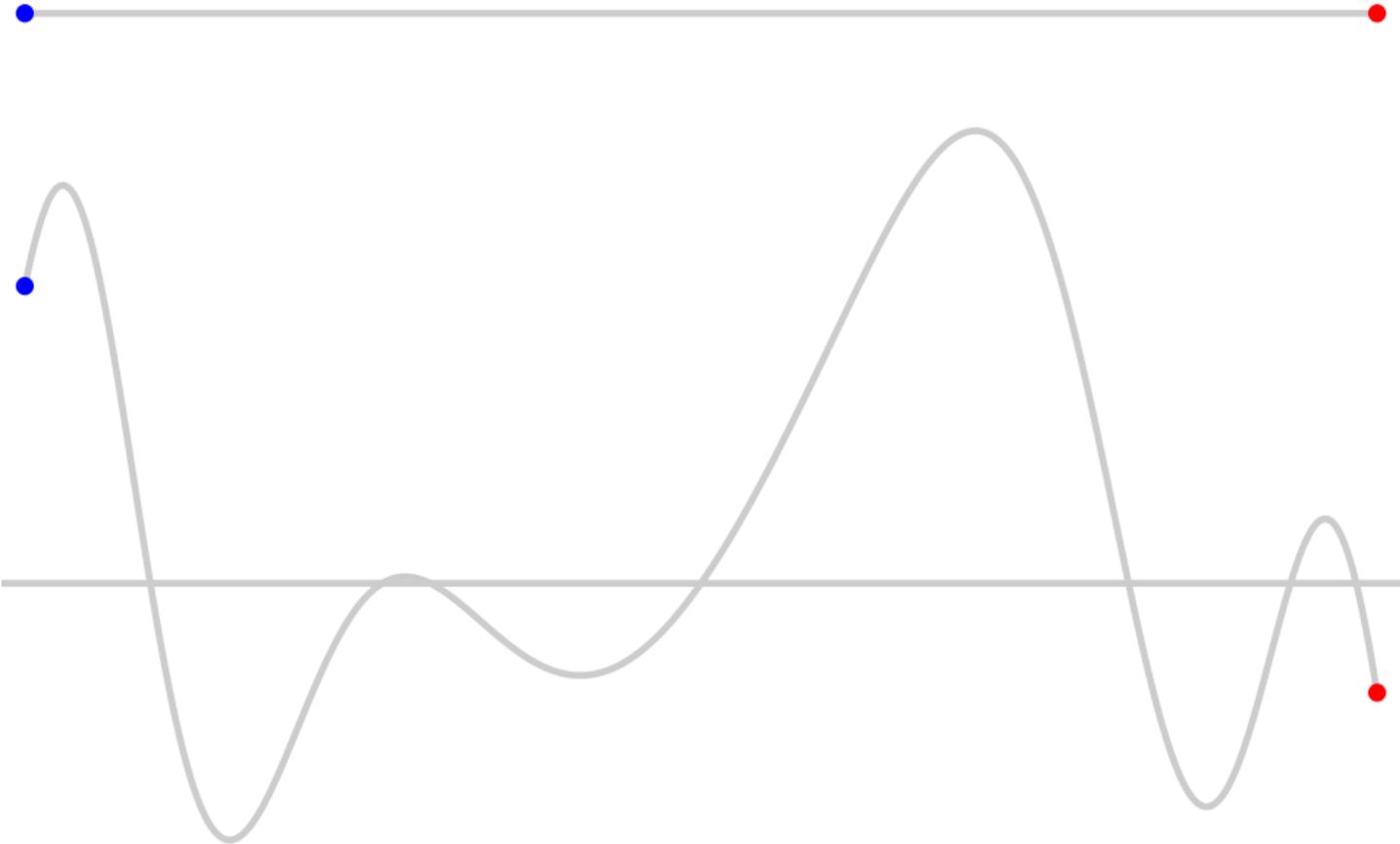
## Método da bisseção

```
function bisect(f,a,b)
    while true do
        local c=(a+b)/2
        if abs(a-b)<delta or c==a or c==b then
            return c
        end
        if f(c)>0 then
            b=c
        else
            a=c
        end
    end
end
```

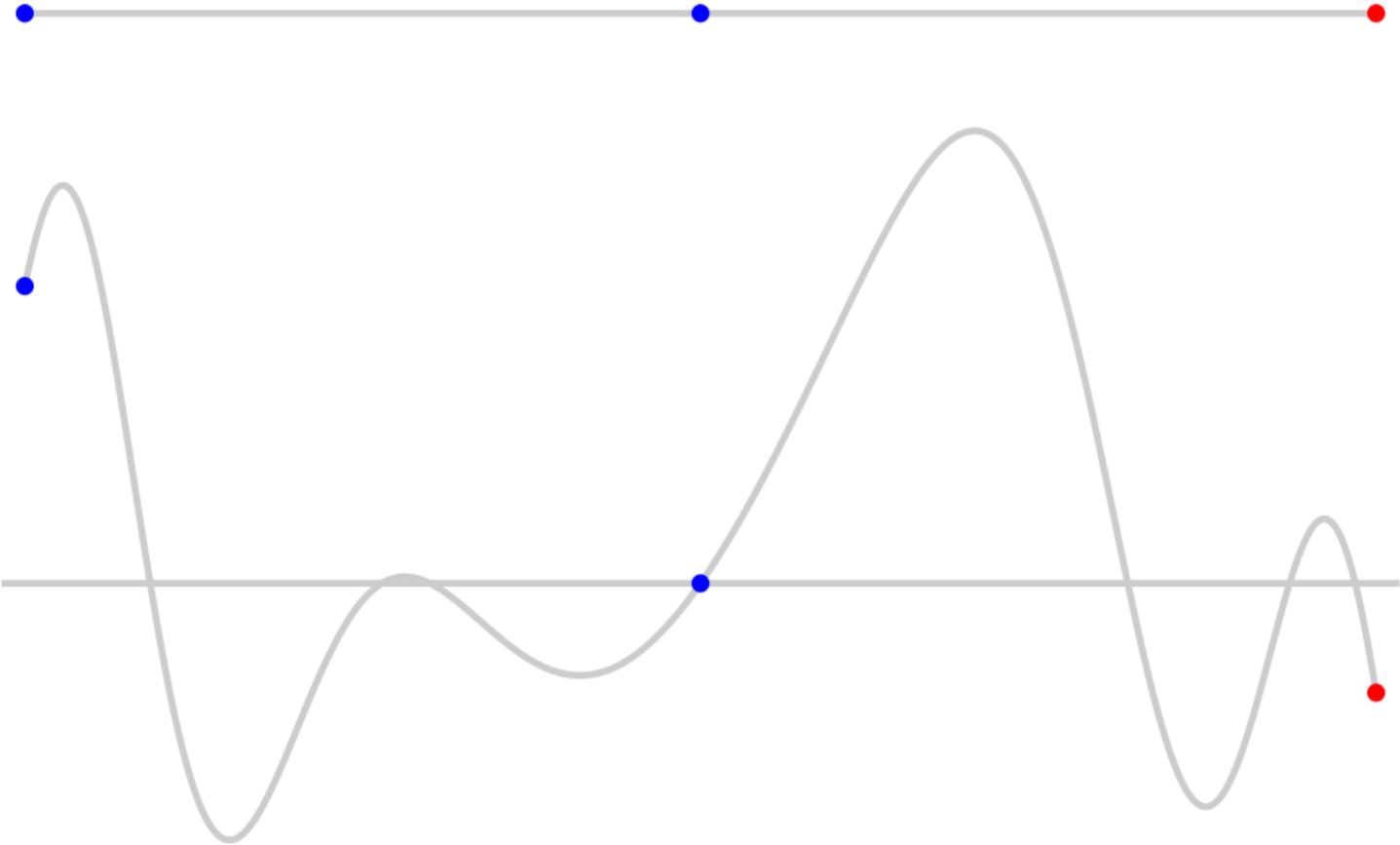
## Amostragem em grade



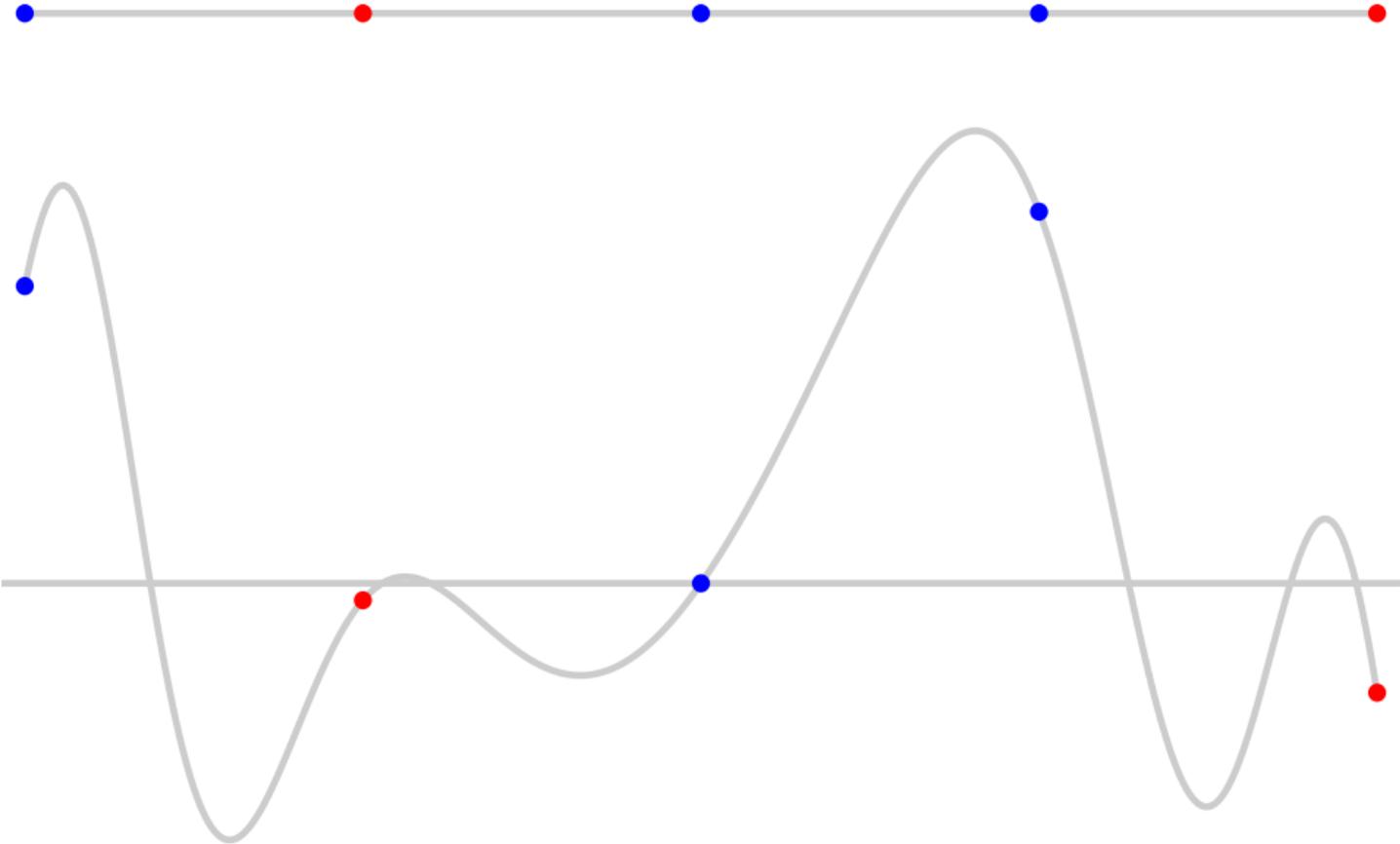
# Amostragem em grade



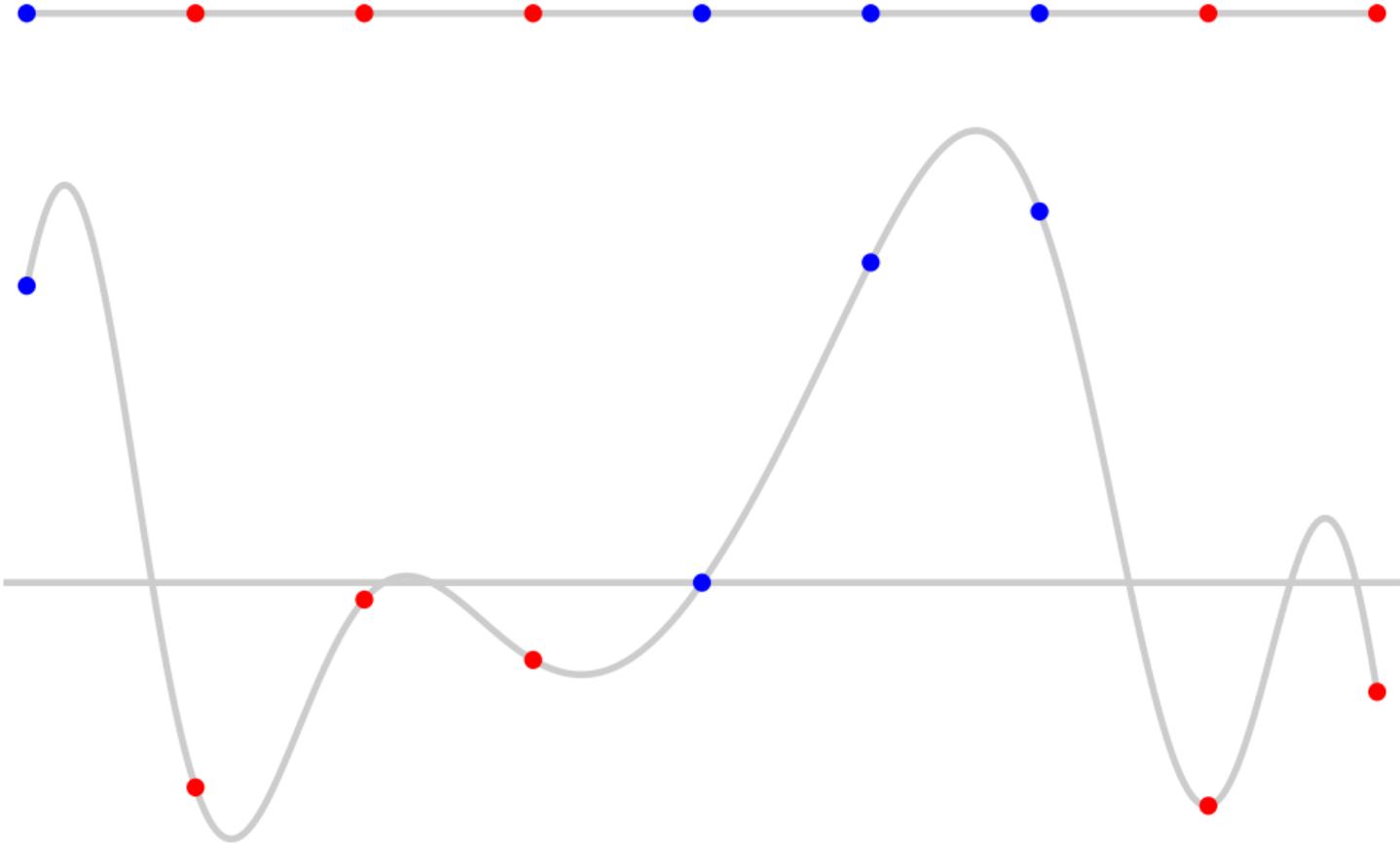
# Amostragem em grade



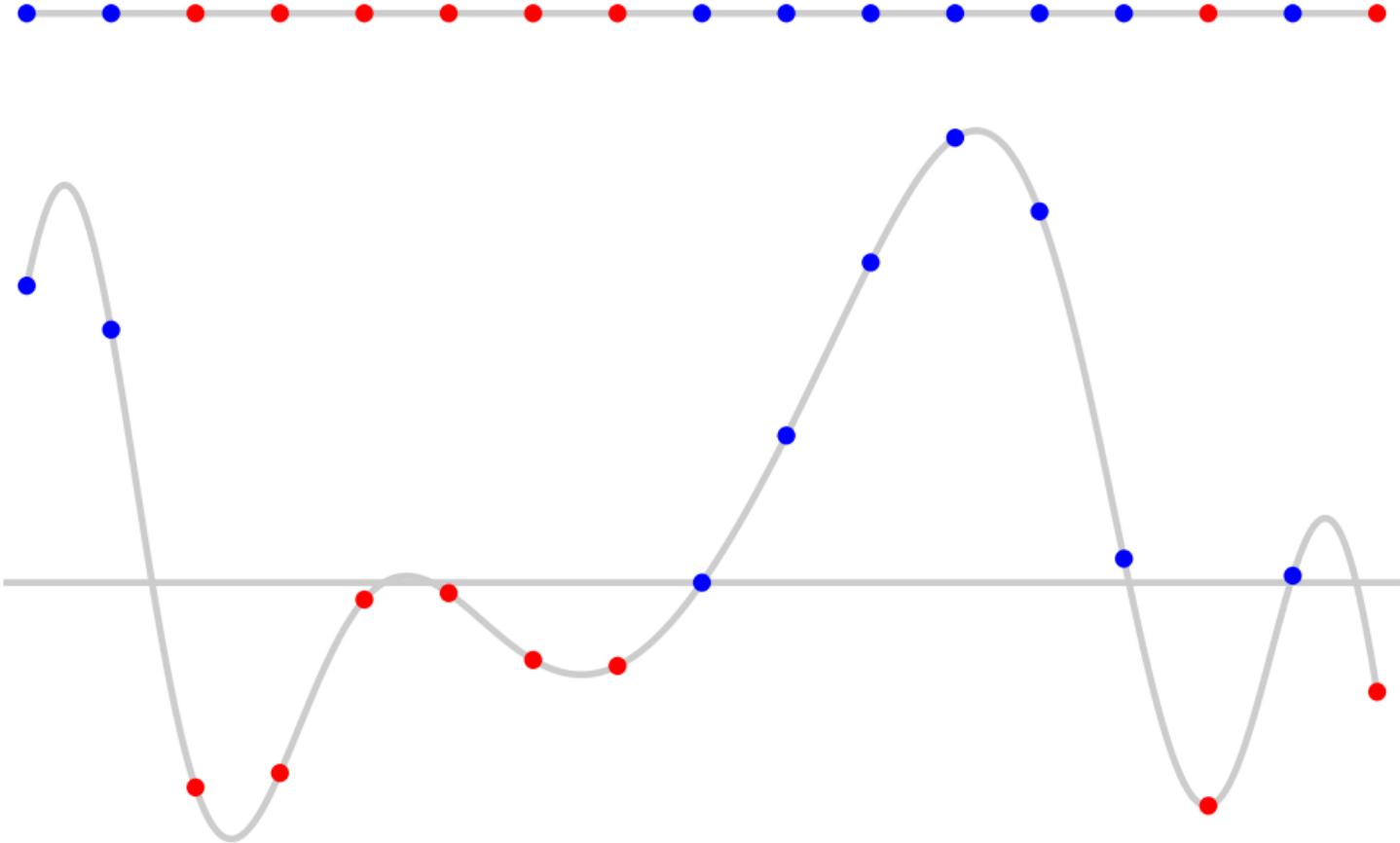
# Amostragem em grade



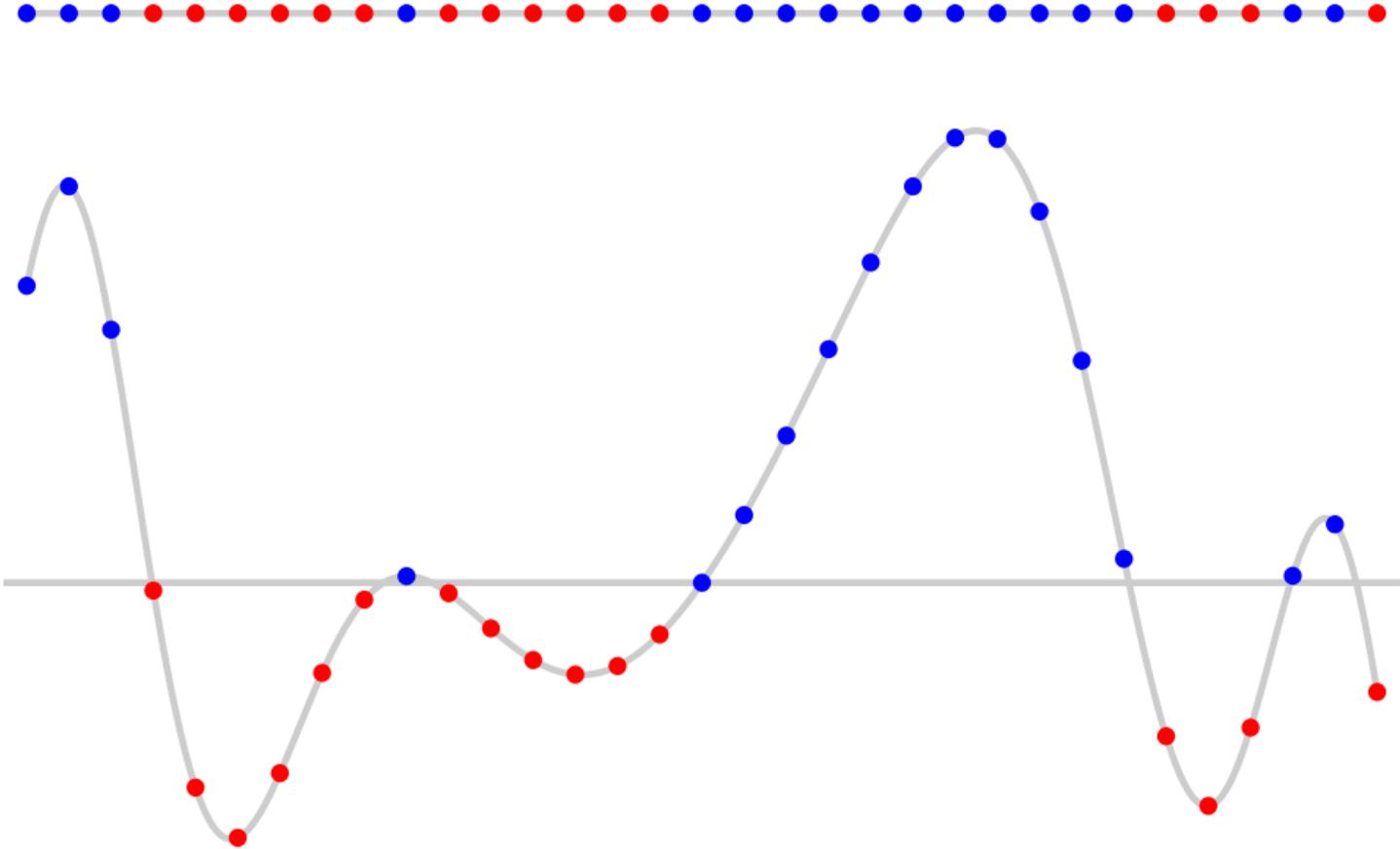
# Amostragem em grade



# Amostragem em grade

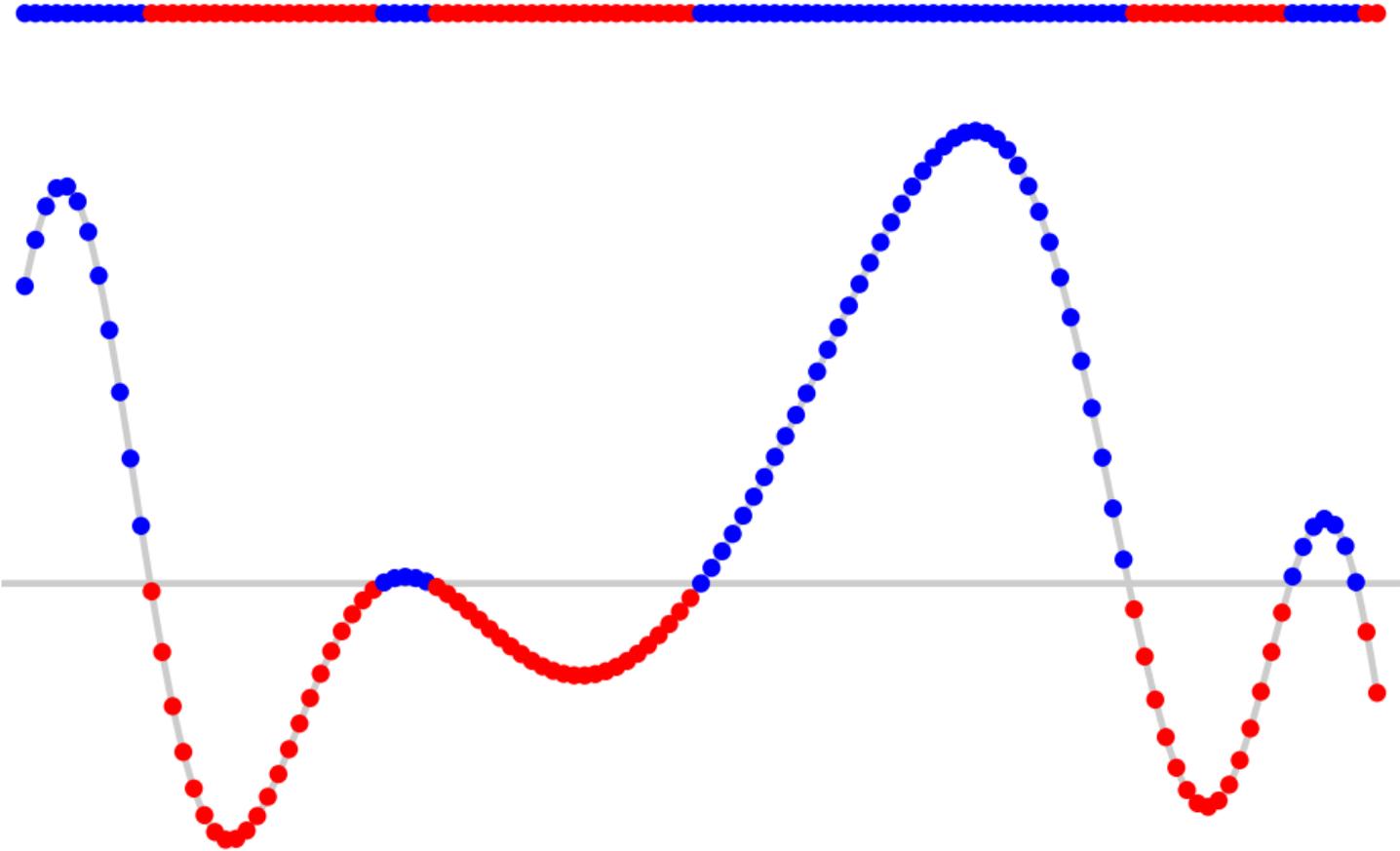


# Amostragem em grade

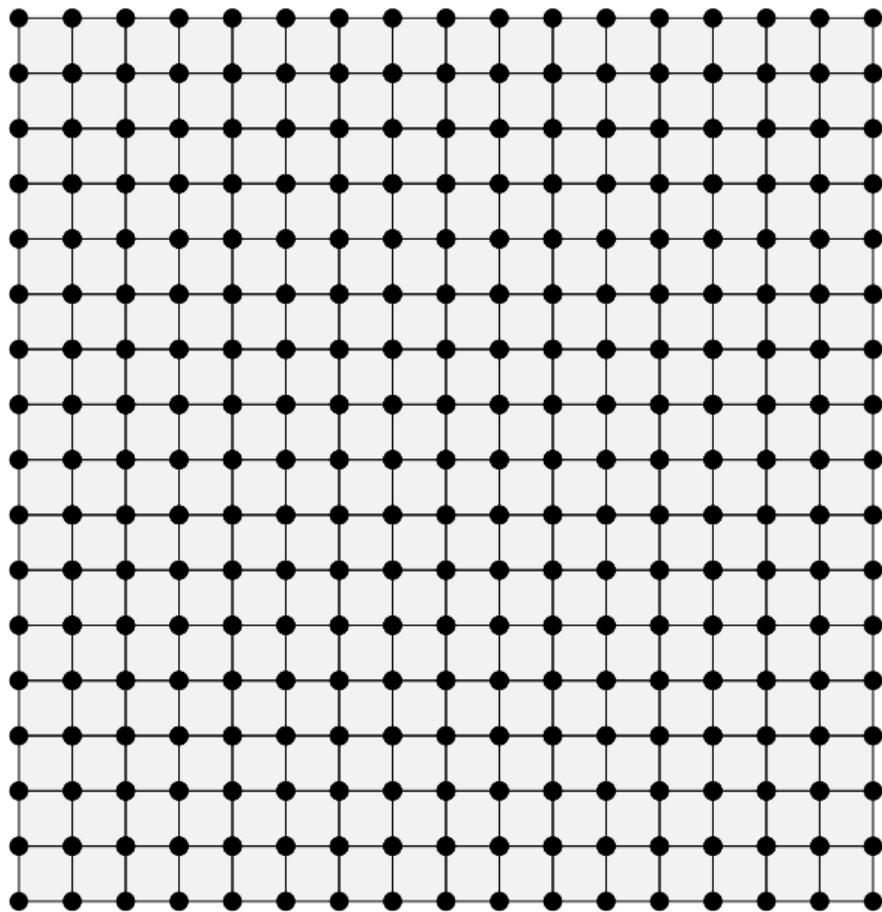




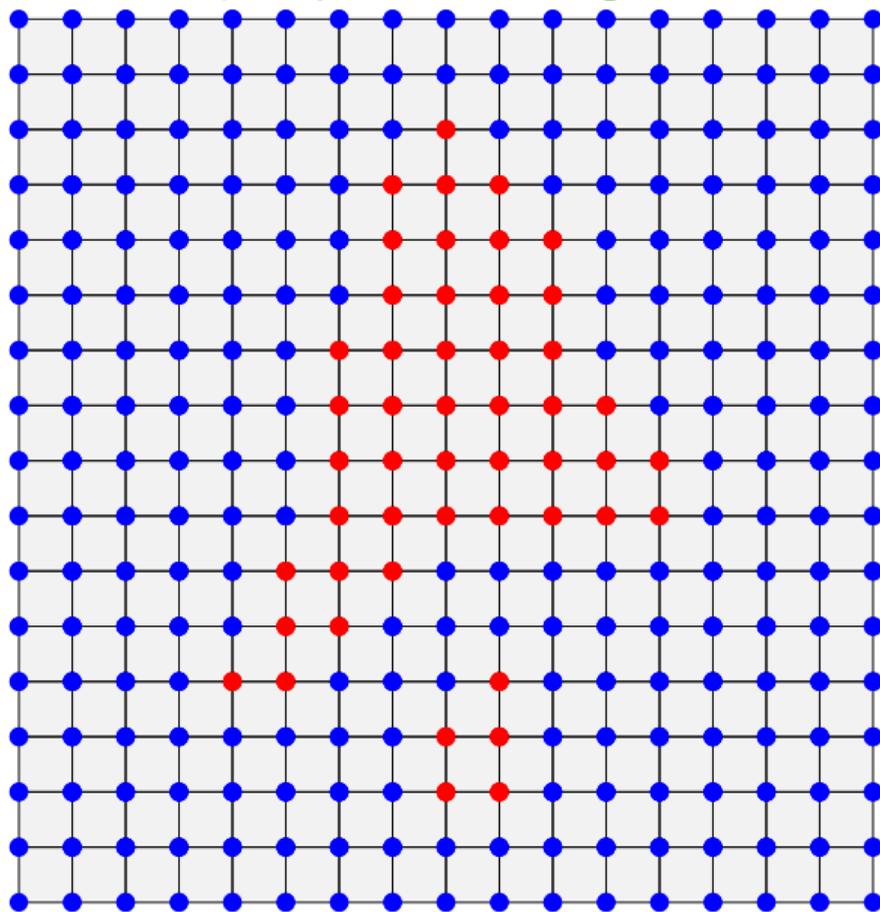
# Amostragem em grade



## Região implícita – classificação por amostragem



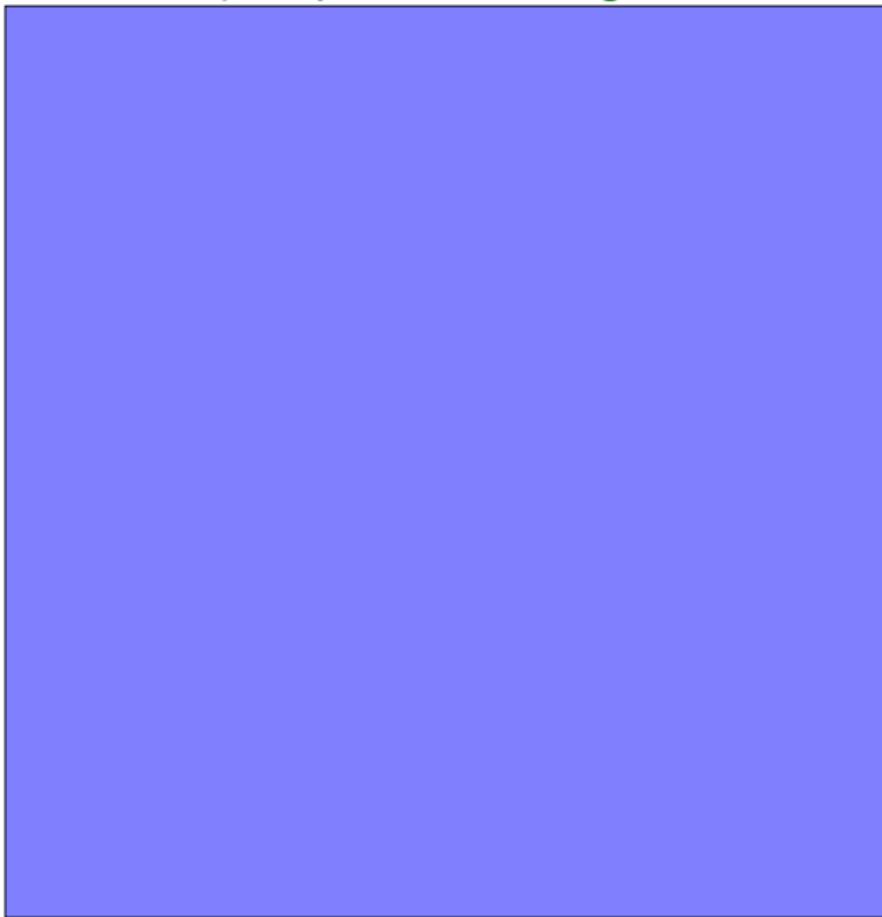
## Região implícita – classificação por amostragem



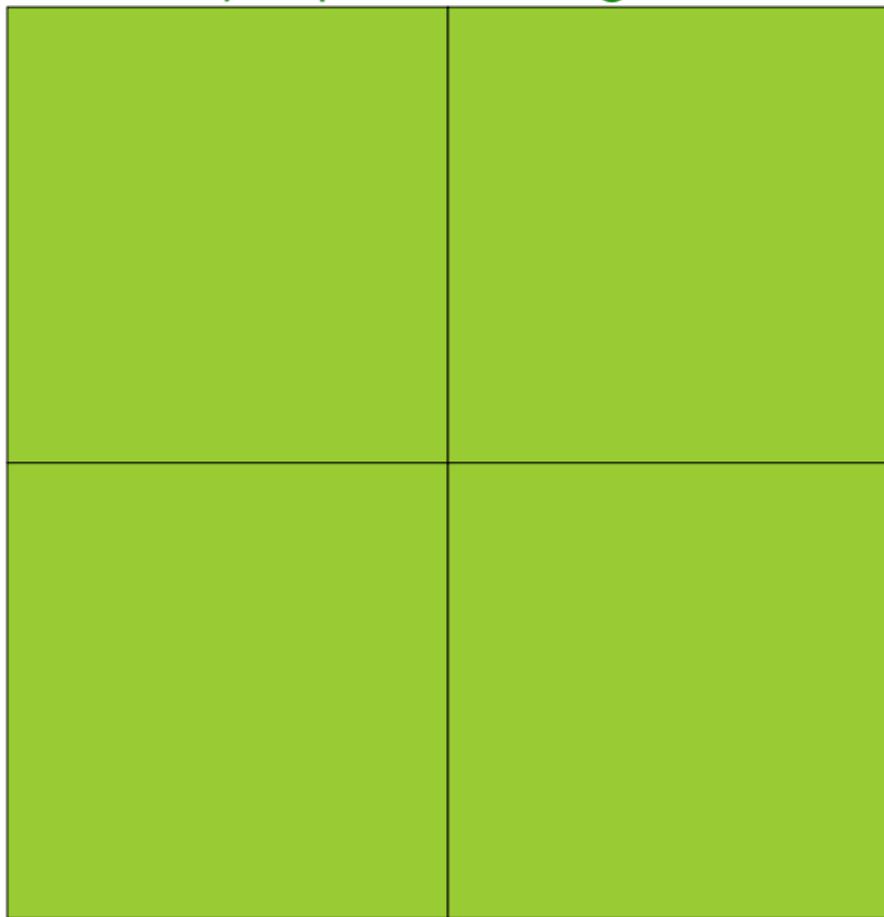




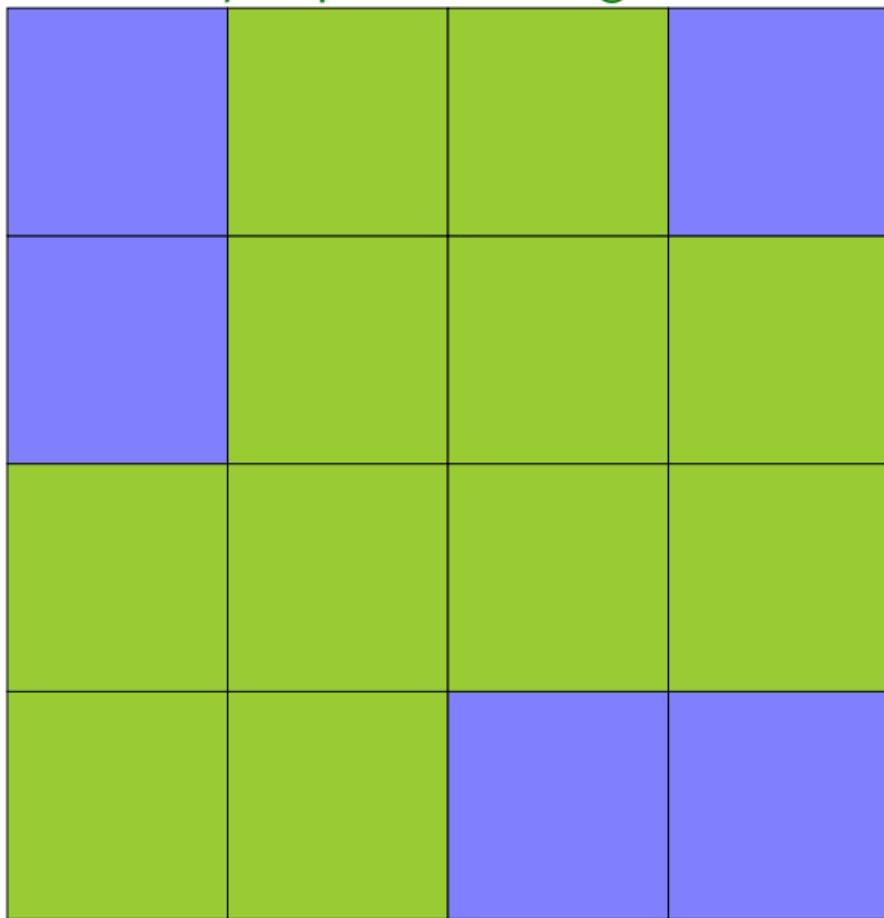
Região implícita – classificação por amostragem



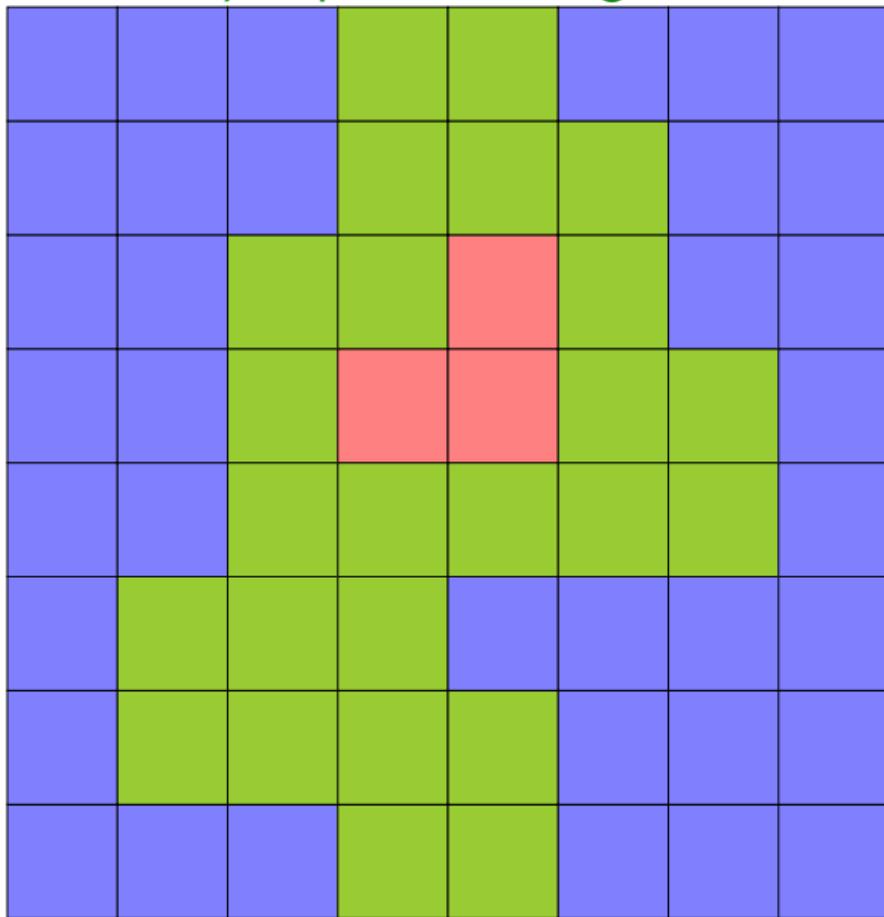
## Região implícita – classificação por amostragem



## Região implícita – classificação por amostragem



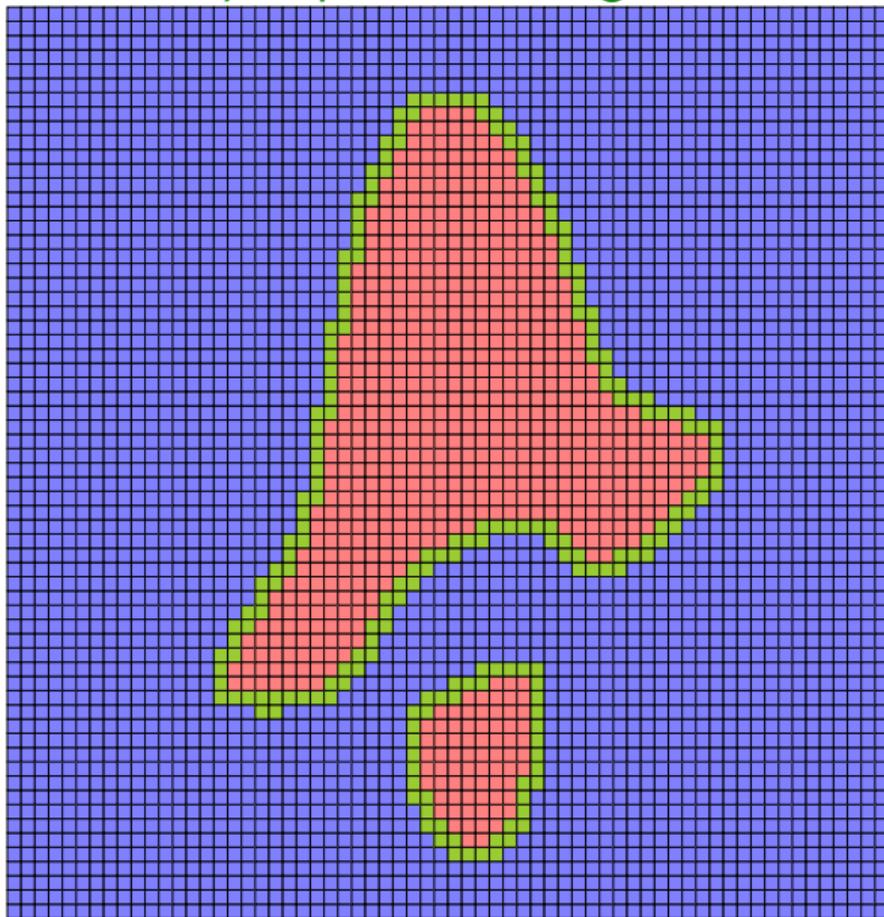
## Região implícita – classificação por amostragem



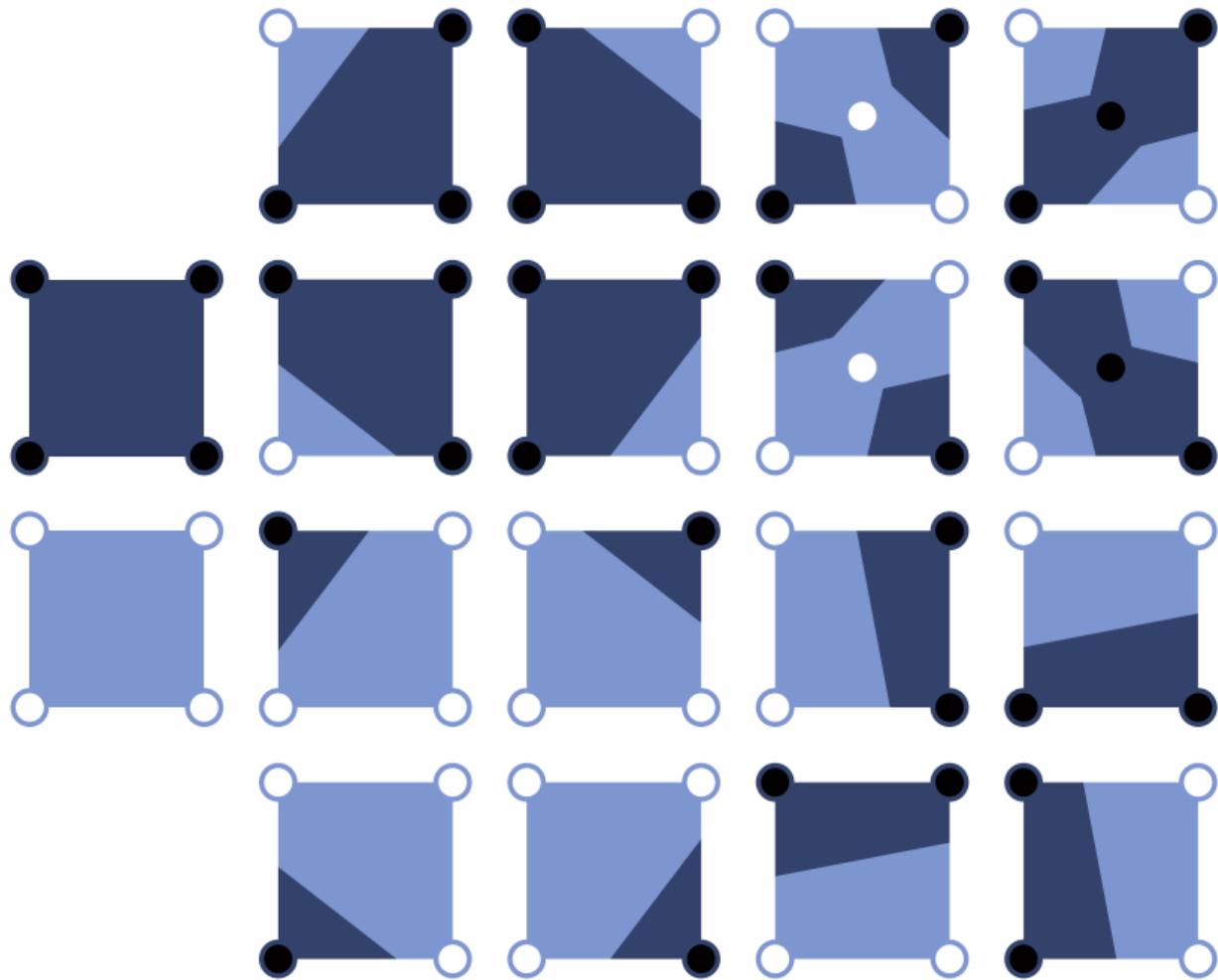




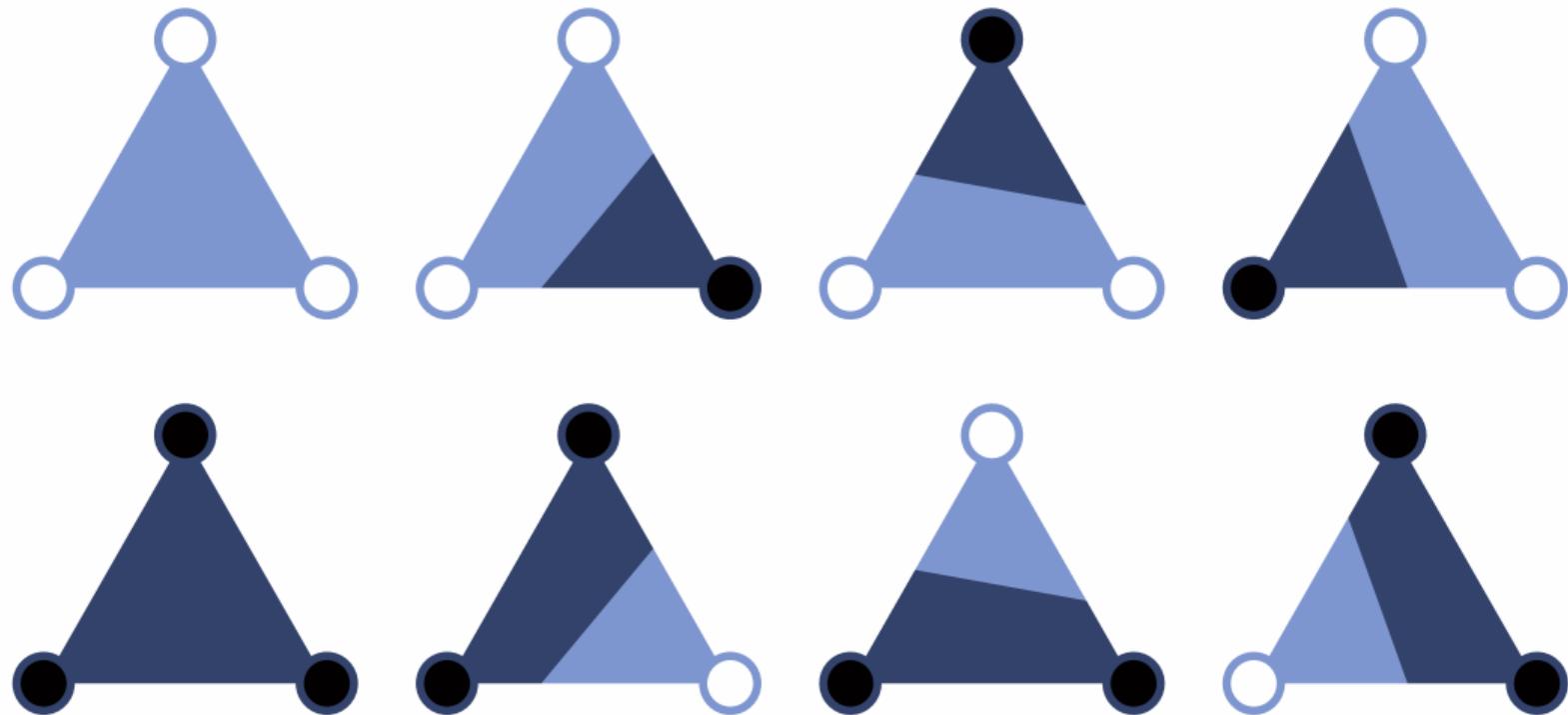
## Região implícita – classificação por amostragem



# Marching squares

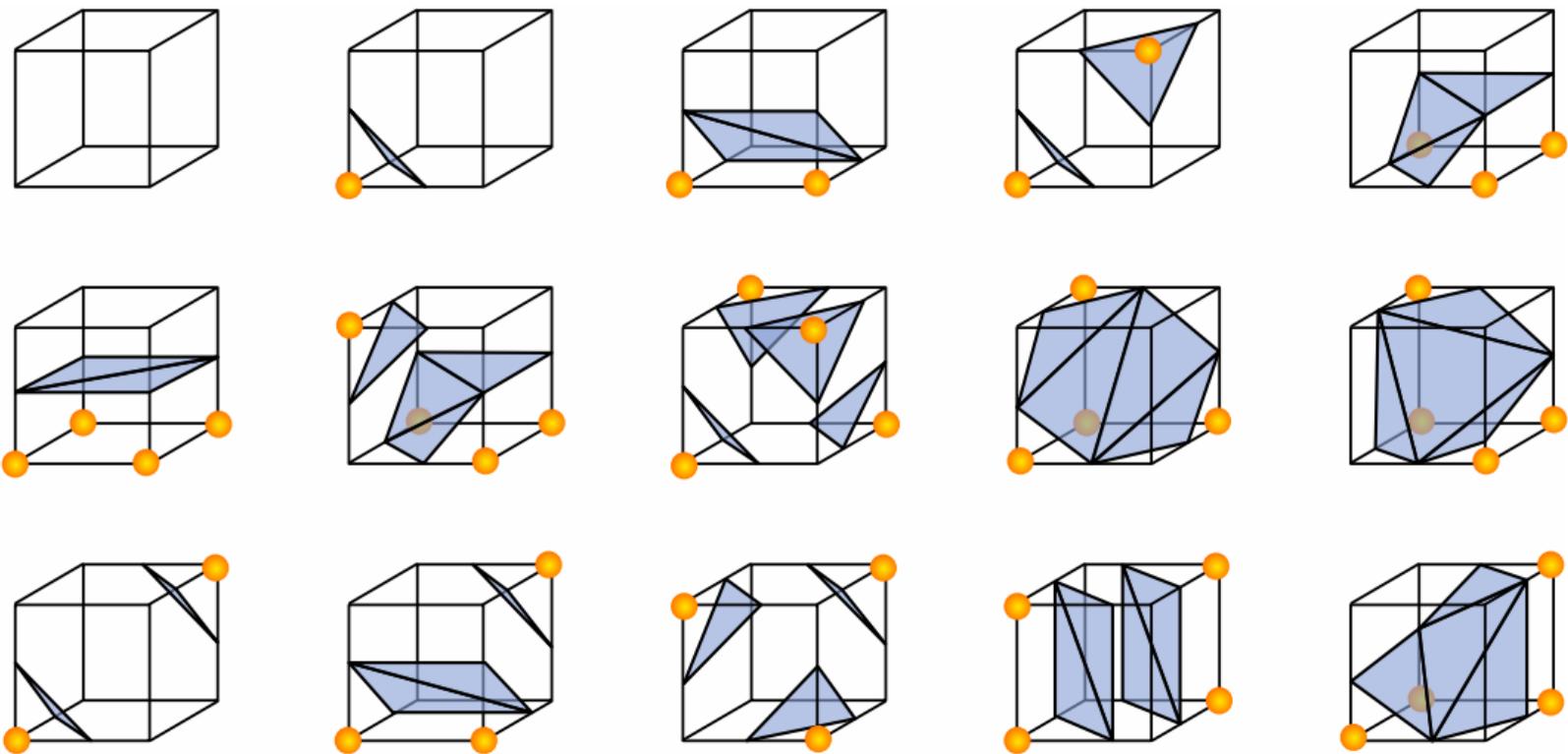


# Marching triangles



# Marching cubes

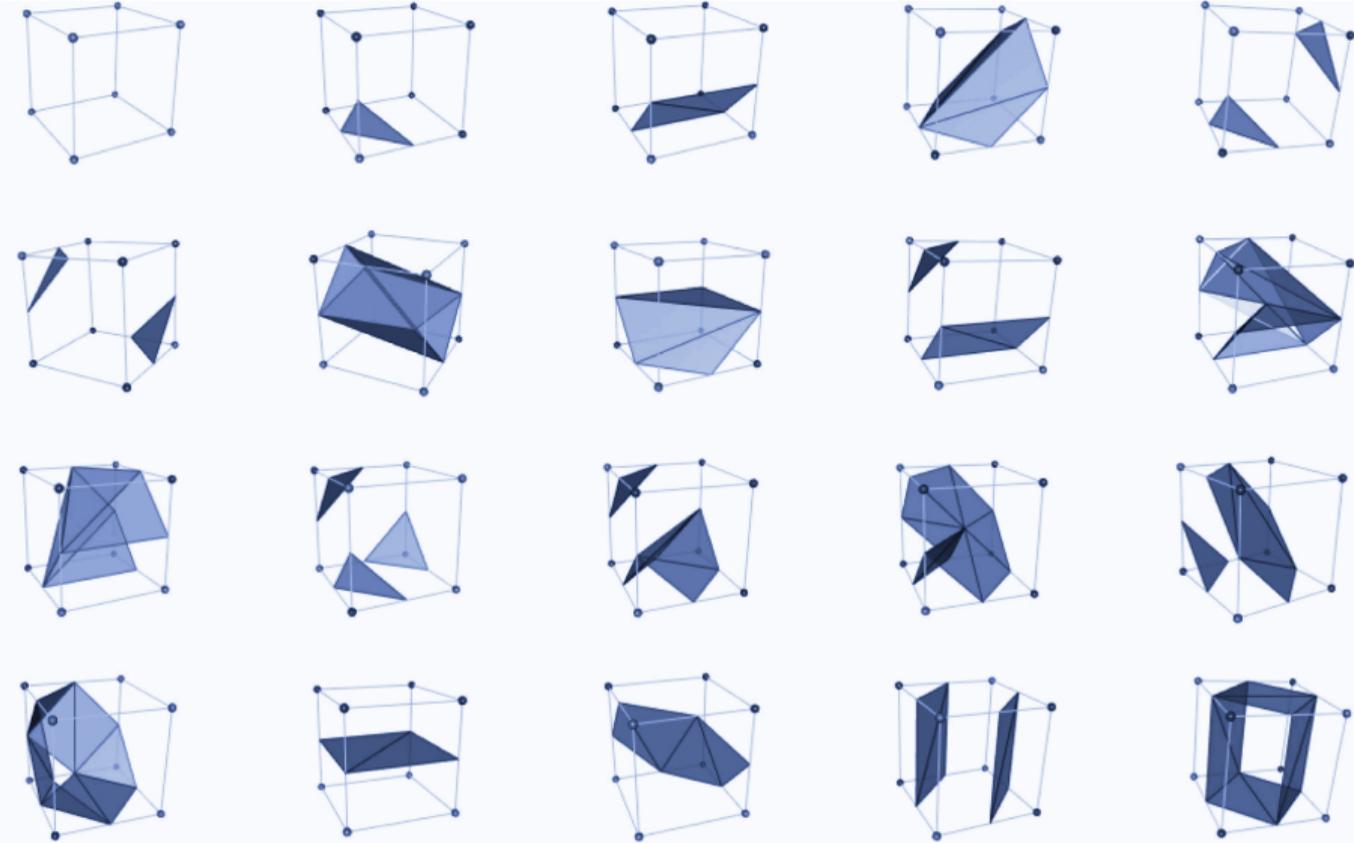
Lorensen–Cline (1987)



# Marching cubes

Chernyaev (1995)

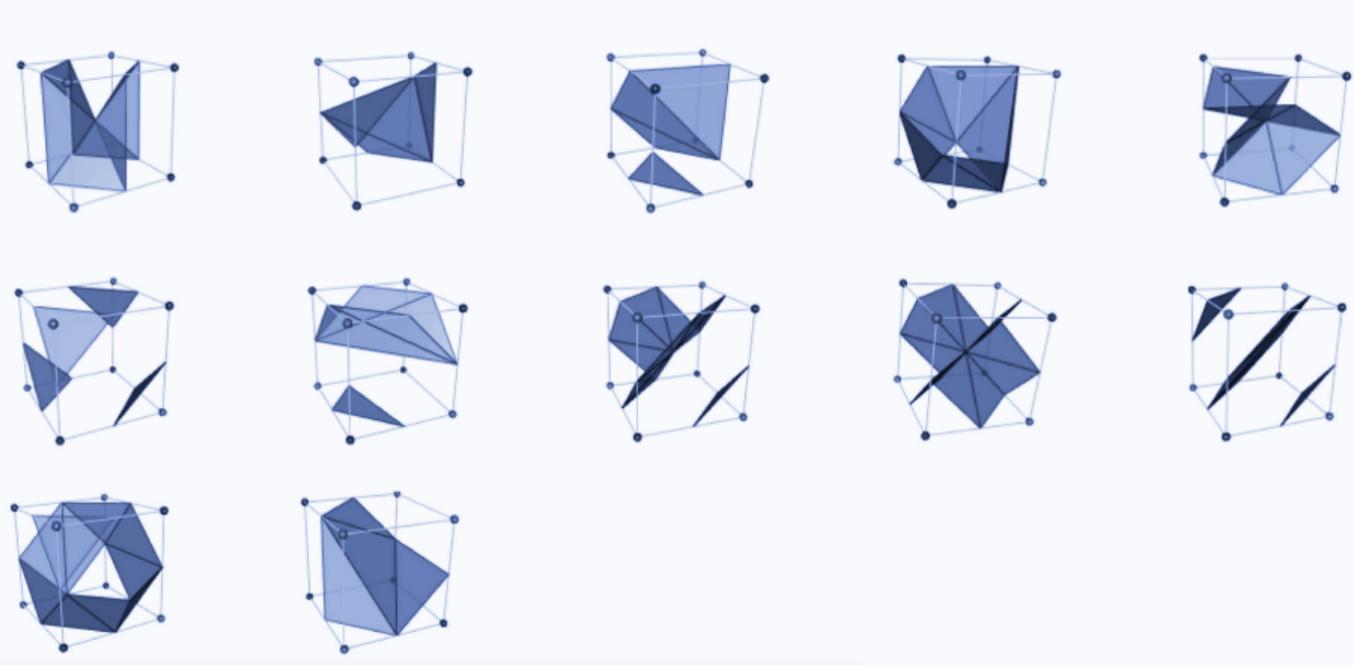
Custódio-Pesco-Silva



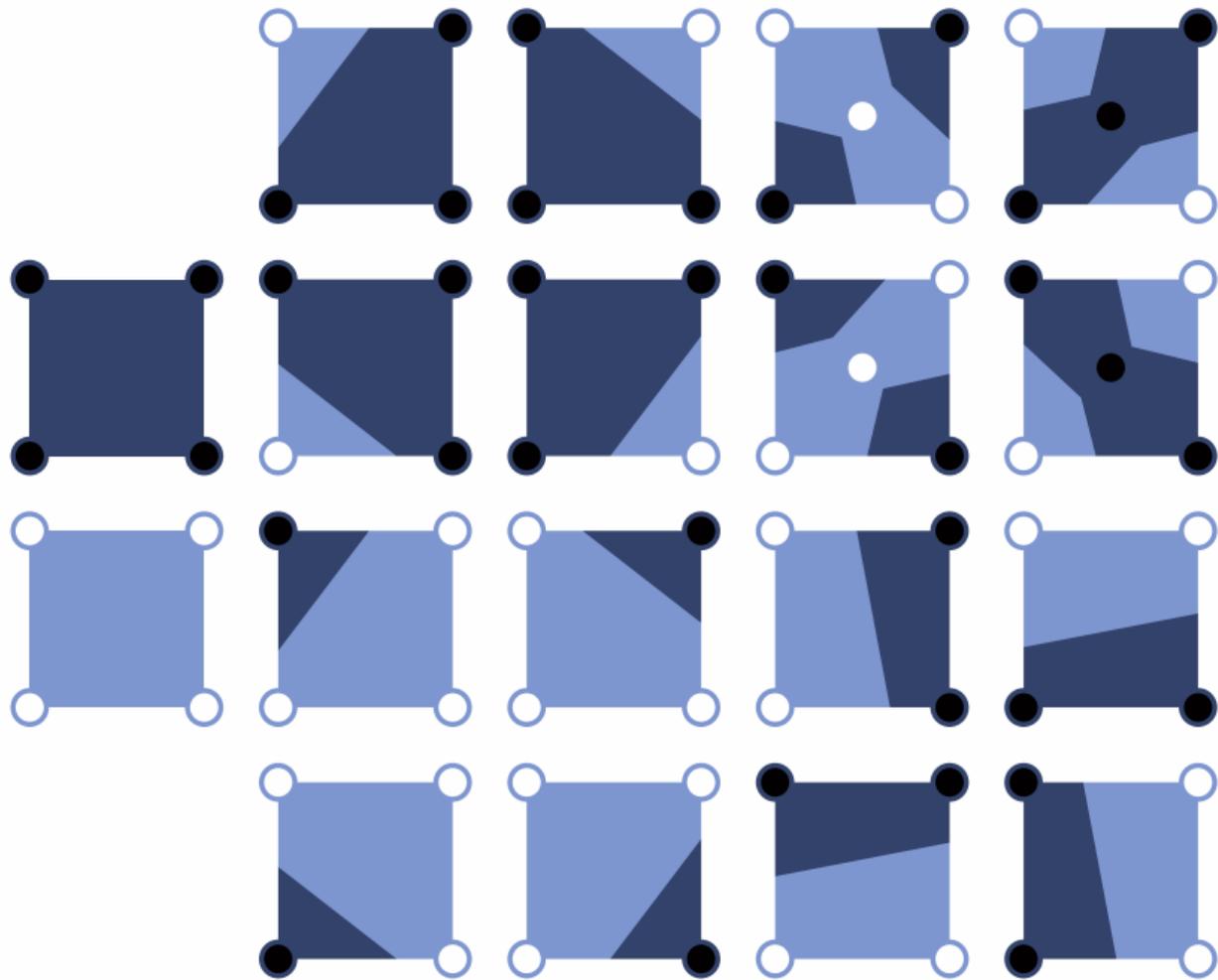
# Marching cubes

Chernyaev (1995)

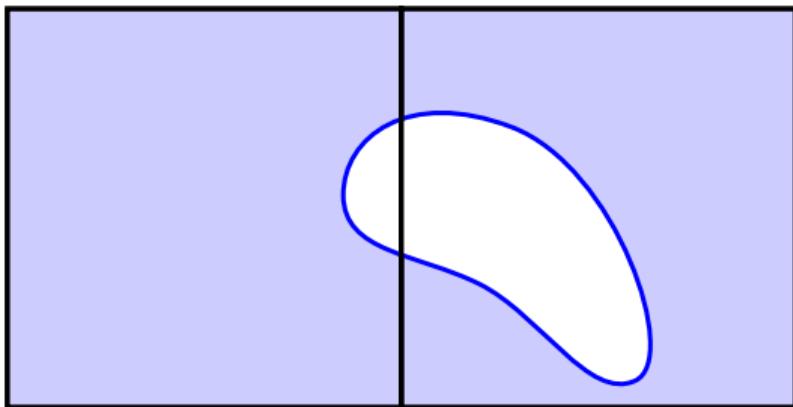
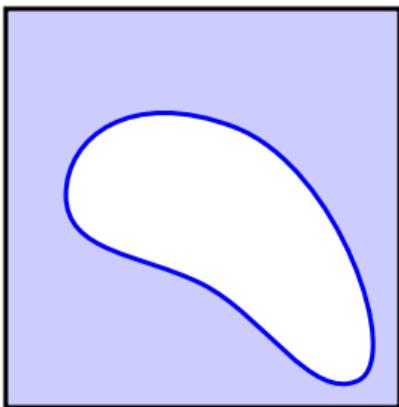
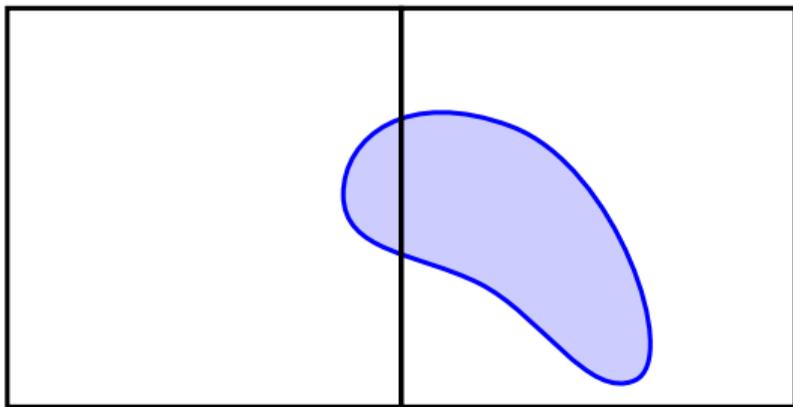
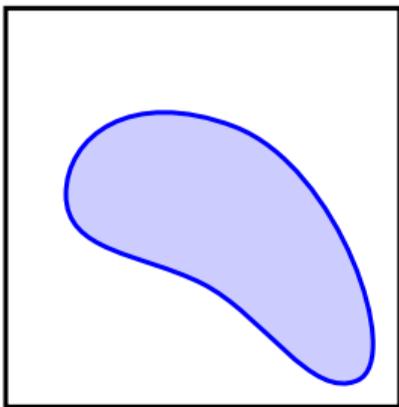
Custódio-Pesco-Silva



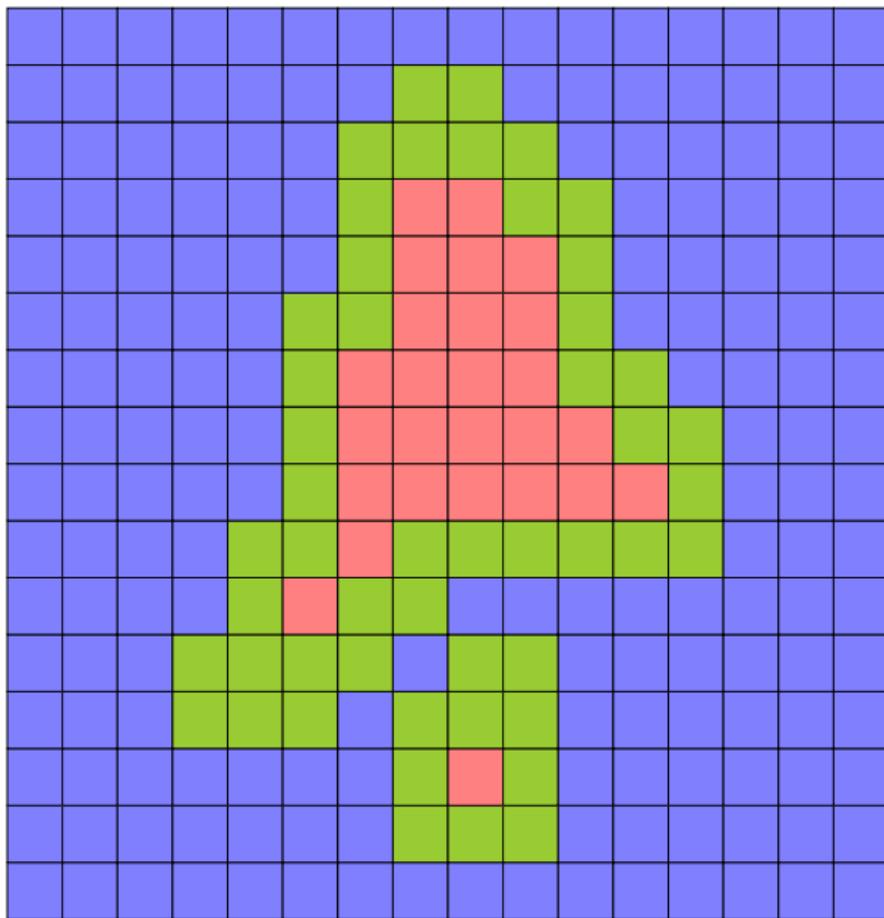
# Marching squares



## Marching squares – casos omissos



## Região implícita – fronteira

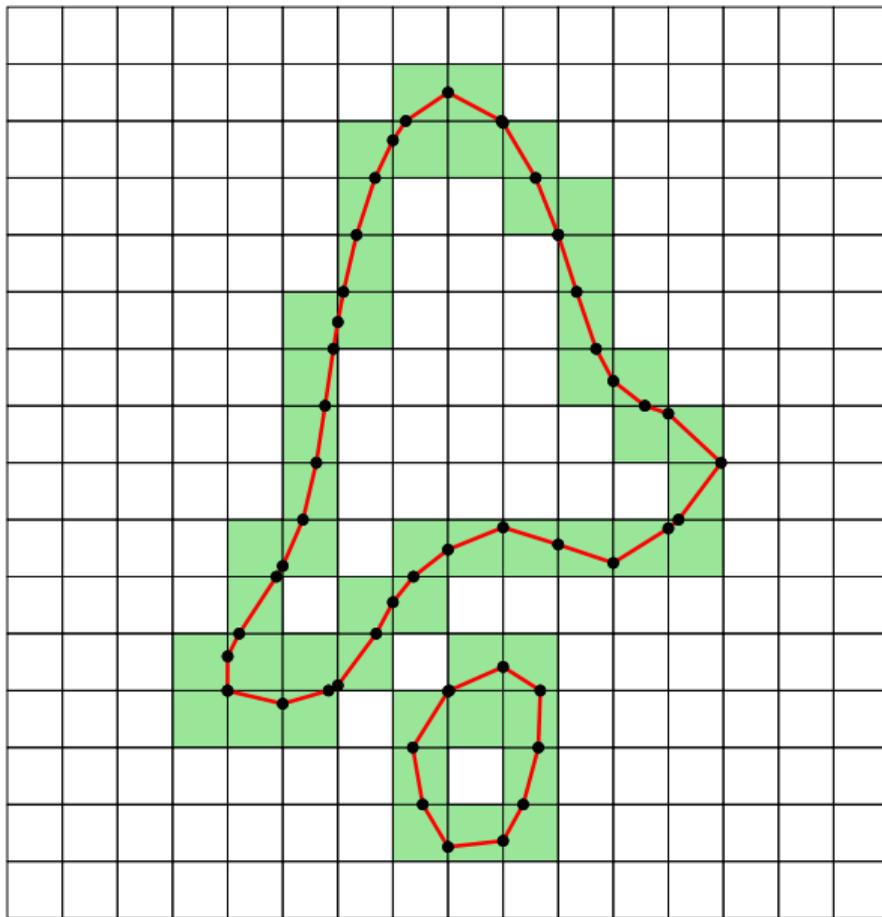






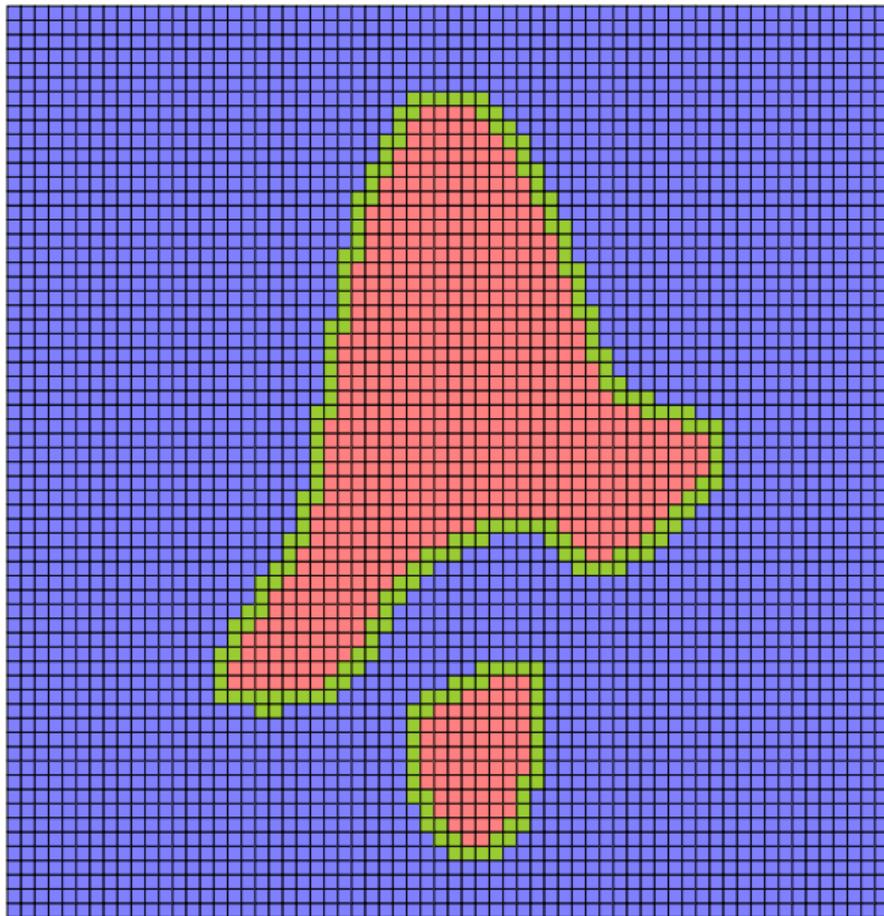
# Região implícita – fronteira

bisseção





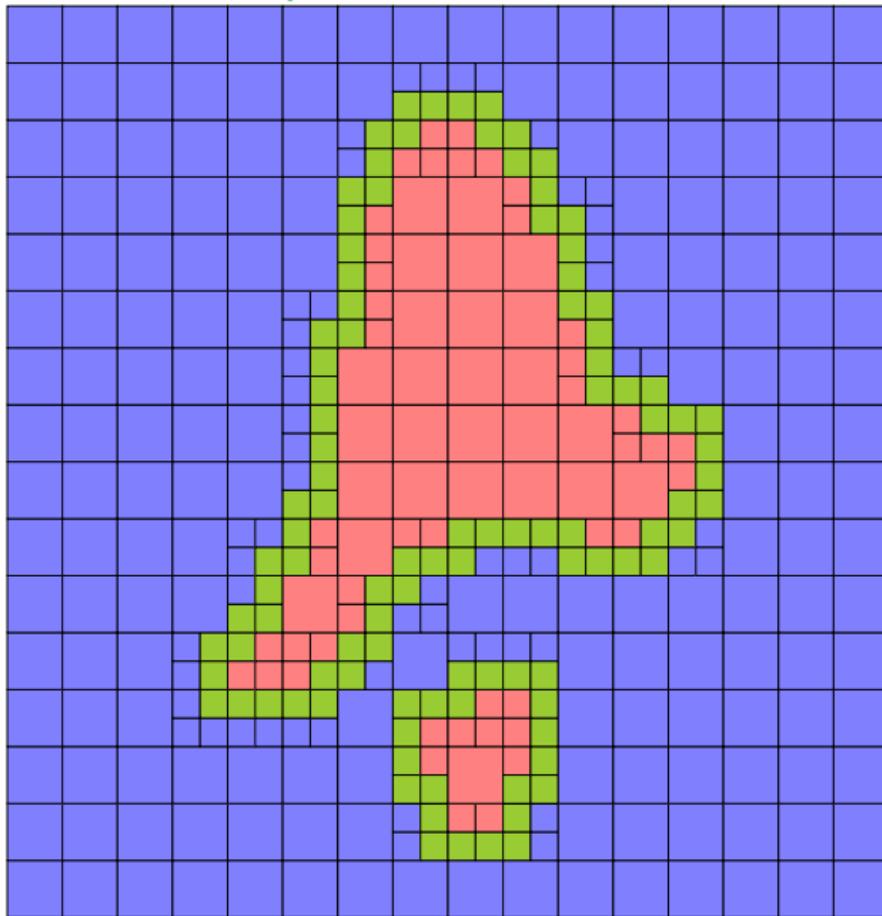
## Enumeração espacial uniforme





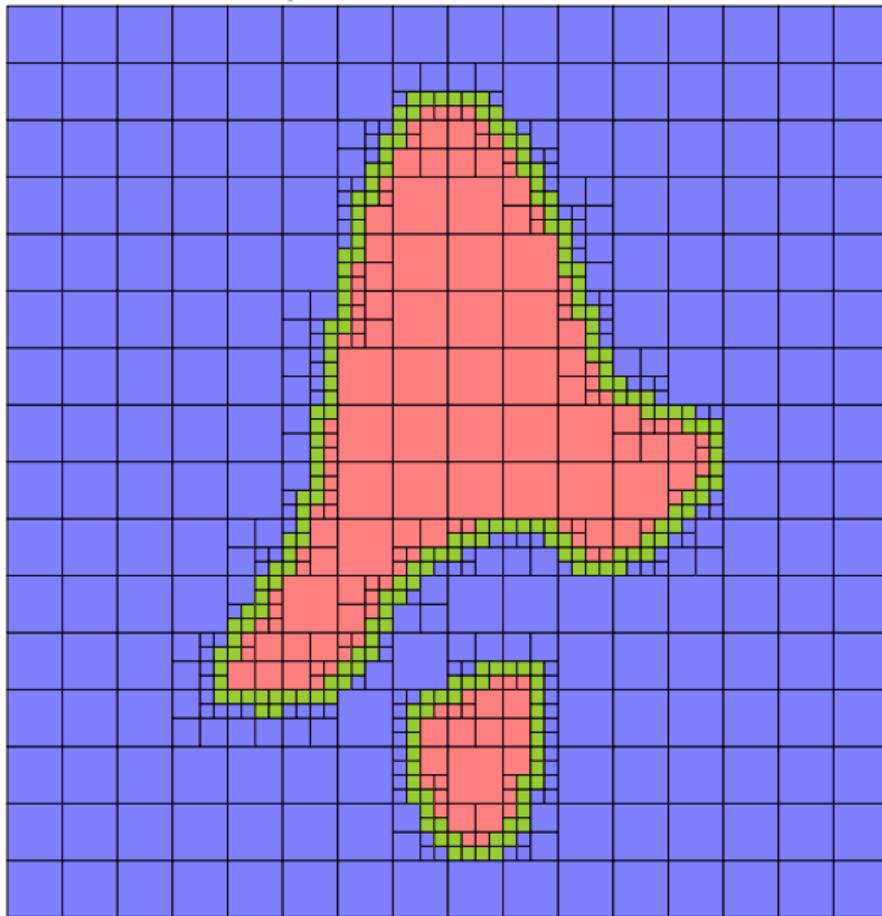
# Enumeração espacial semi-adaptativa

Suffern (1990)



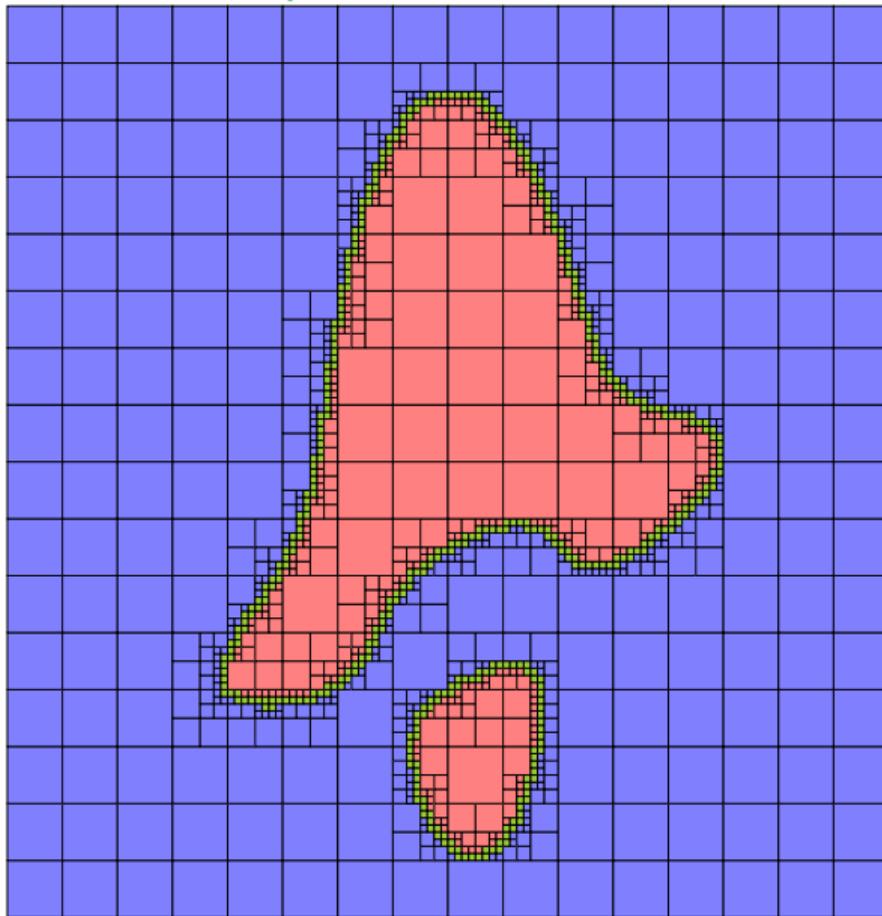
# Enumeração espacial semi-adaptativa

Suffern (1990)



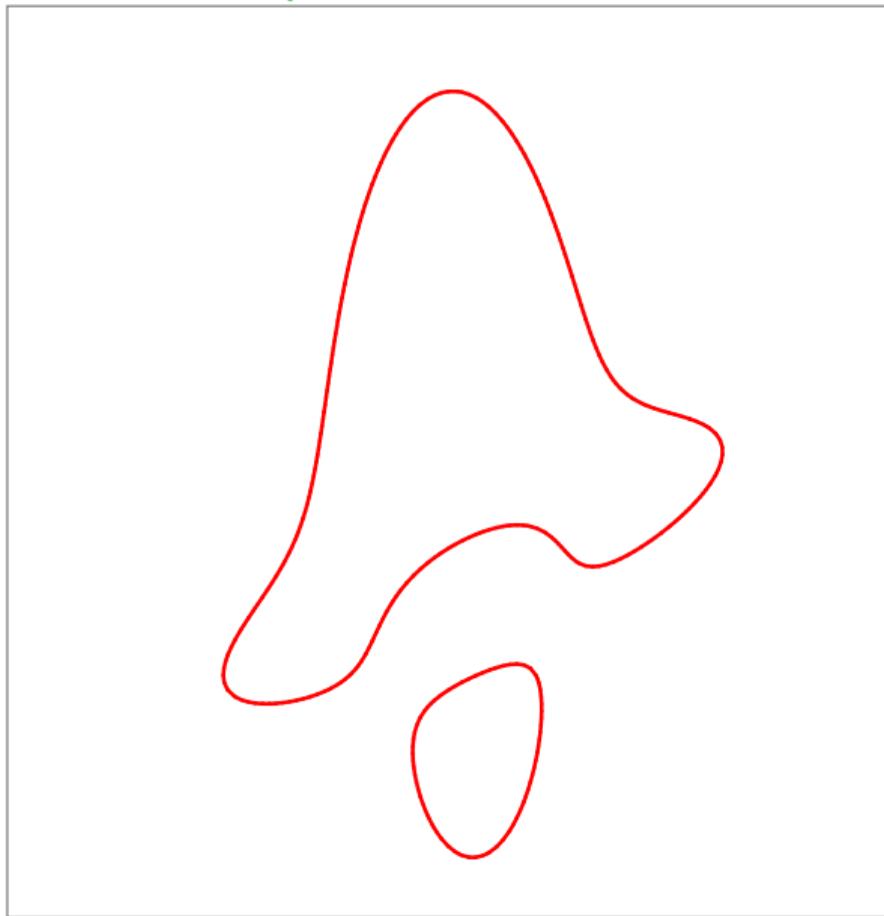
# Enumeração espacial semi-adaptativa

Suffern (1990)



## Enumeração espacial semi-adaptativa

Suffern (1990)

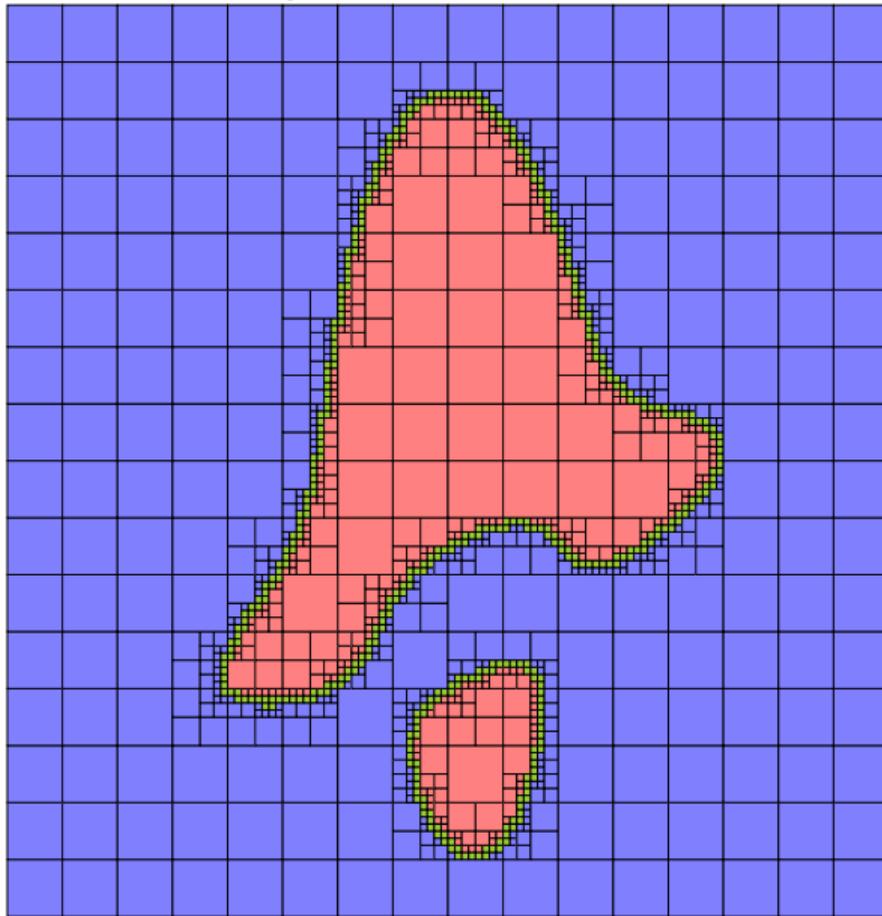


```
function explore(xmin,xmax,ymin,ymax,level)
  if level < searchlevel then
    subdivide(xmin,xmax,ymin,ymax,level)
  elseif contour_present(xmin,xmax,ymin,ymax) then
    if level < plotlevel then
      subdivide(xmin,xmax,ymin,ymax,level)
    else
      draw(xmin,xmax,ymin,ymax)
    end
  else
    draw(xmin,xmax,ymin,ymax)
  end
end
end
```

```
function subdivide(xmin,xmax,ymin,ymax,level)
    local xmid=(xmin+xmax)/2
    local ymid=(ymin+ymax)/2
    explore(xmin,xmid,ymin,ymid,level+1)
    explore(xmid,xmax,ymin,ymid,level+1)
    explore(xmin,xmid,ymid,ymax,level+1)
    explore(xmid,xmax,ymid,ymax,level+1)
end
```

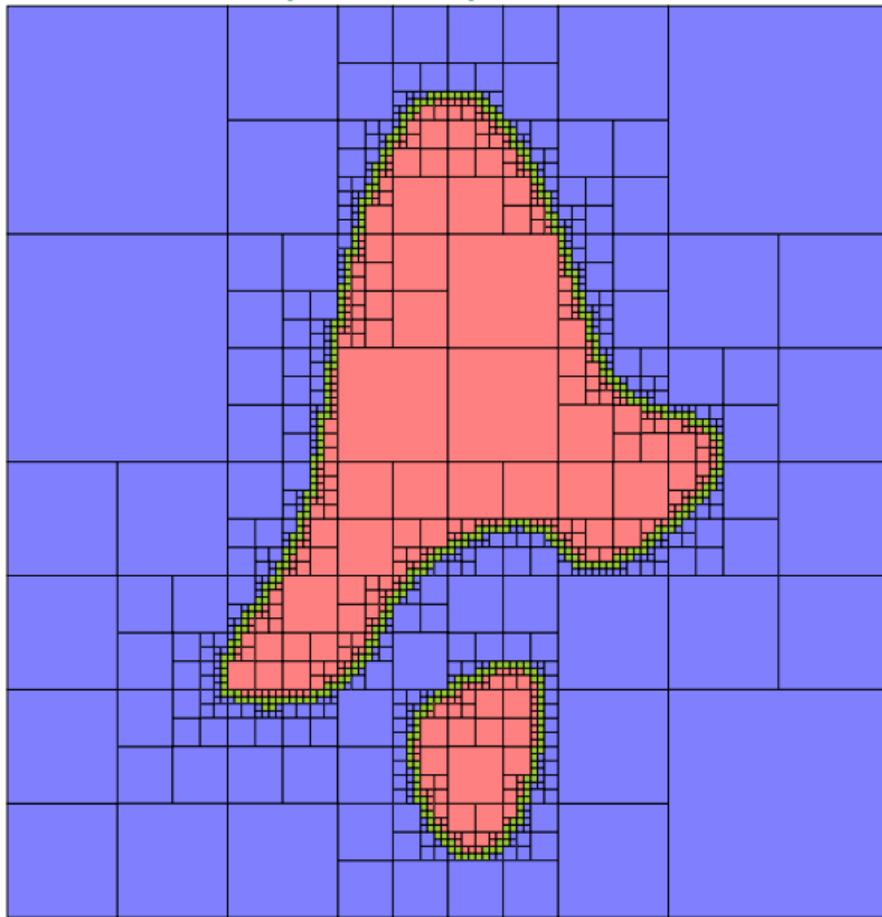
# Enumeração espacial semi-adaptativa

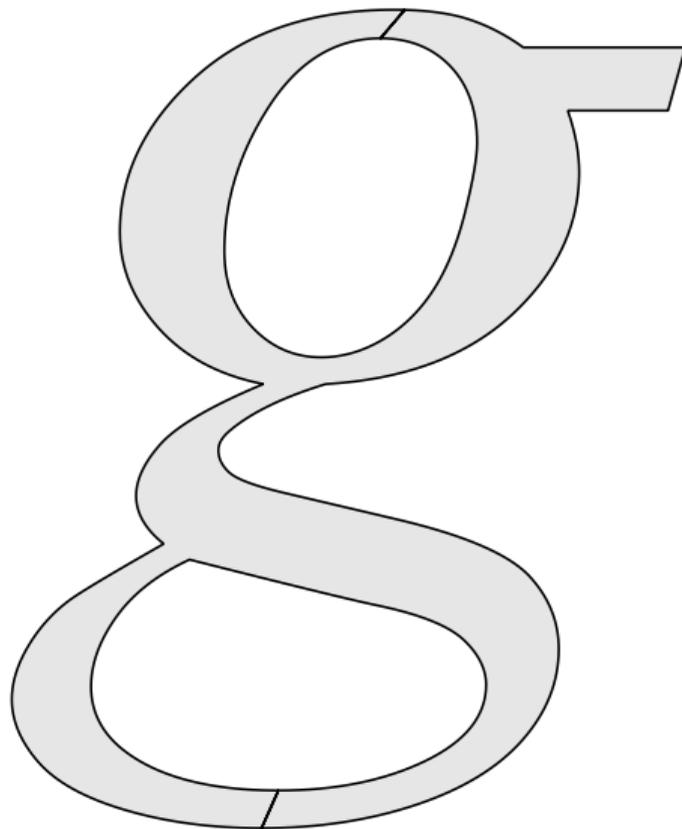
Suffern (1990)



# Enumeração espacial semi-adaptativa podada

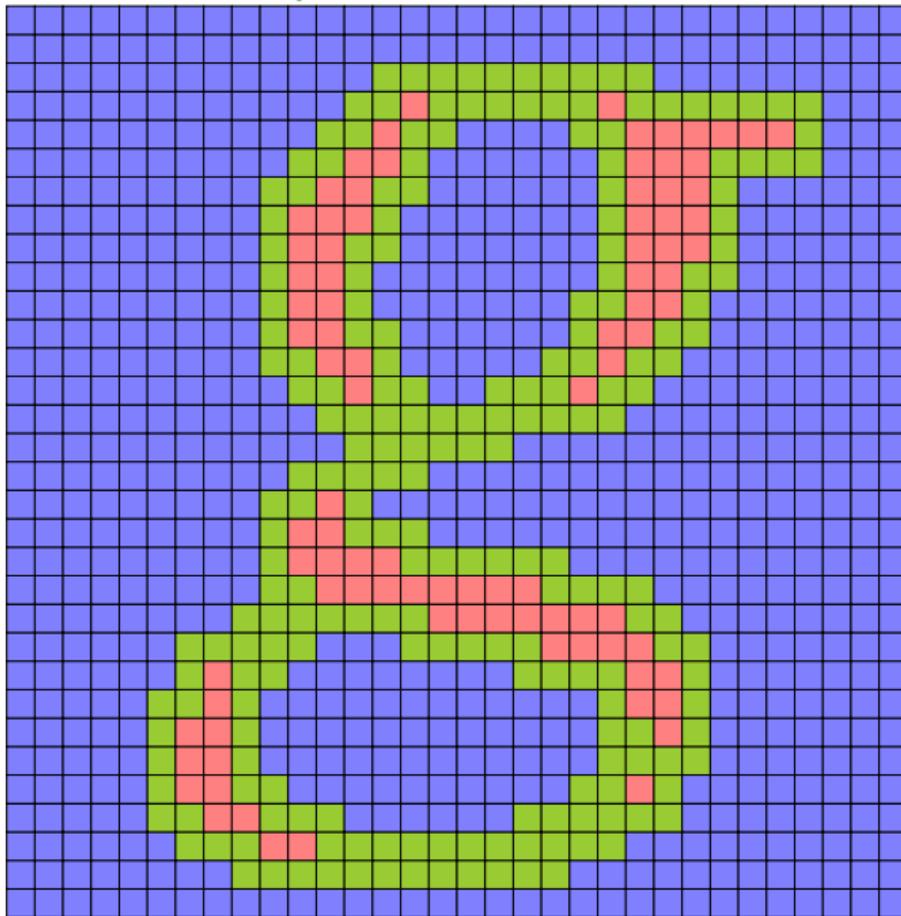
Suffern (1990)





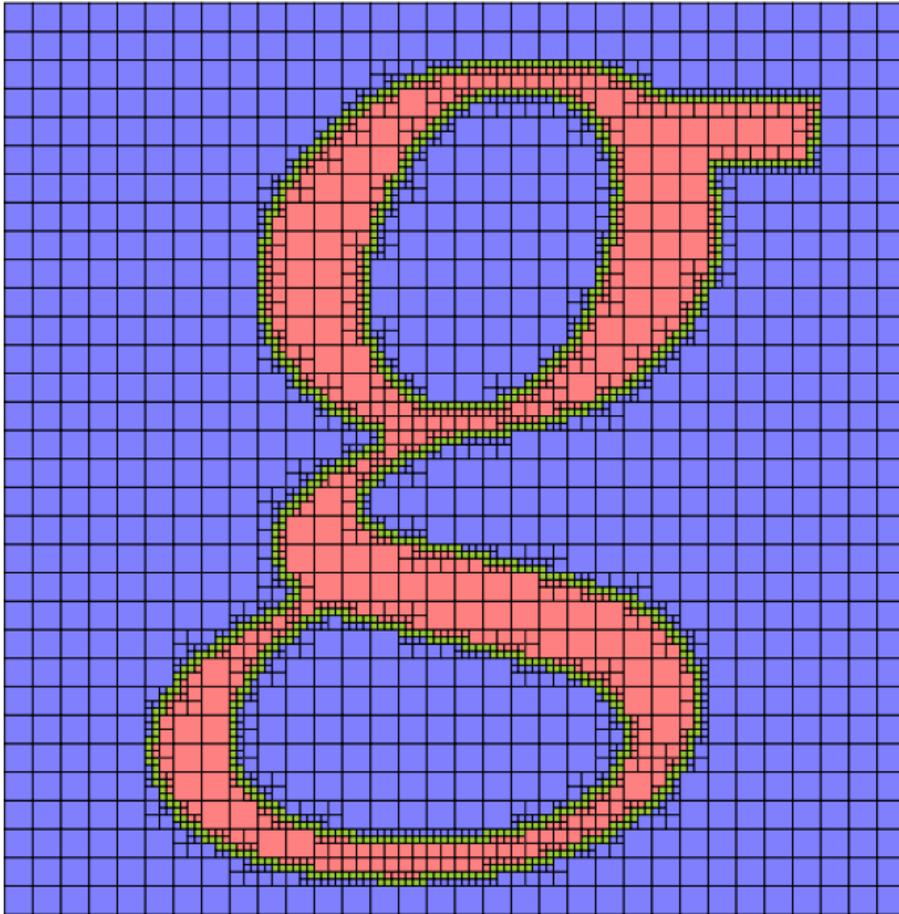
# Enumeração espacial semi-adaptativa

Suffern (1990)



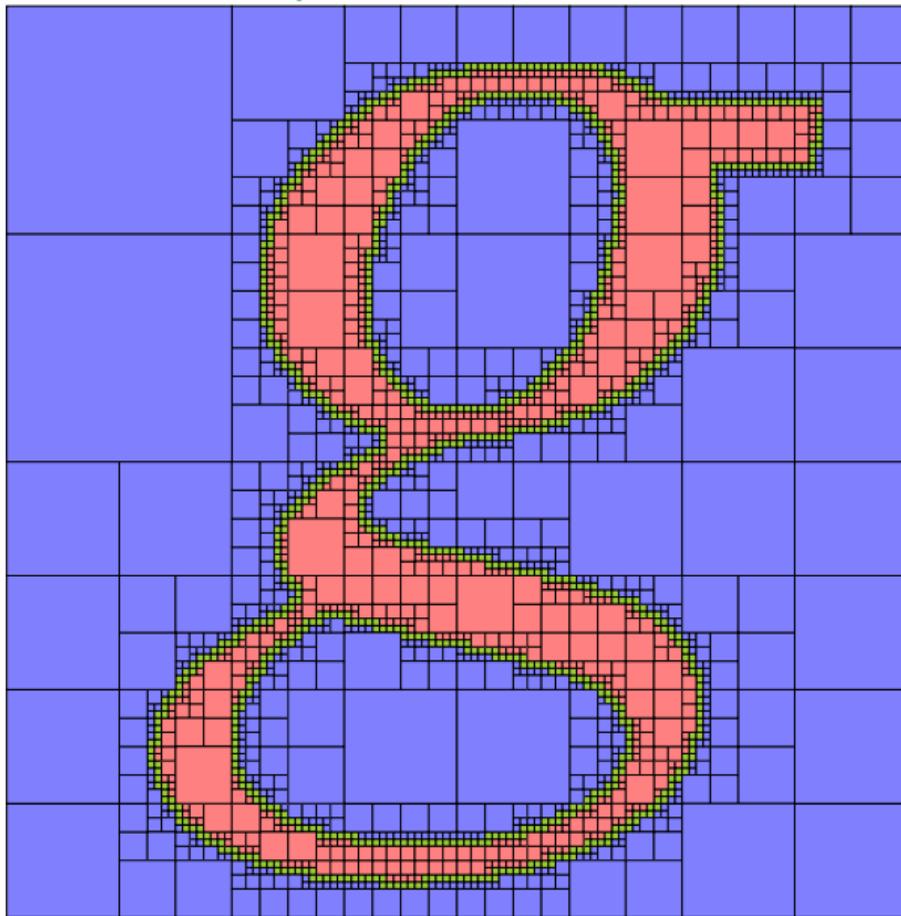
# Enumeração espacial semi-adaptativa

Suffern (1990)



# Enumeração espacial semi-adaptativa

Suffern (1990)



# Enumeração espacial semi-adaptativa

Suffern (1990)



## Localizando ponto em relação a polígono

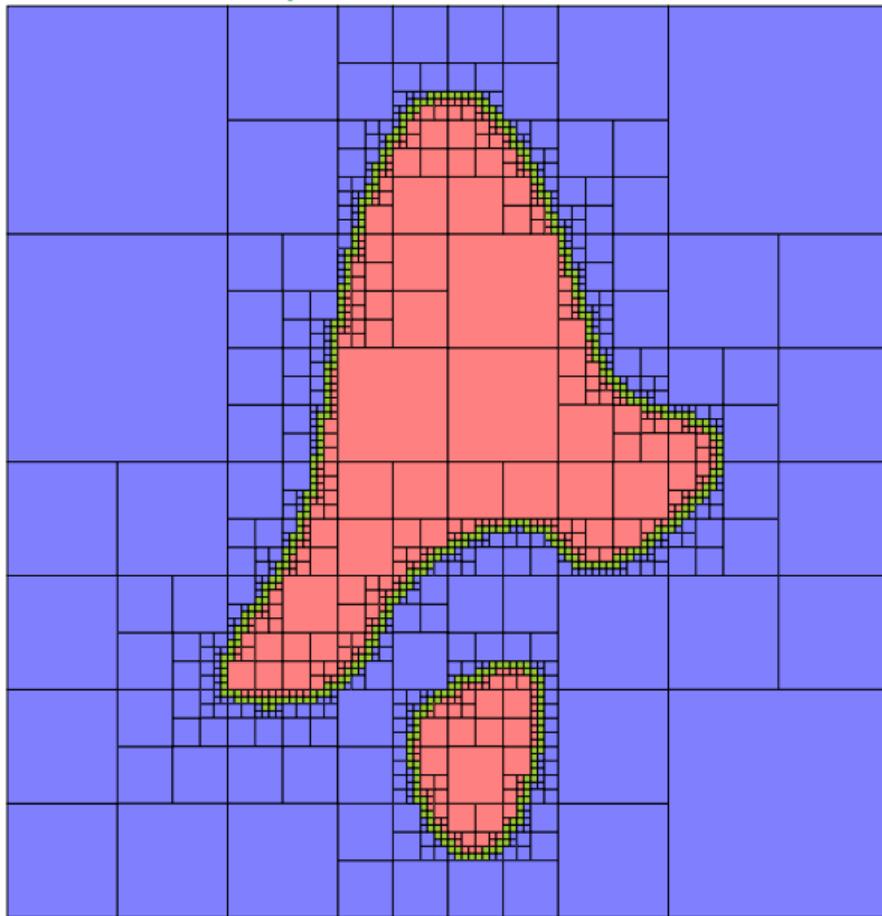
W. Randolph Franklin

```
function f(x,y)
    return inside(x,y) and -1 or 1
end

function inside(x,y)
    local inside=false
    for i=1,N do
        local j=i+1
        if ((Y[i]>y)~=(Y[j]>y)) and
            (x<((X[j]-X[i])*(y-Y[i])/(Y[j]-Y[i])+X[i])) then
            inside = not inside
        end
    end
    return inside
end
```

# Enumeração espacial semi-adaptativa

Suffern (1990)



## Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$

- $X = X_1 \times \cdots \times X_n \subseteq \Omega$

- **oráculo:** intervalo  $F(X) \subseteq \mathbf{R}$

$$F(X) \supseteq f(X) = \{f(\mathbf{x}) : \mathbf{x} \in X\}$$

$$x_i \in X_i \implies f(x_1, \dots, x_n) \in F(X)$$

## Oráculos intervalares

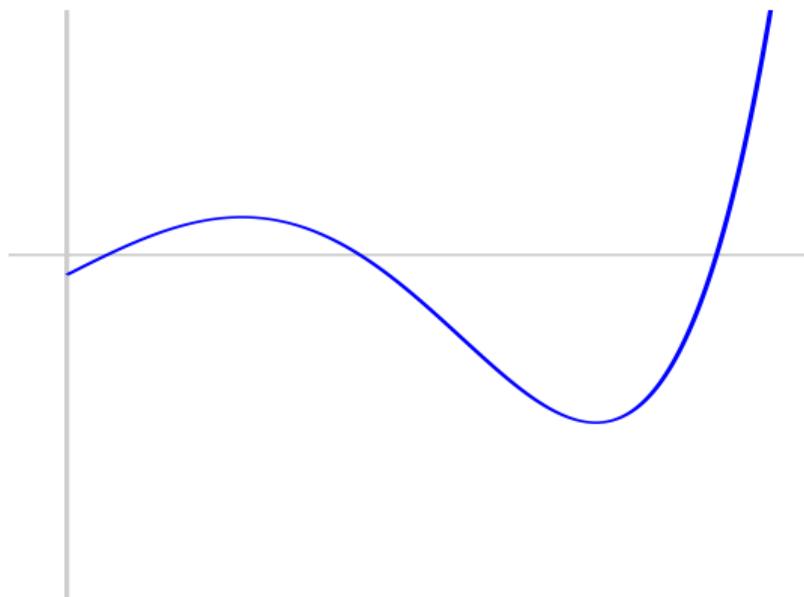
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$f$

## Oráculos intervalares

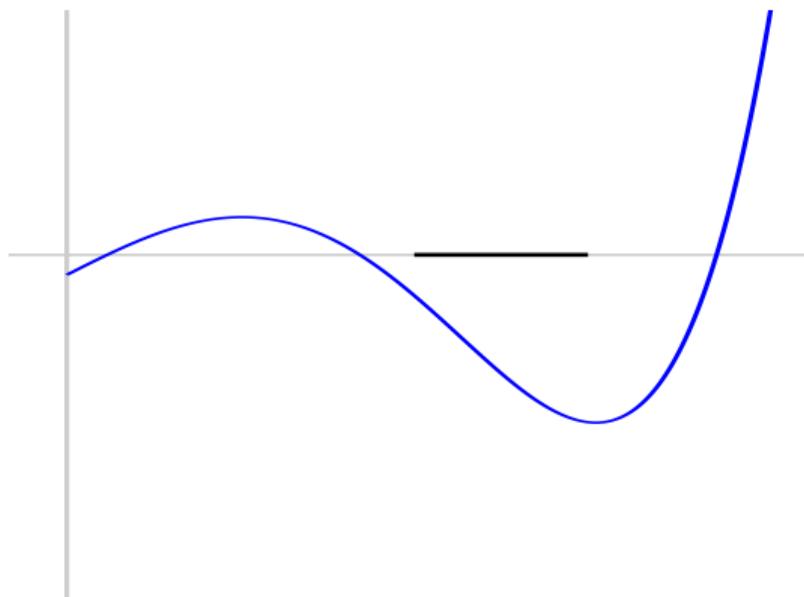
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- **oráculo:** intervalo  $F(X) \subseteq \mathbf{R}$

$$F(X) \supseteq f(X) = \{f(x) : x \in X\}$$

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$x$

## Oráculos intervalares

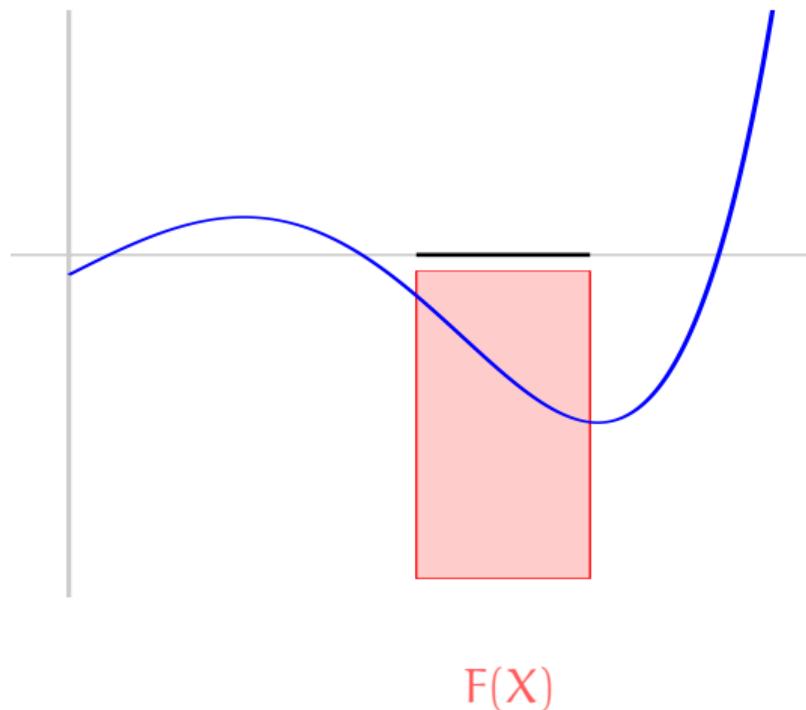
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## Oráculos intervalares

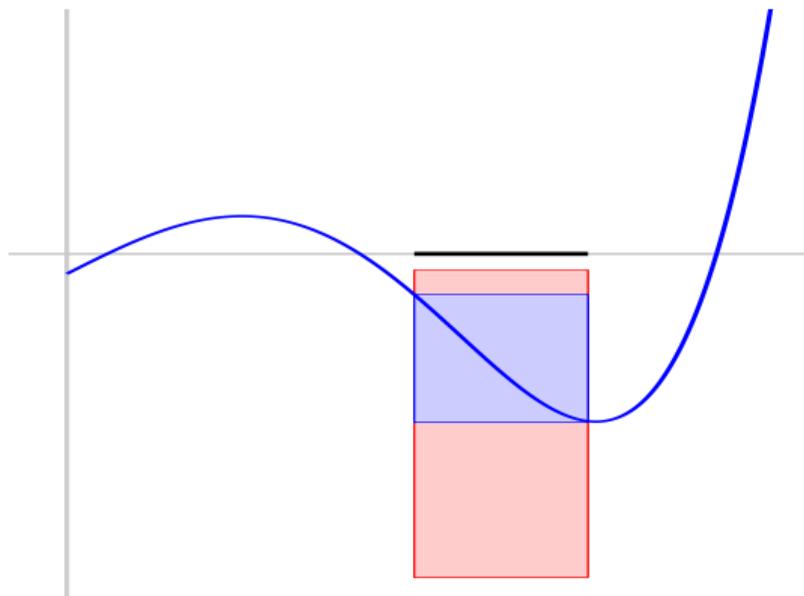
■  $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$

■  $X = X_1 \times \cdots \times X_n \subseteq \Omega$

■ **oráculo:** intervalo  $F(X) \subseteq \mathbf{R}$

$$F(X) \supseteq f(X) = \{f(x) : x \in X\}$$

$$x_i \in X_i \implies f(x_1, \dots, x_n) \in F(X)$$



$$F(X) \supseteq f(X)$$

## Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$

- $X = X_1 \times \cdots \times X_n \subseteq \Omega$

- **oráculo:** intervalo  $F(X) \subseteq \mathbf{R}$

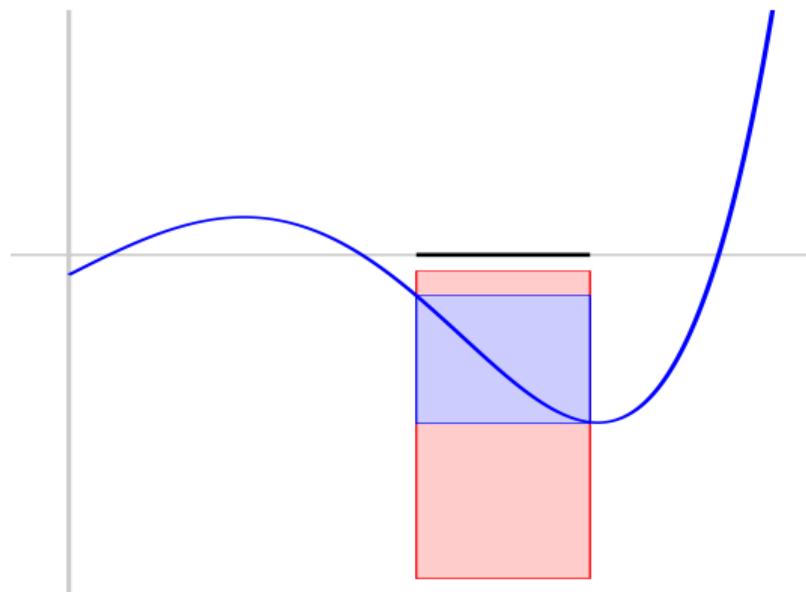
$$F(X) \supseteq f(X) = \{f(x) : x \in X\}$$

$$x_i \in X_i \implies f(x_1, \dots, x_n) \in F(X)$$

- $0 \notin F(X) \implies 0 \notin f(X)$

- **prova computacional**

$f(x) = 0$  não tem solução em  $X$



$$F(X) \supsetneq f(X)$$

## Oráculos intervalares

- aritmética intervalar

Moore (1966)

- formas de valor médio

Caprani–Madsen (1980)

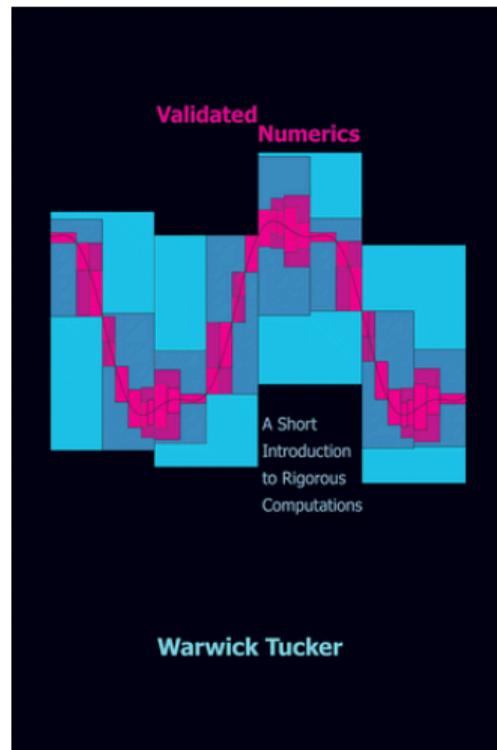
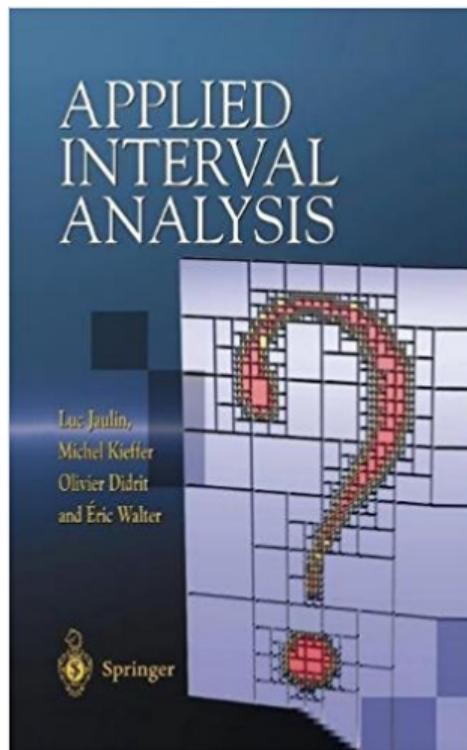
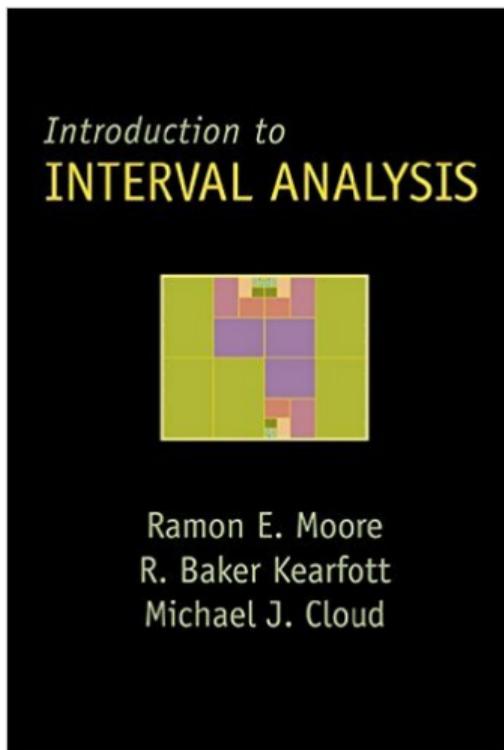
- aritmética afim

Comba–Stolfi (1993)

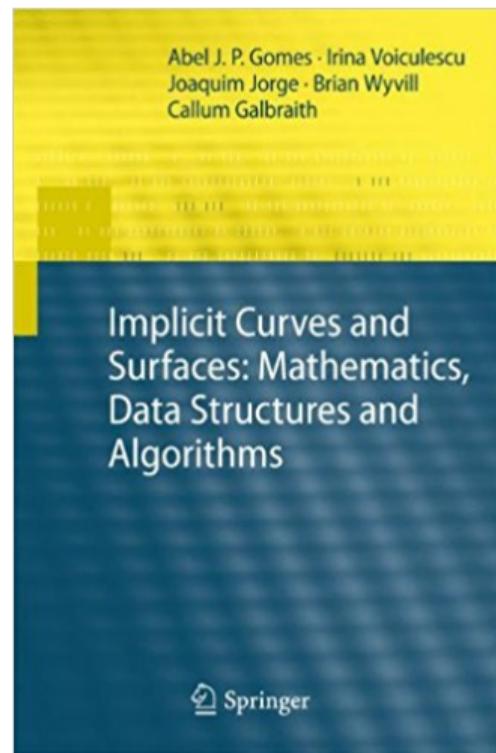
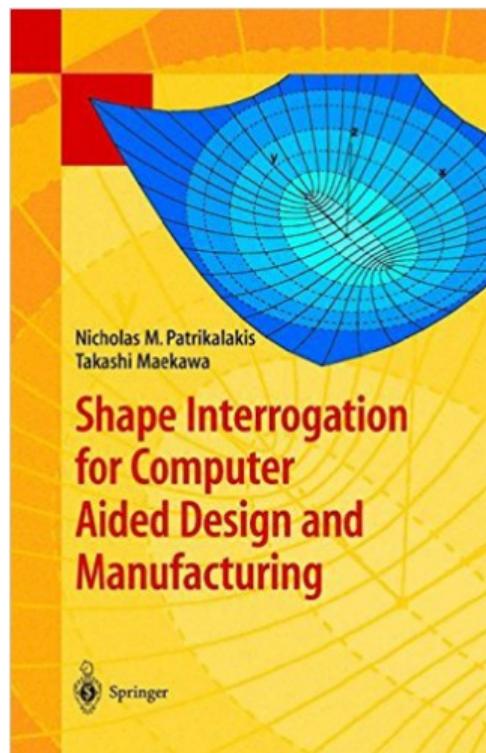
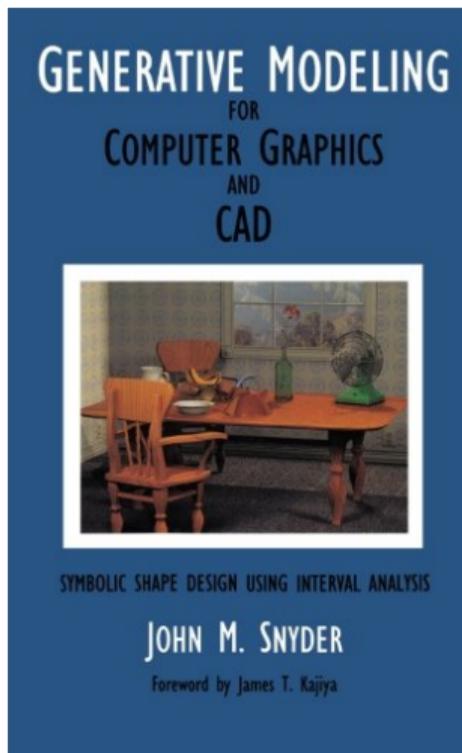
- formas de Taylor

Berz et al. (1996)

# Métodos intervalares



# Métodos intervalares



## Aritmética intervalar

$$\begin{aligned} [a, b] \star [c, d] &= \{x \star y : x \in [a, b], y \in [c, d]\} \\ &= [\min(x \star y), \max(x \star y)] \end{aligned}$$

## Aritmética intervalar

$$\begin{aligned} [a, b] \star [c, d] &= \{x \star y : x \in [a, b], y \in [c, d]\} \\ &= [\min(x \star y), \max(x \star y)] \end{aligned}$$

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

$$[a, b] / [c, d] = [a, b] \times [1/d, 1/c]$$

$$[a, b]^2 = [\min(a^2, b^2), \max(a^2, b^2)] \quad \text{se } 0 \notin [a, b]$$

$$[a, b]^2 = [0, \max(a^2, b^2)] \quad \text{se } 0 \in [a, b]$$

$$\exp [a, b] = [\exp(a), \exp(b)]$$

## Aritmética intervalar

$$\begin{aligned} [a, b] \star [c, d] &= \{x \star y : x \in [a, b], y \in [c, d]\} \\ &= [\min(x \star y), \max(x \star y)] \end{aligned}$$

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

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$$[a, b]^2 = [0, \max(a^2, b^2)] \quad \text{se } 0 \in [a, b]$$

$$\exp [a, b] = [\exp(a), \exp(b)]$$

uso conveniente via sobrecarga de operadores

```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return show(x,y,"O") end
    if z<0 then return show(x,y,"I") end
    if level==0 then return show(x,y,"U") end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    explore(xlo,ylo,level-1)
    explore(xhi,ylo,level-1)
    explore(xlo,yhi,level-1)
    explore(xhi,yhi,level-1)
    return nil
end
```

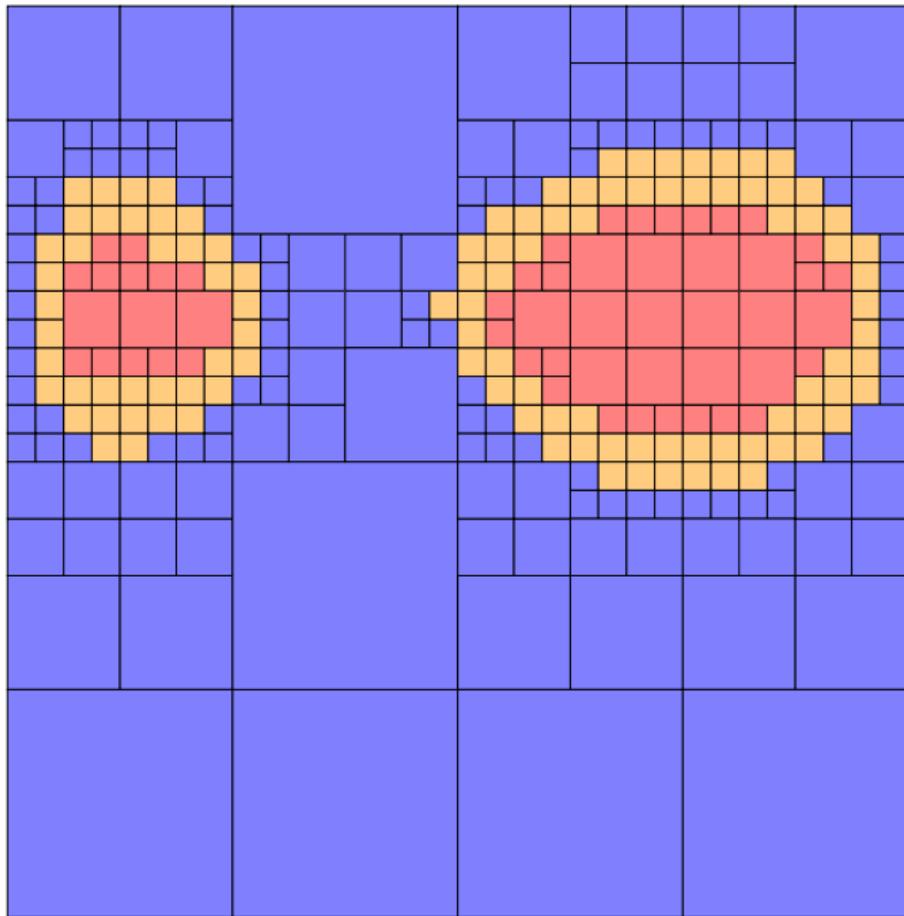
## Região implícita – método intervalar

“When you have eliminated the impossible,  
whatever remains, however improbable,  
must be the truth.”

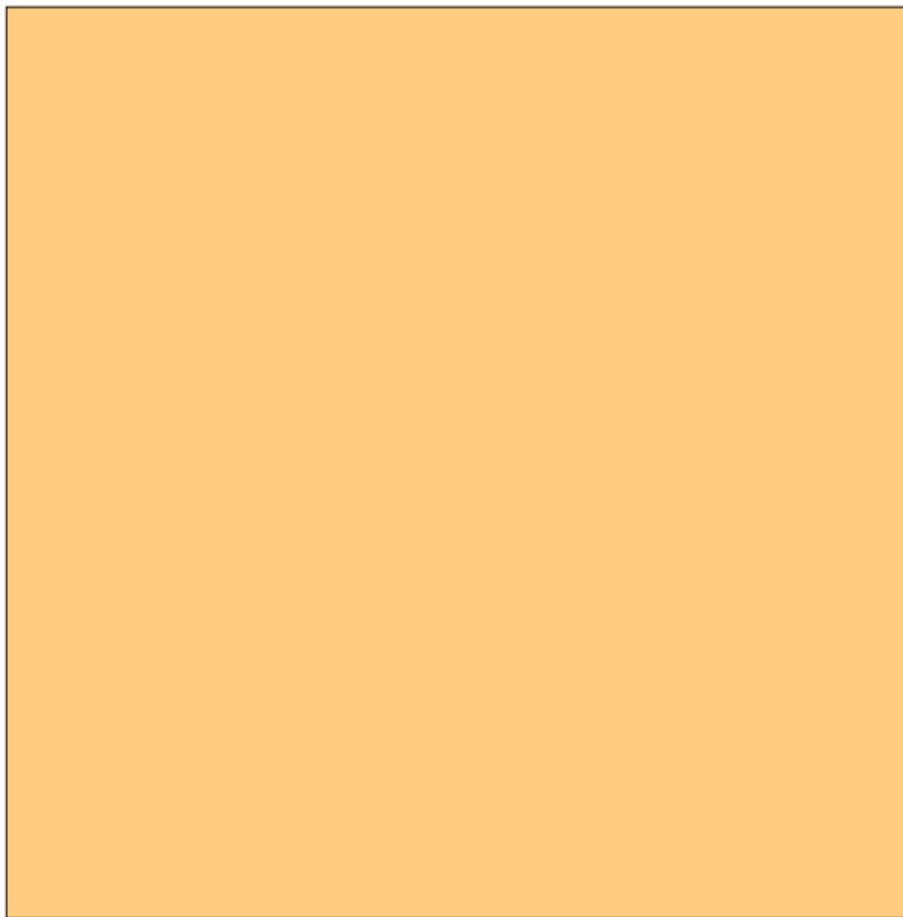
— Sherlock Holmes in *The Sign of Four*



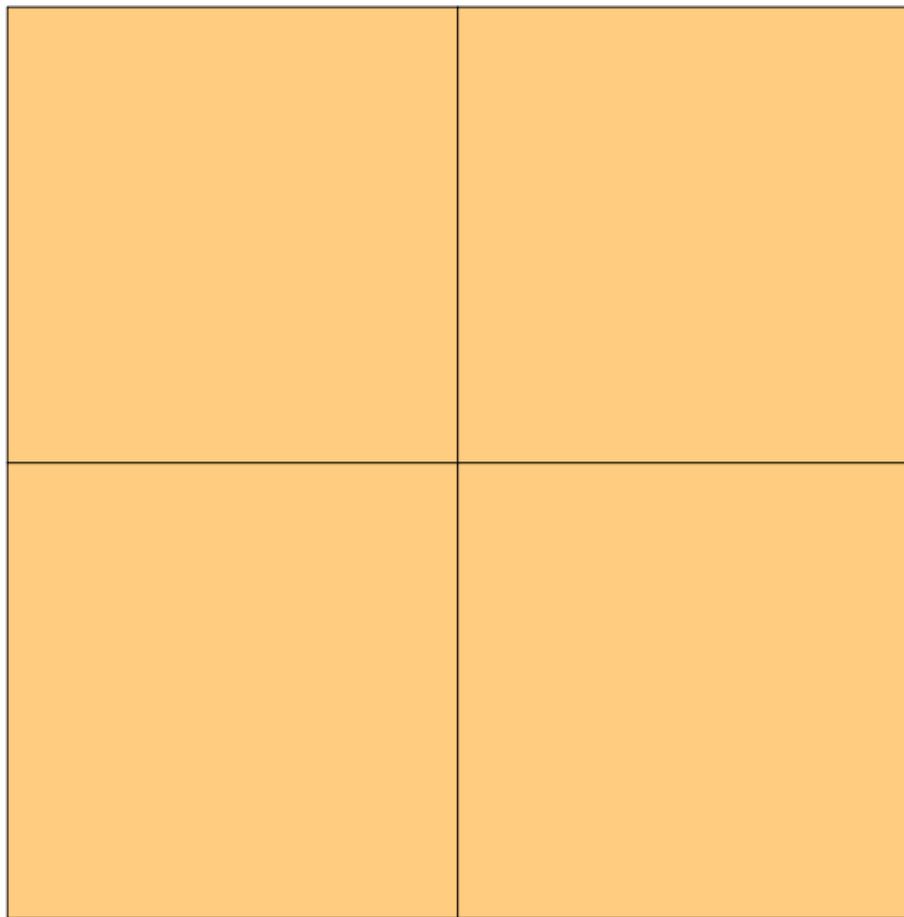
## Região implícita – método intervalar



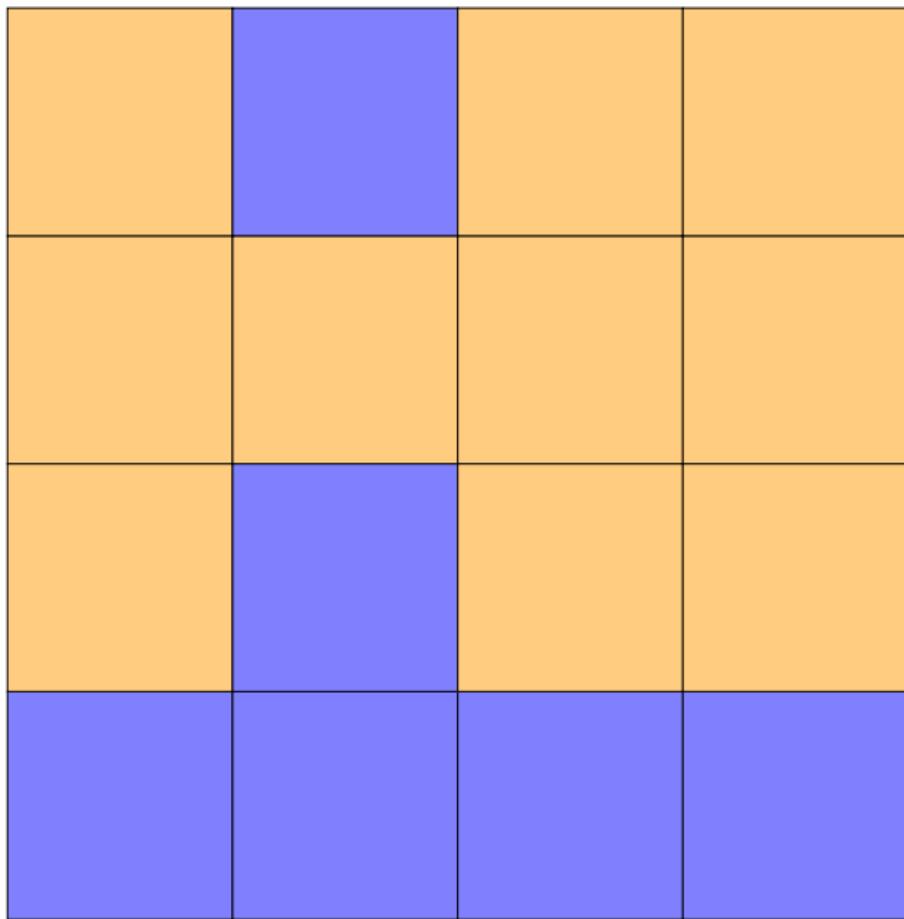
## Região implícita – método intervalar



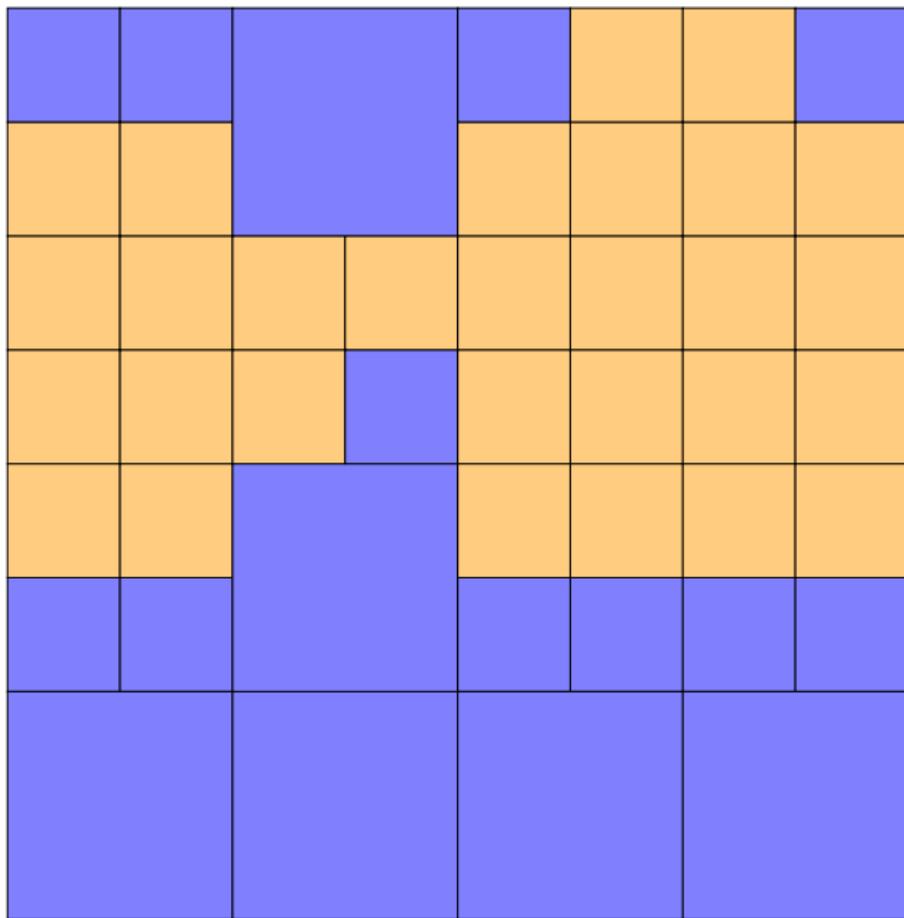
## Região implícita – método intervalar



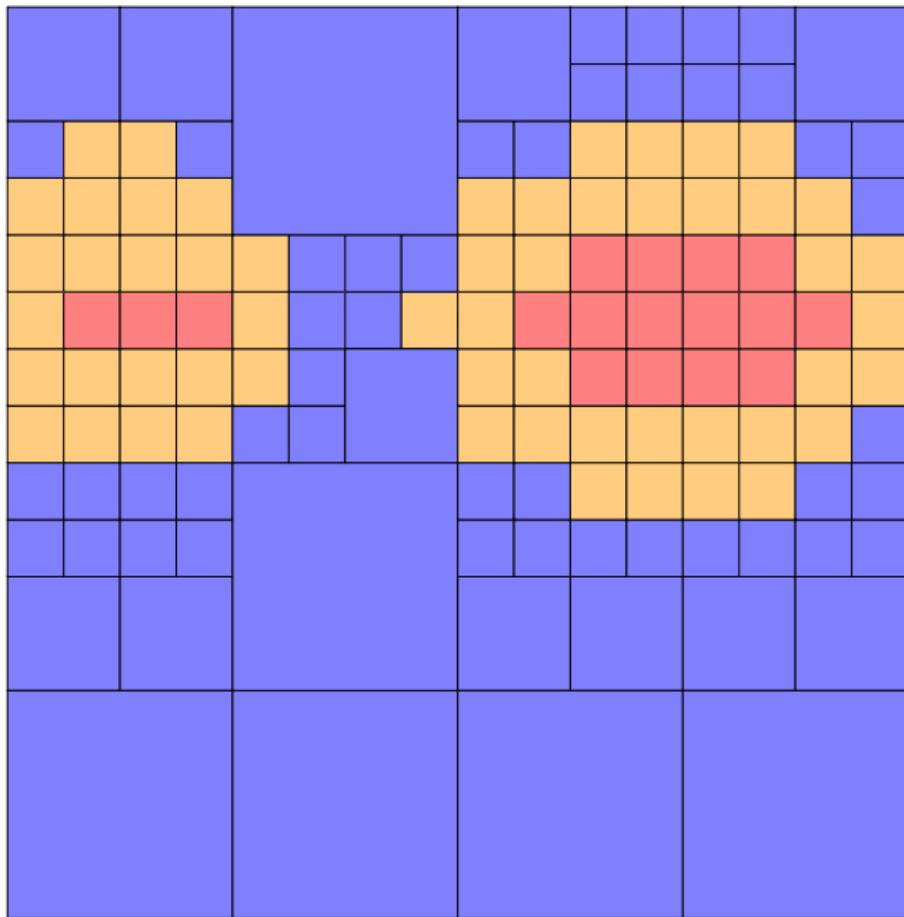
## Região implícita – método intervalar



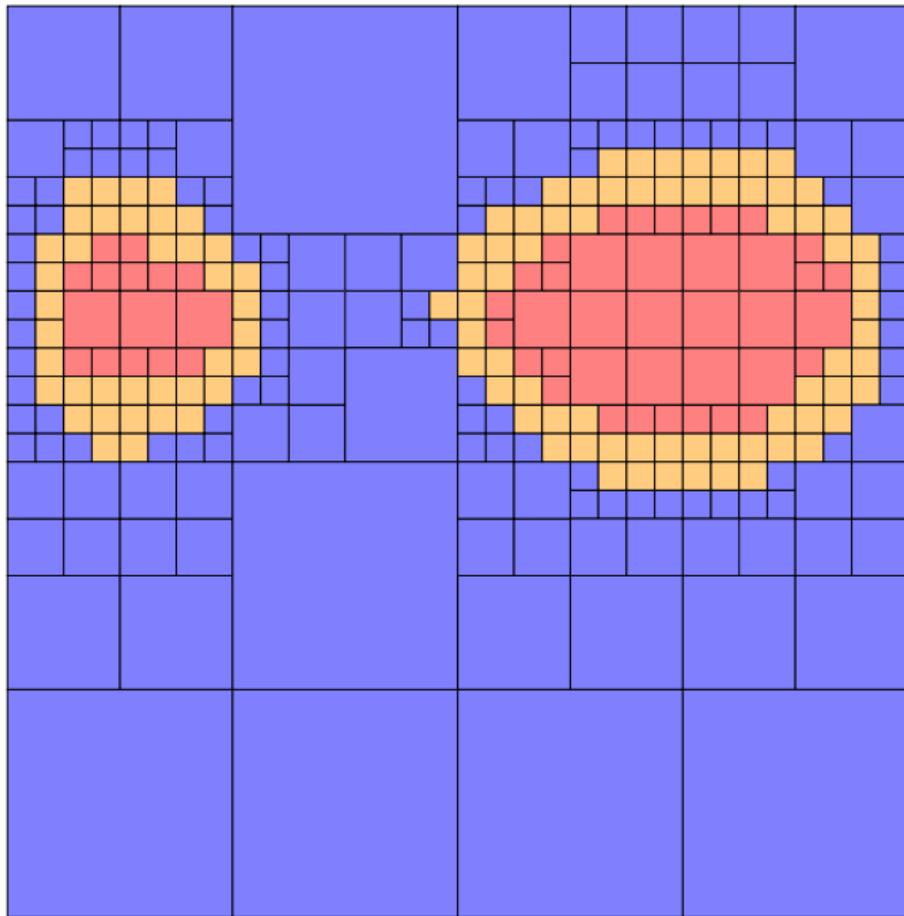
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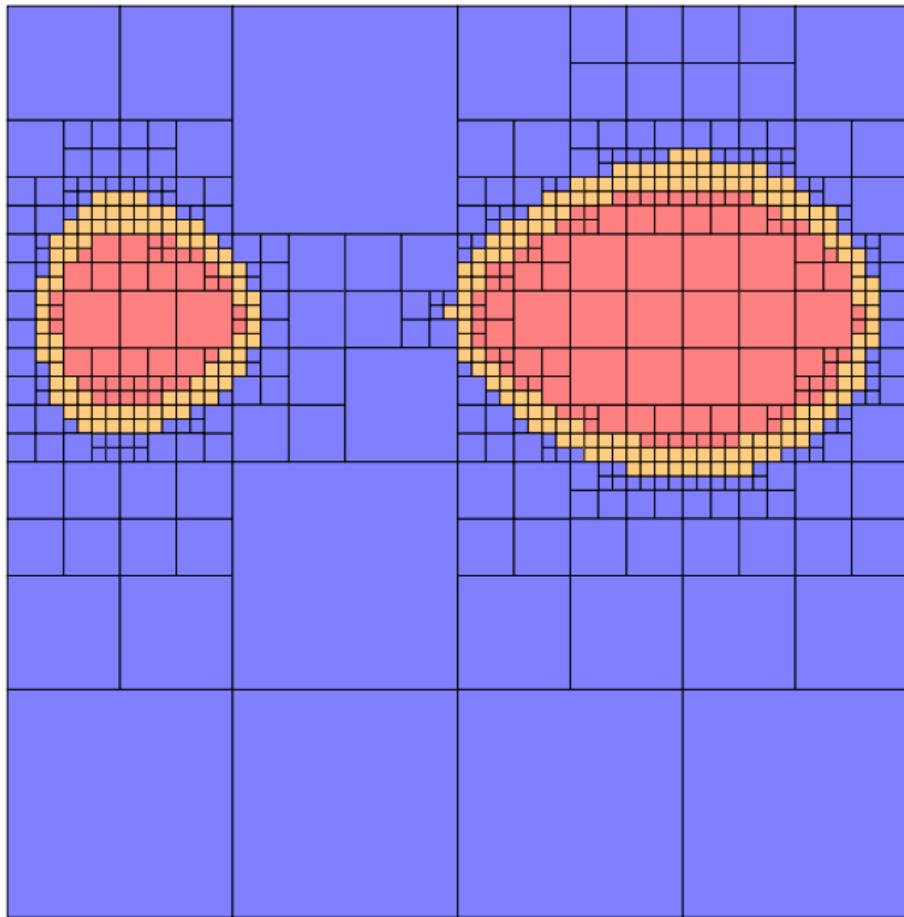
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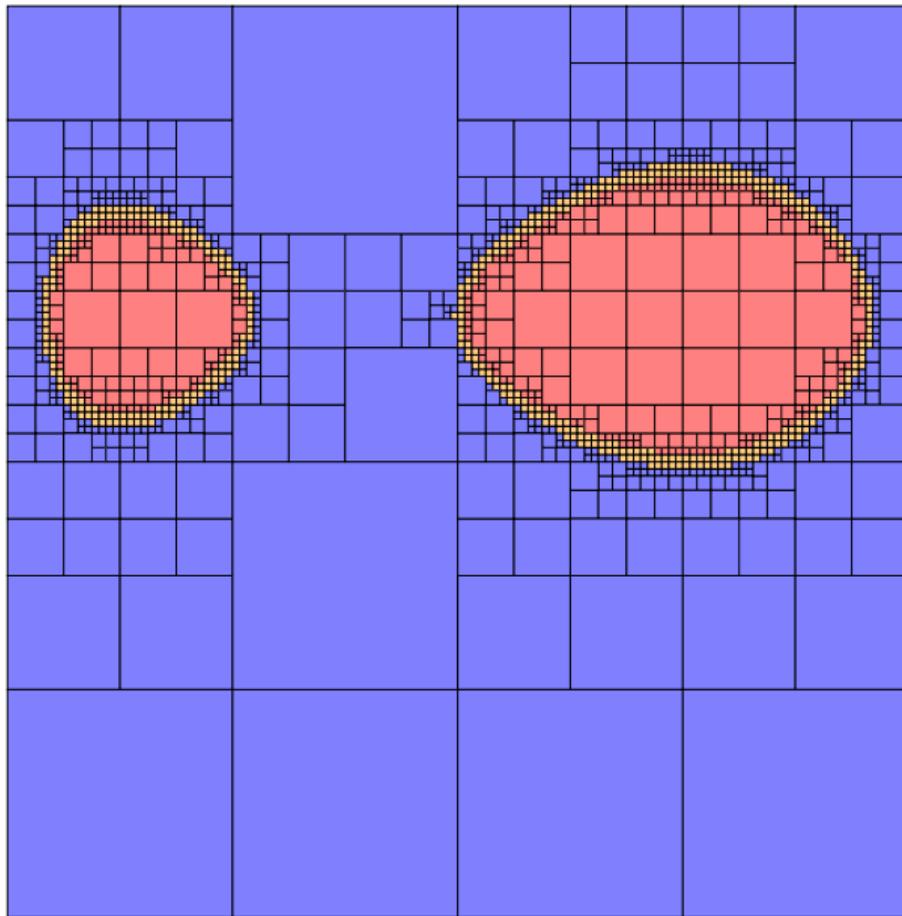
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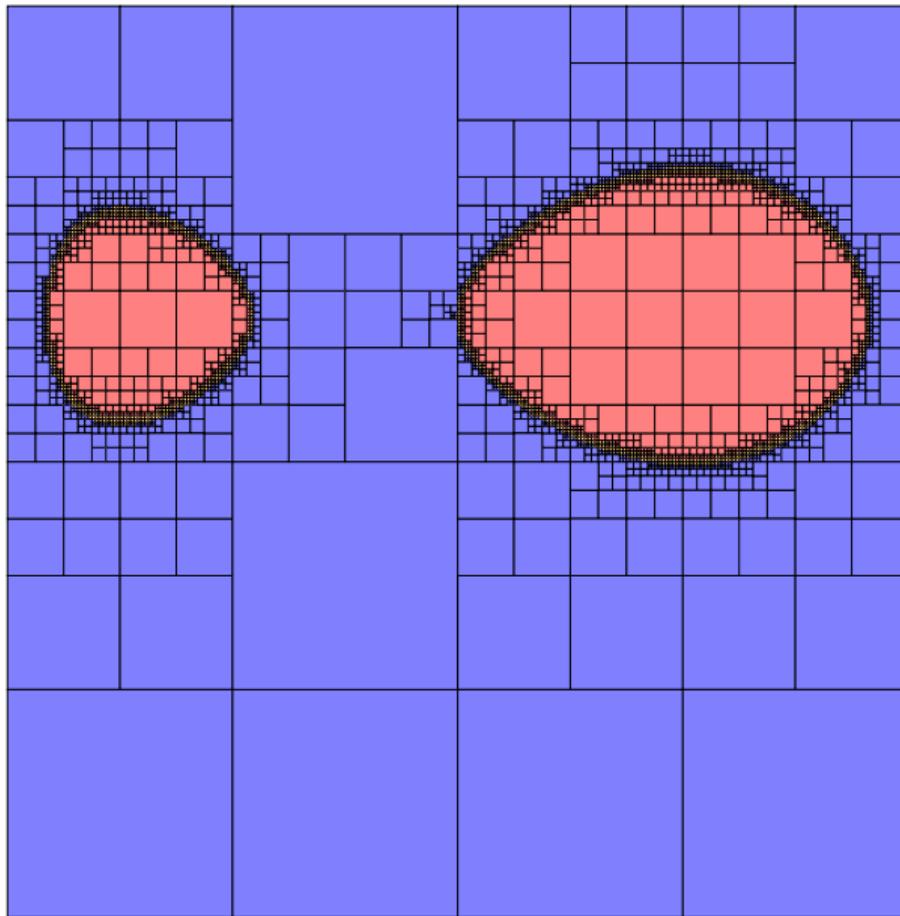
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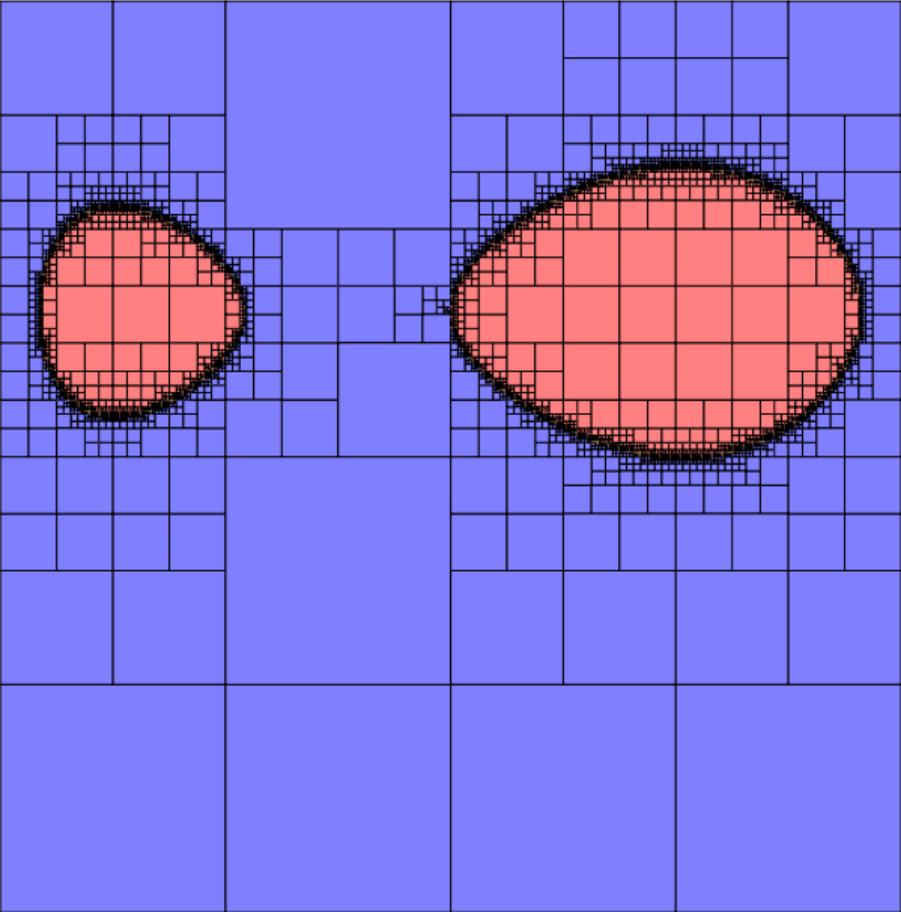
## Região implícita – método intervalar



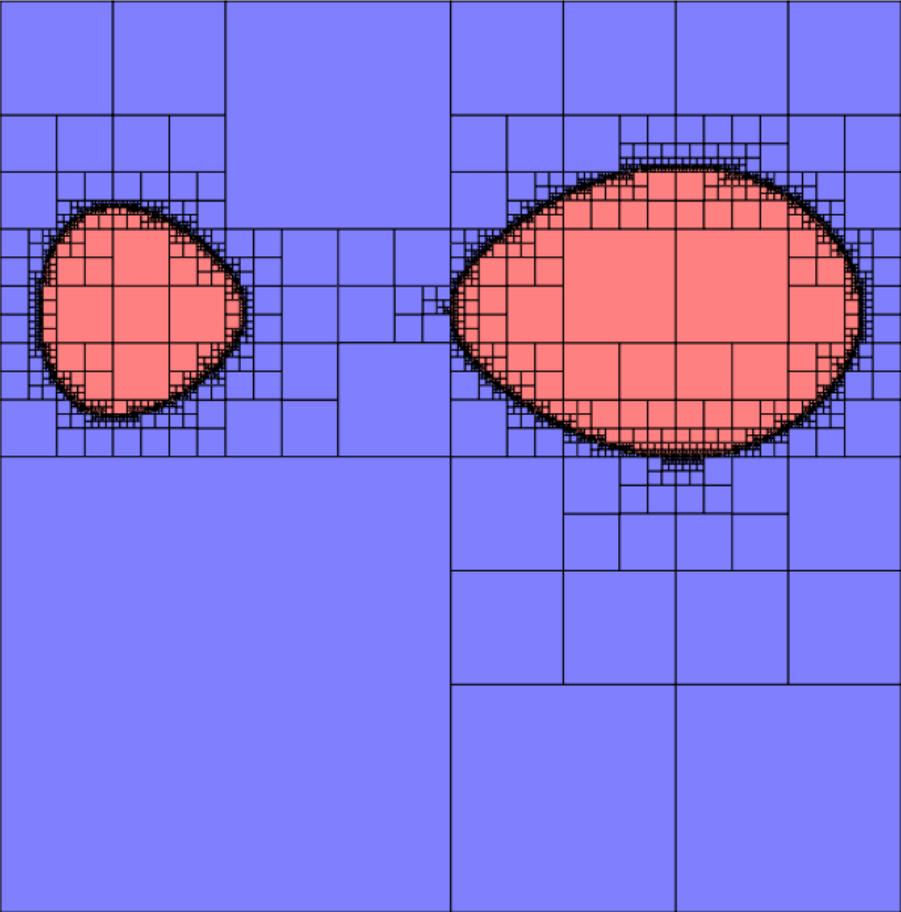
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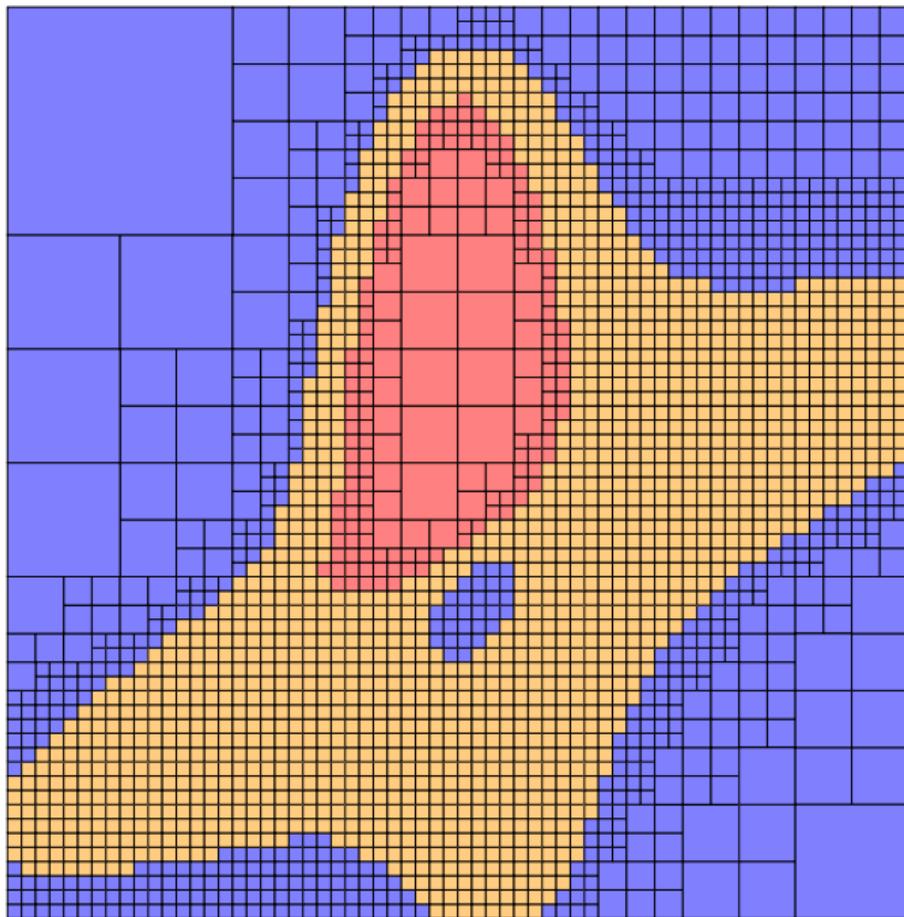
# Região implícita – método intervalar



# Região implícita – método intervalar

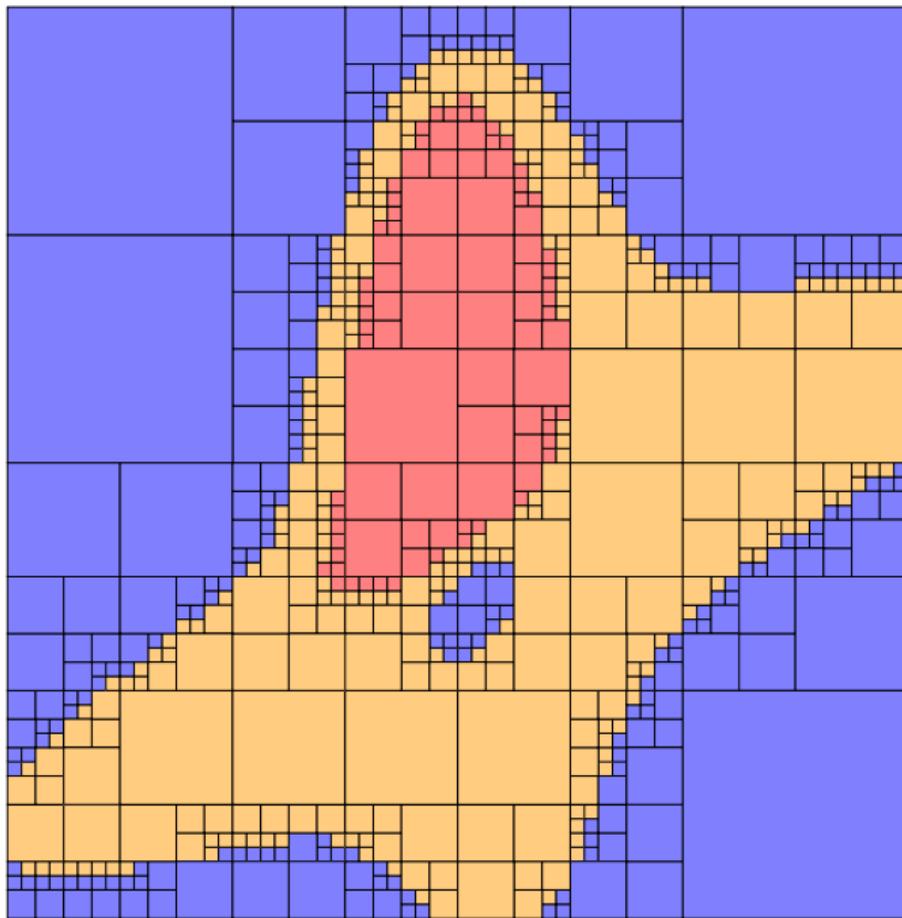


## Região implícita – método intervalar



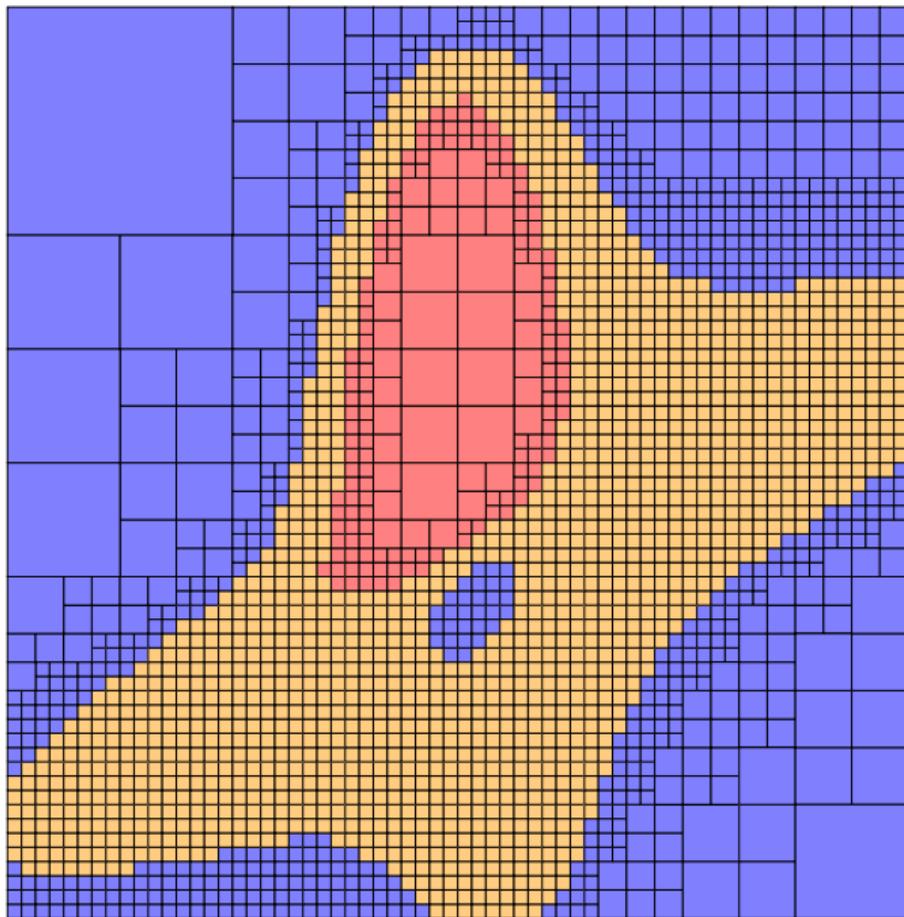
2423 leaves

## Região implícita – método intervalar



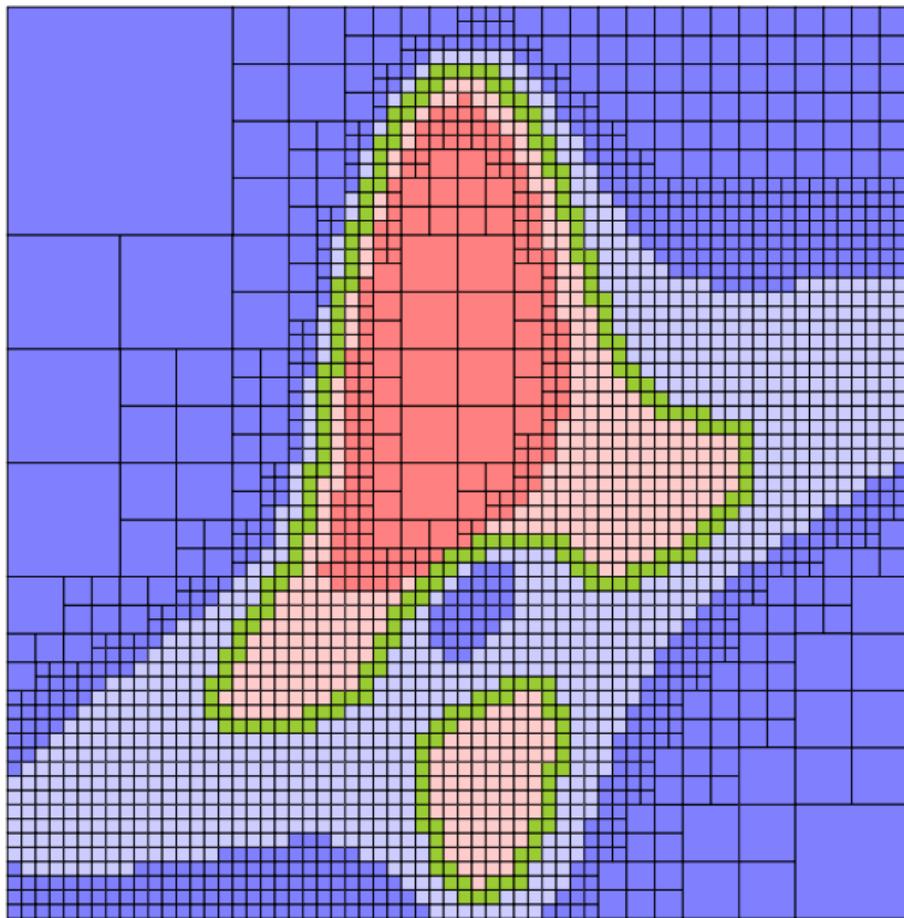
635 leaves

## Região implícita – método intervalar



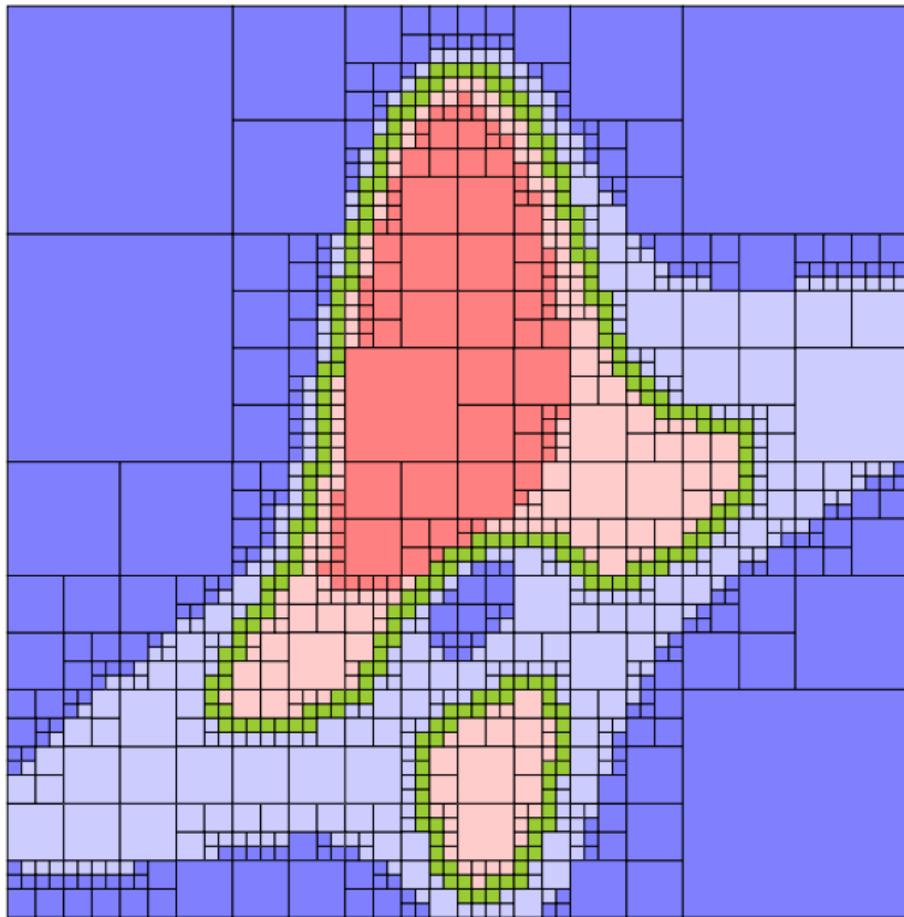
2423 leaves

## Região implícita – método intervalar híbrido



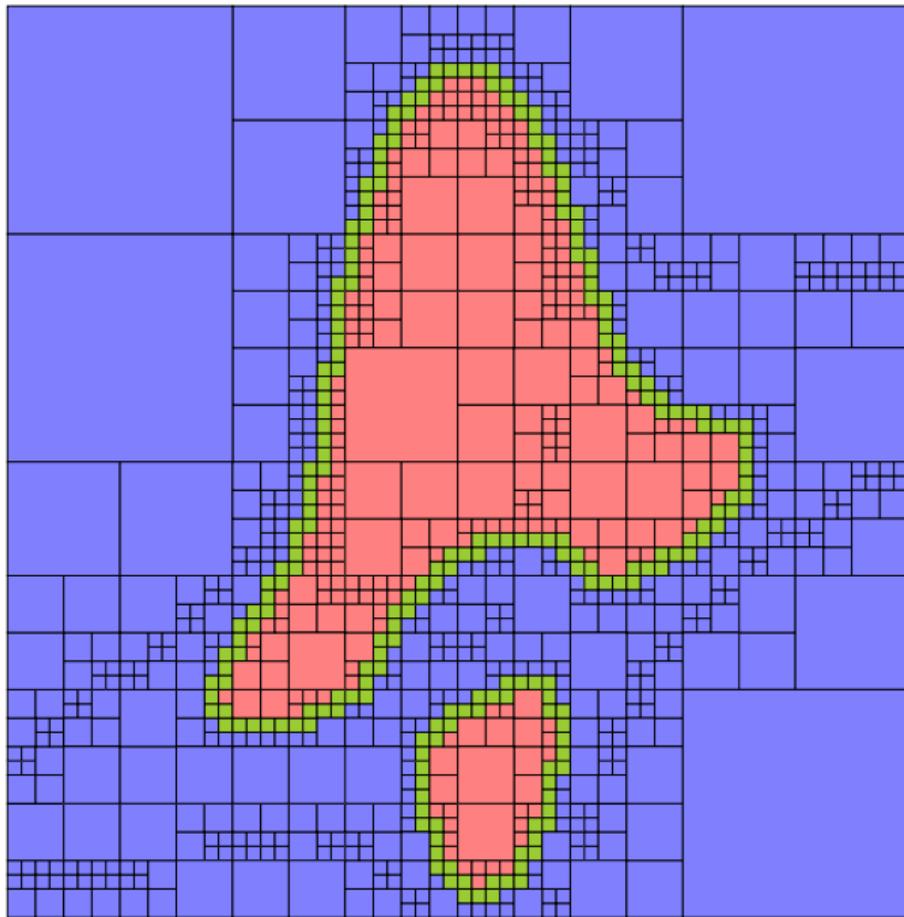
2423 leaves

## Região implícita – método intervalar híbrido



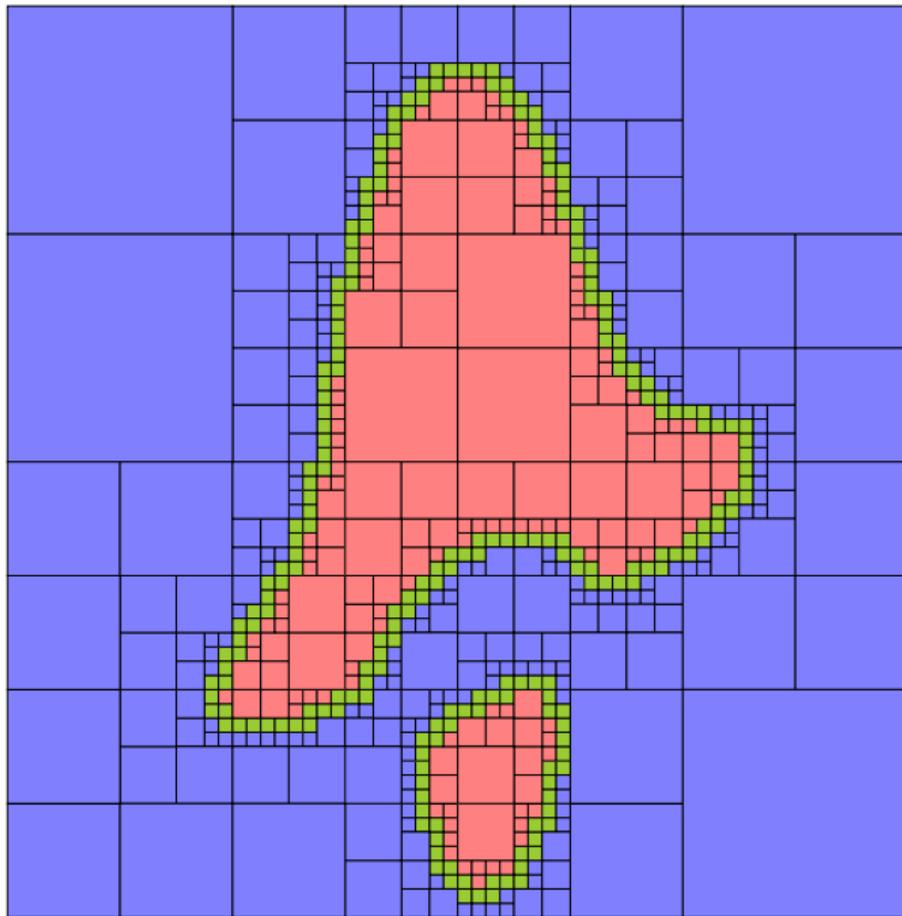
1061 leaves

## Região implícita – método intervalar híbrido otimista



1061 leaves

## Região implícita – método intervalar híbrido otimista



644 leaves

## Região implícita – método intervalar

```
function explore(x,y,level)
  local z=f(x,y)
  if z>0 then return show(x,y,"O") end
  if z<0 then return show(x,y,"I") end
  if level==0 then return show(x,y,"U") end
  local xlo,xhi=split(x)
  local ylo,yhi=split(y)
  explore(xlo,ylo,level-1)
  explore(xhi,ylo,level-1)
  explore(xlo,yhi,level-1)
  explore(xhi,yhi,level-1)
  return nil
end
```

## Região implícita – método intervalar híbrido otimista

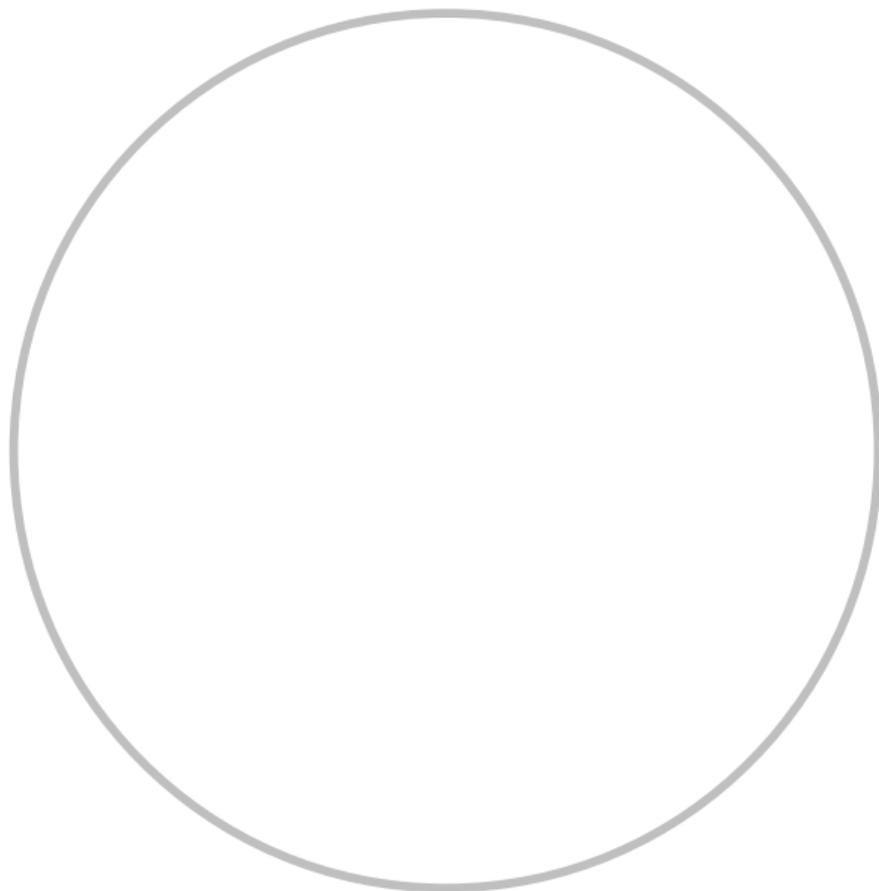
```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return "O" end
    if z<0 then return "I" end
    if level==0 then return classify(x,y) end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    local c1=explore(xlo,ylo,level-1)
    local c2=explore(xhi,ylo,level-1)
    local c3=explore(xlo,yhi,level-1)
    local c4=explore(xhi,yhi,level-1)
    if c1==c2 and c1==c3 and c1==c4 then
        return c1
    else
        show(xlo,ylo,c1)
        show(xhi,ylo,c2)
        show(xlo,yhi,c3)
        show(xhi,yhi,c4)
        return nil
    end
end
end
```

## Região implícita – método intervalar híbrido otimista

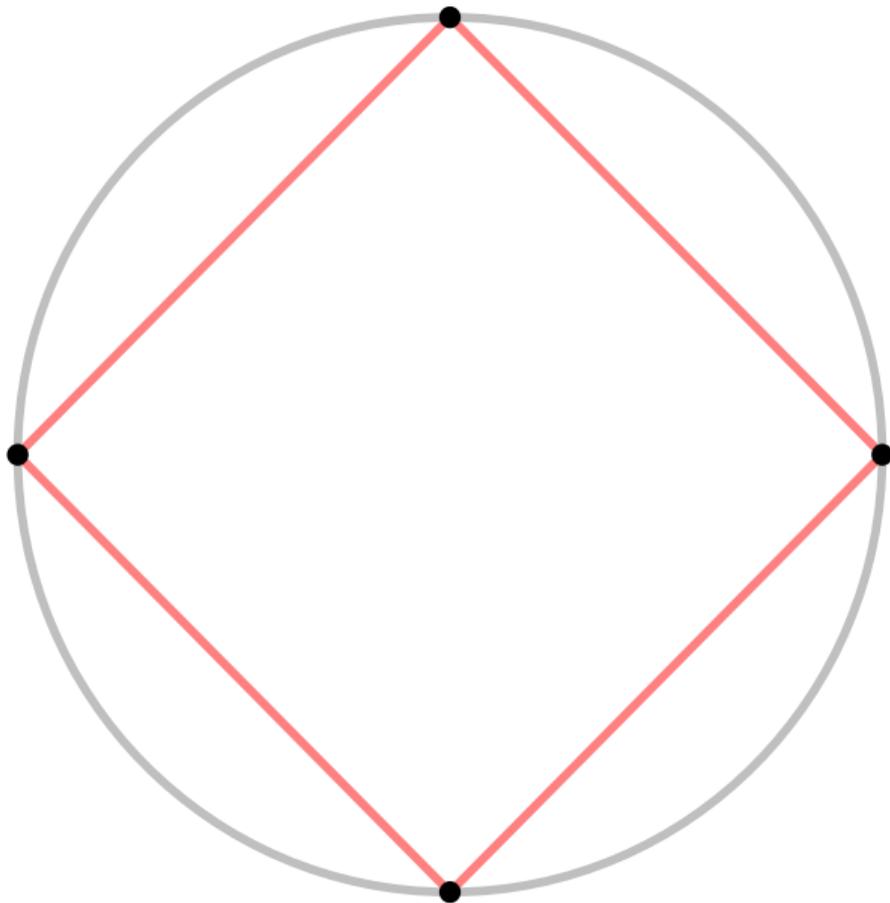
```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return "O" end
    if z<0 then return "I" end
    if level==0 then return classify(x,y) end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    local c1=explore(xlo,ylo,level-1)
    local c2=explore(xhi,ylo,level-1)
    local c3=explore(xlo,yhi,level-1)
    local c4=explore(xhi,yhi,level-1)
    if c1==c2 and c1==c3 and c1==c4 then
        return c1
    else
        show(xlo,ylo,c1)
        show(xhi,ylo,c2)
        show(xlo,yhi,c3)
        show(xhi,yhi,c4)
        return nil
    end
end
```

```
function classify(x,y)
    local xmin,xmax=x:extremes()
    local ymin,ymax=y:extremes()
    local f1=f(xmin,ymin)
    local f2=f(xmax,ymin)
    local f3=f(xmax,ymax)
    local f4=f(xmin,ymax)
    if (f1>0 and f2>0 and f3>0 and f4>0) then
        return optimistic and "O" or "PO"
    end
    if (f1<0 and f2<0 and f3<0 and f4<0) then
        return optimistic and "I" or "PI"
    end
    return "V"
end
```

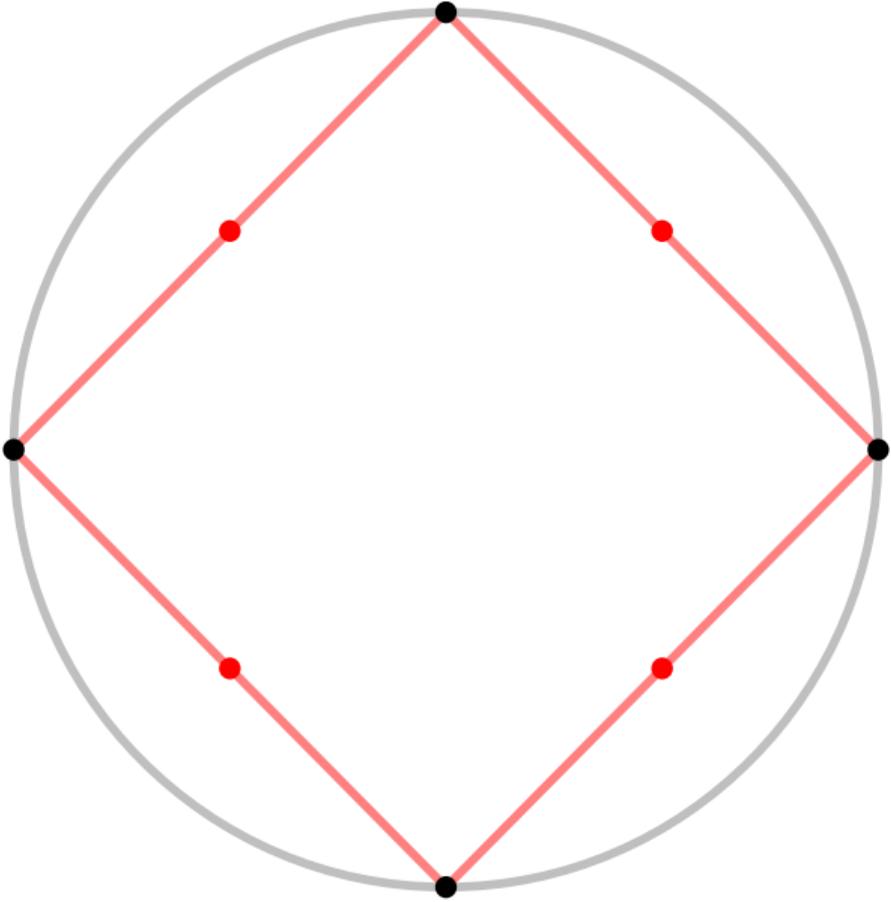
## Curva implícita – refinamento intrínstico



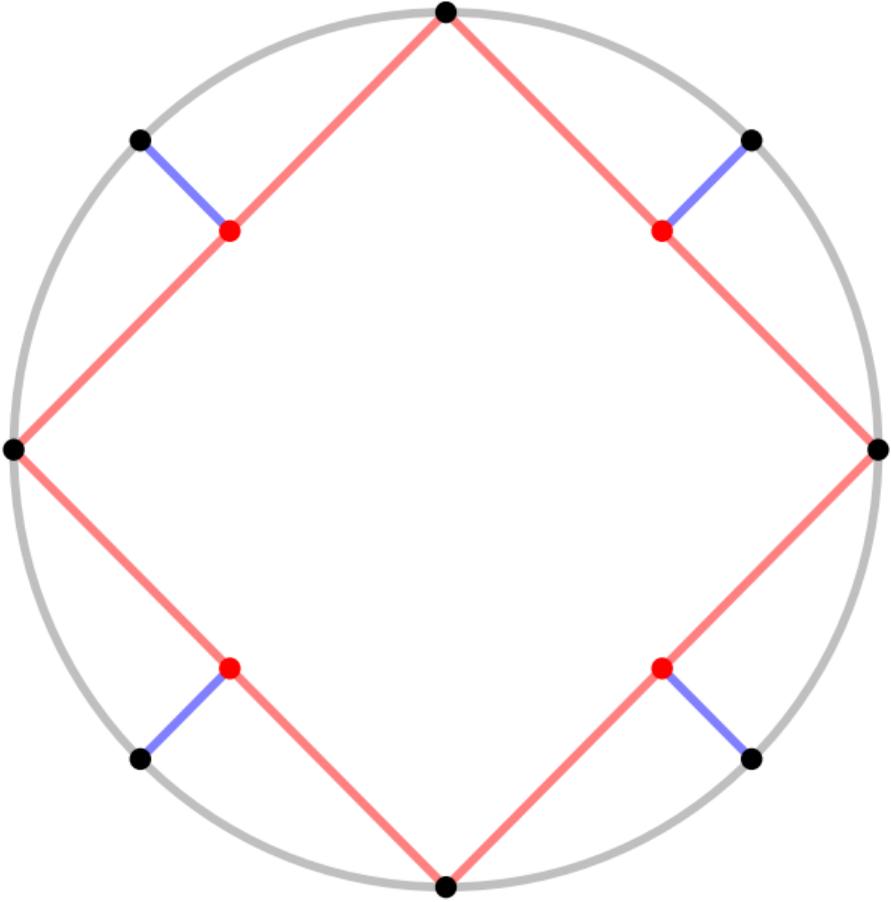
## Curva implícita – refinamento intrínstico



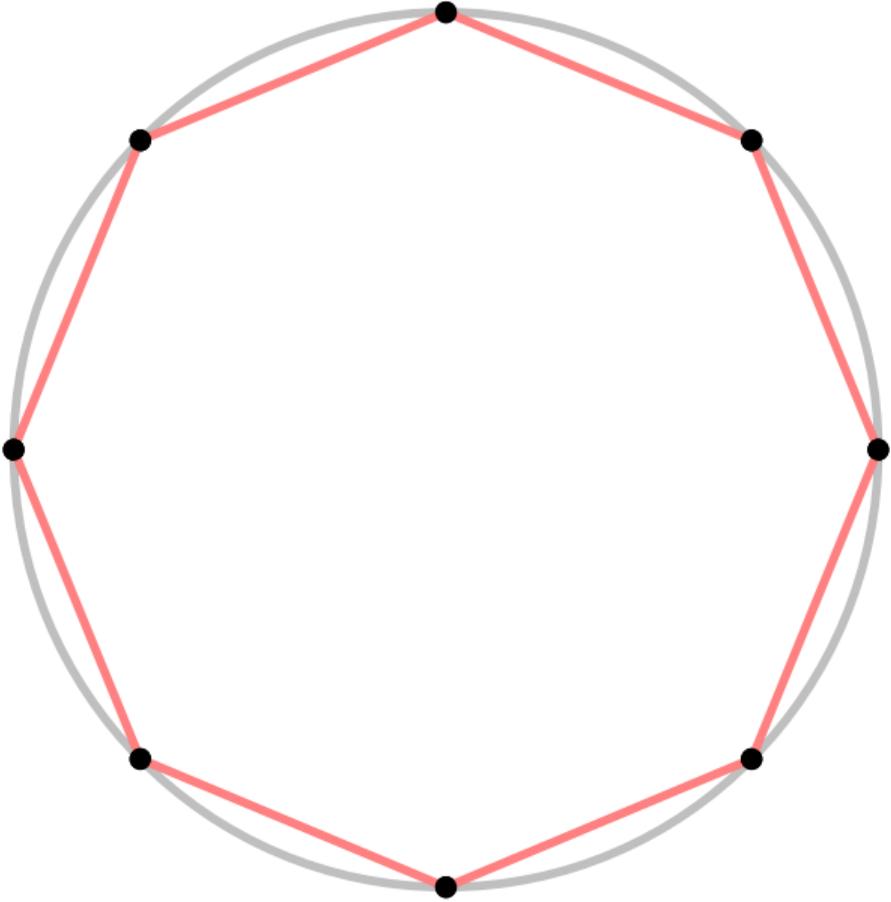
# Curva implícita – refinamento intríntrico



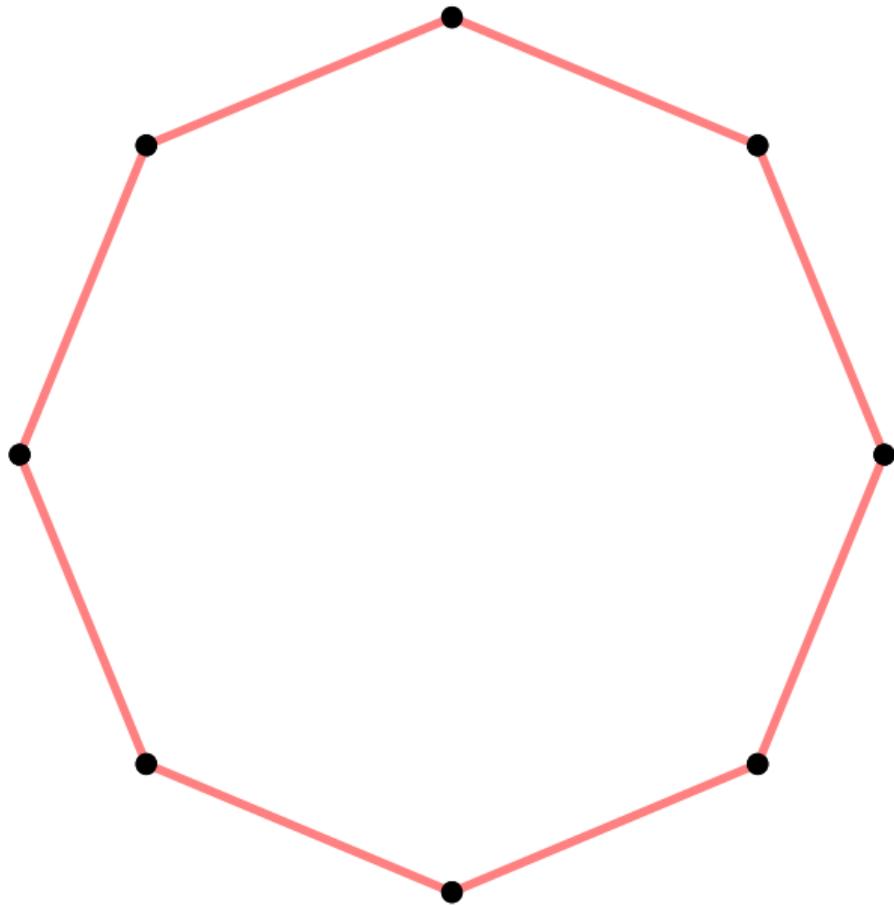
# Curva implícita – refinamento intríntrico



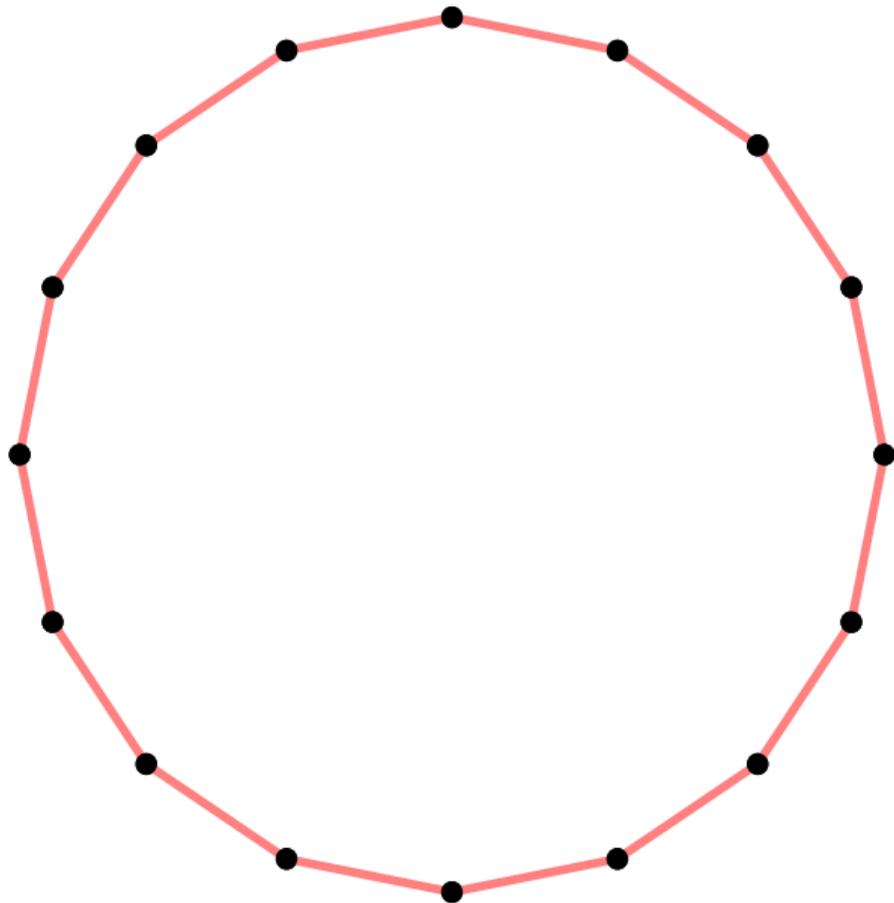
Curva implícita – refinamento intríntrico



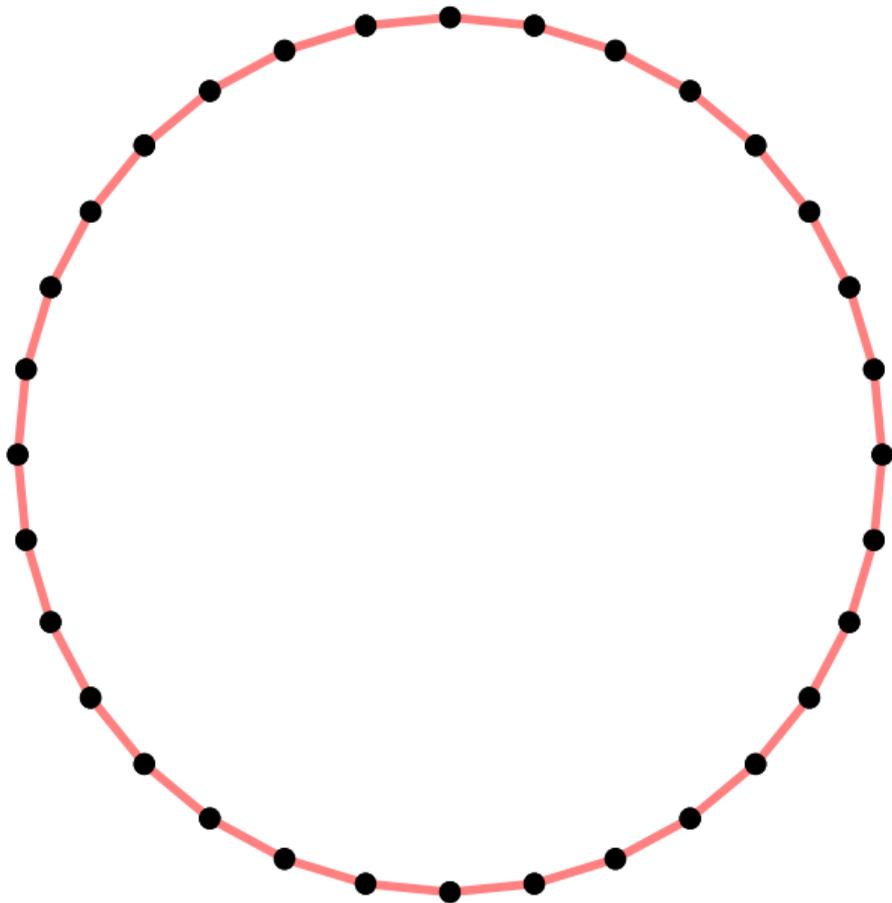
## Curva implícita – refinamento intrínstico



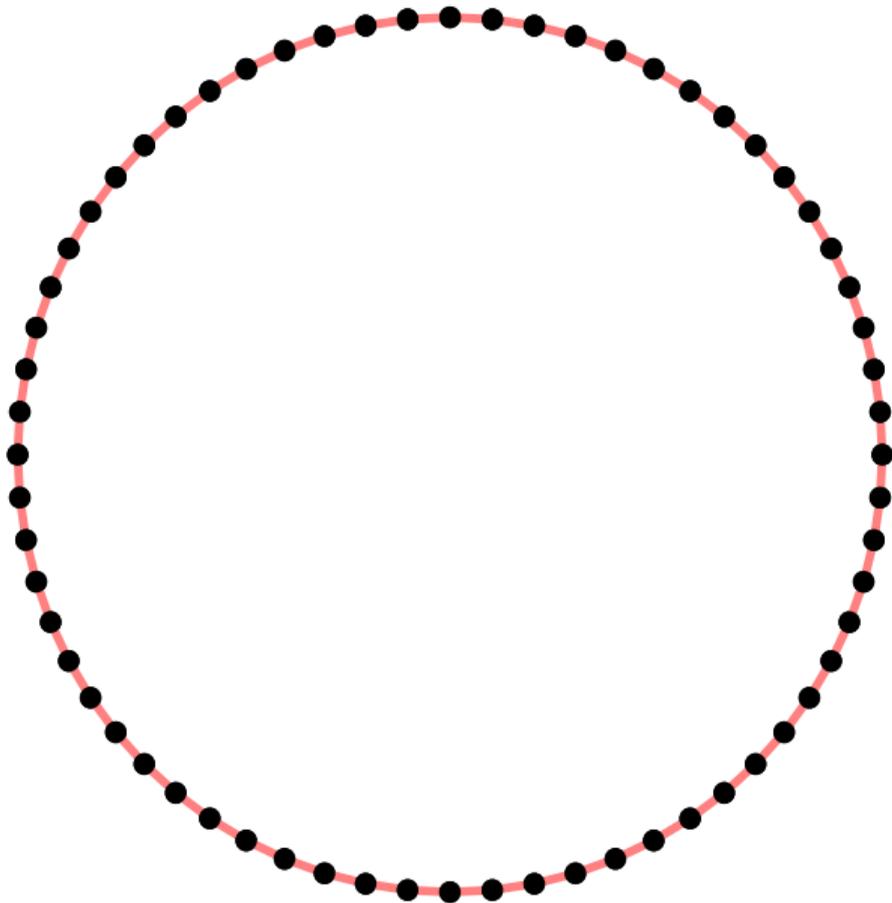
## Curva implícita – refinamento intrínstico



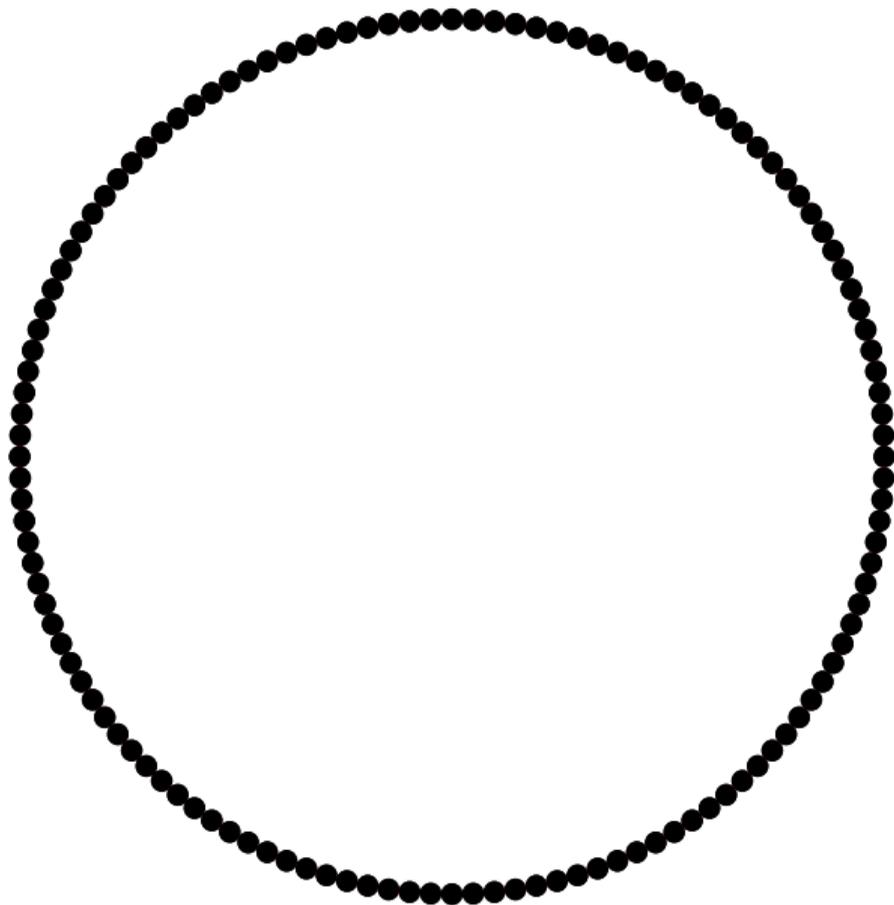
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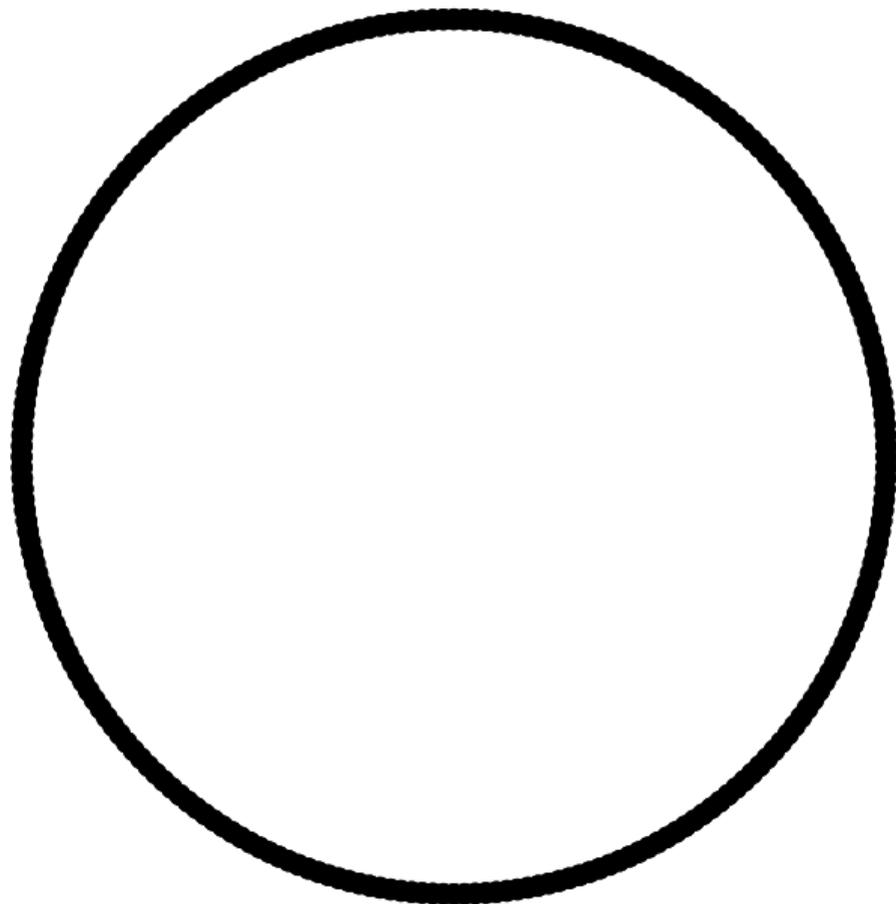
## Curva implícita – refinamento intrínstico



## Curva implícita – refinamento intrínstico



## Curva implícita – refinamento intrínstico



objetos paramétricos

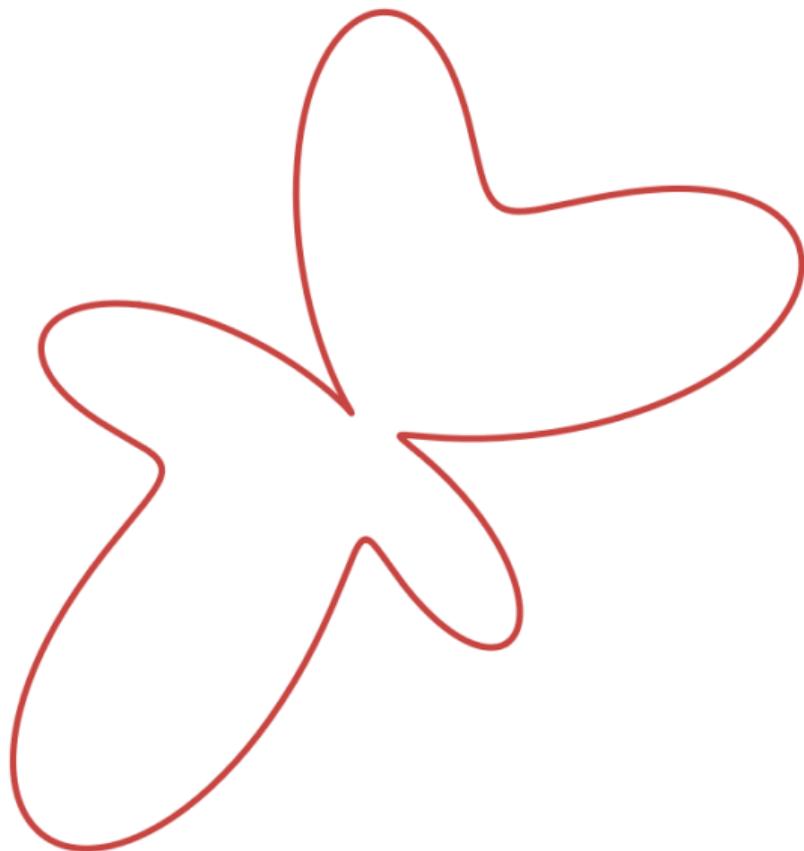
## Curva paramétrica

curva = fronteira da região

$$\gamma: [a, b] \rightarrow \mathbf{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



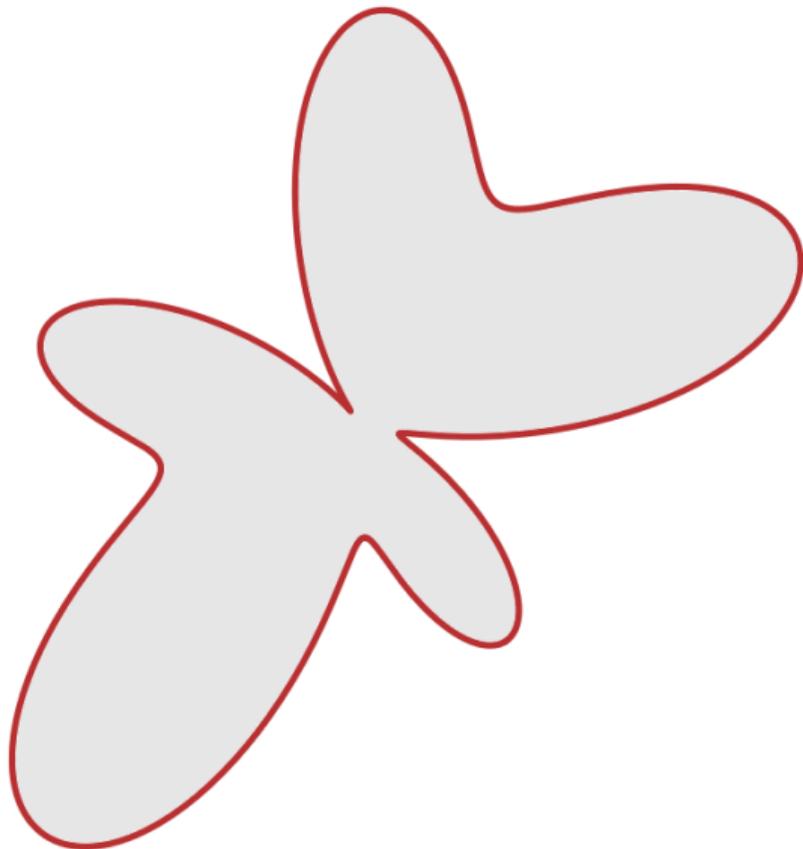
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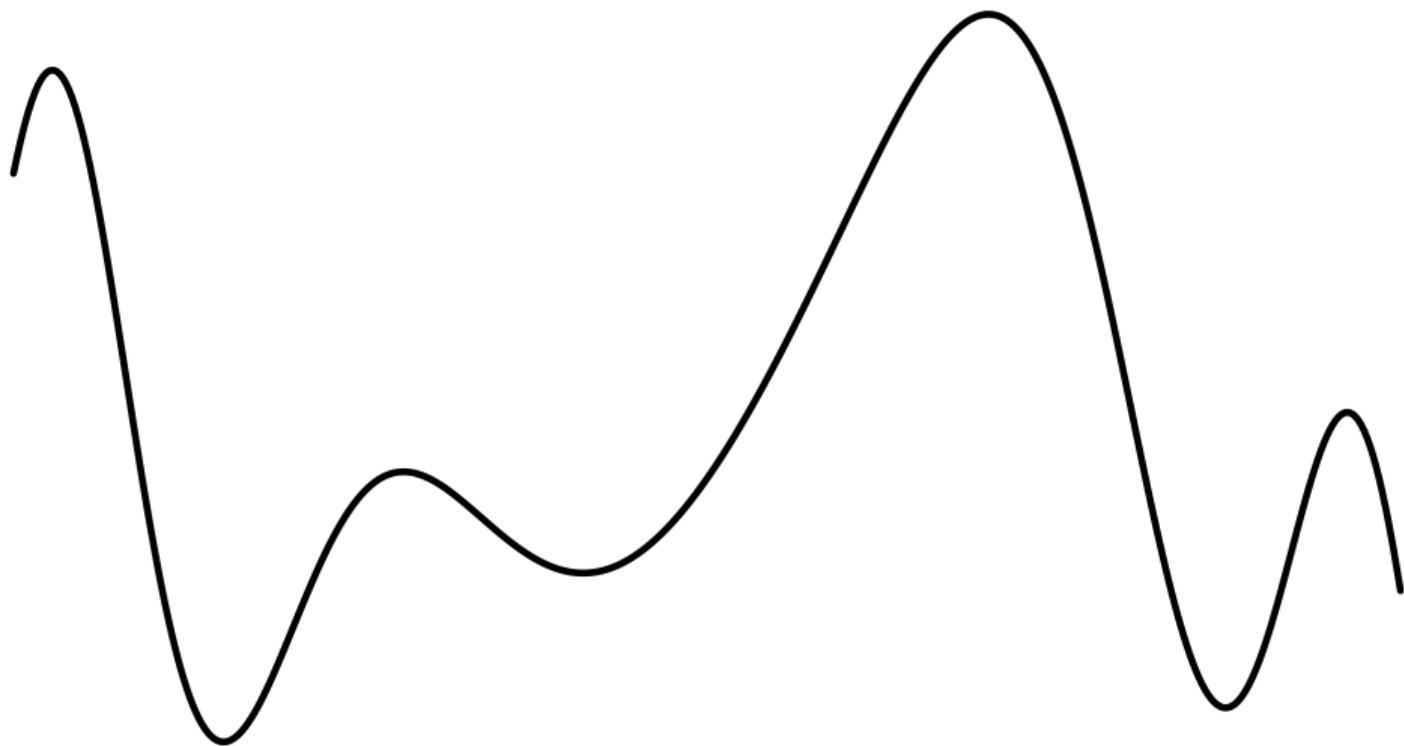
$$\gamma(t) = (x(t), y(t))$$

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- classificação de pontos
- classificação de caixas

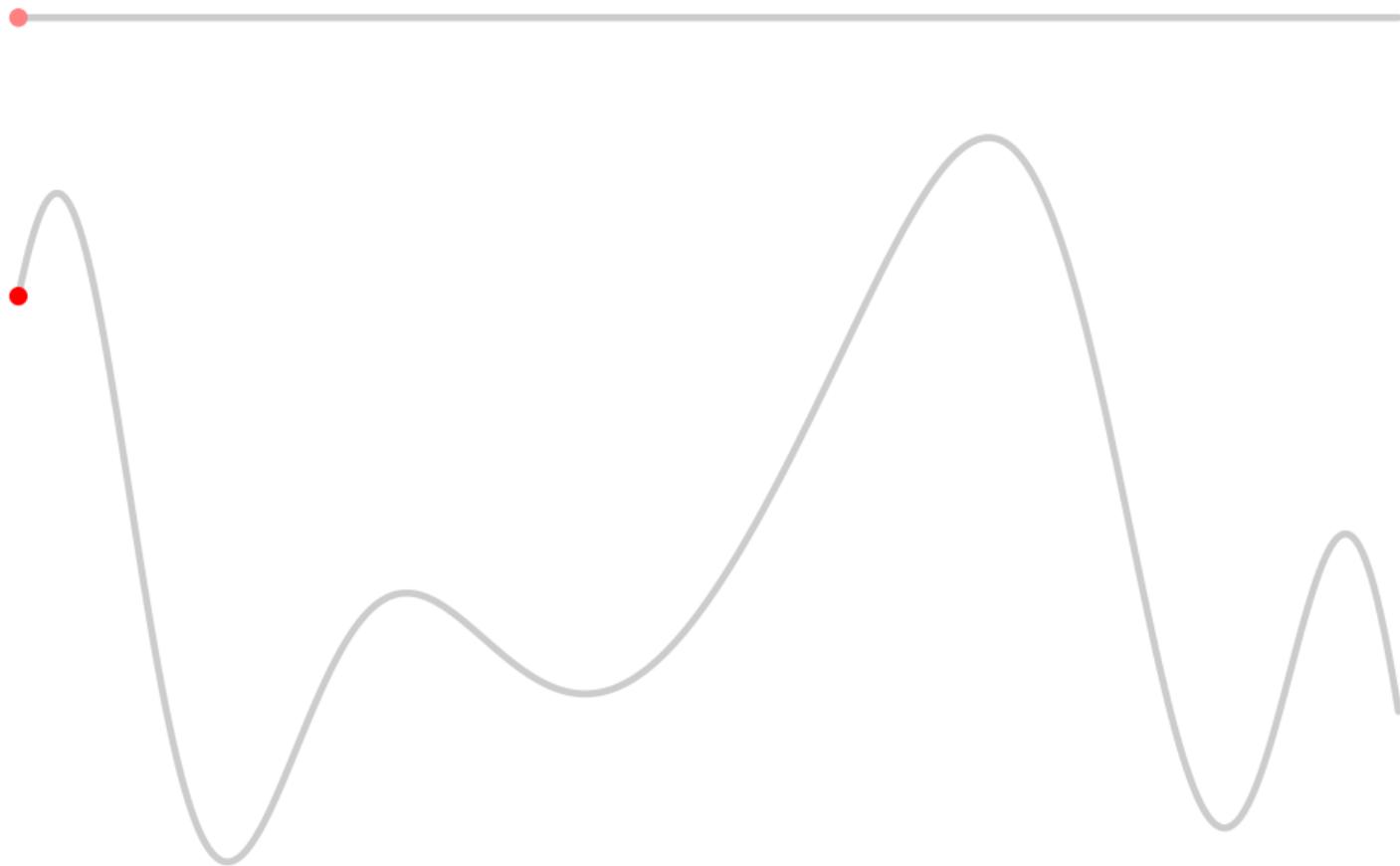


## Curva paramétrica – amostragem uniforme no domínio

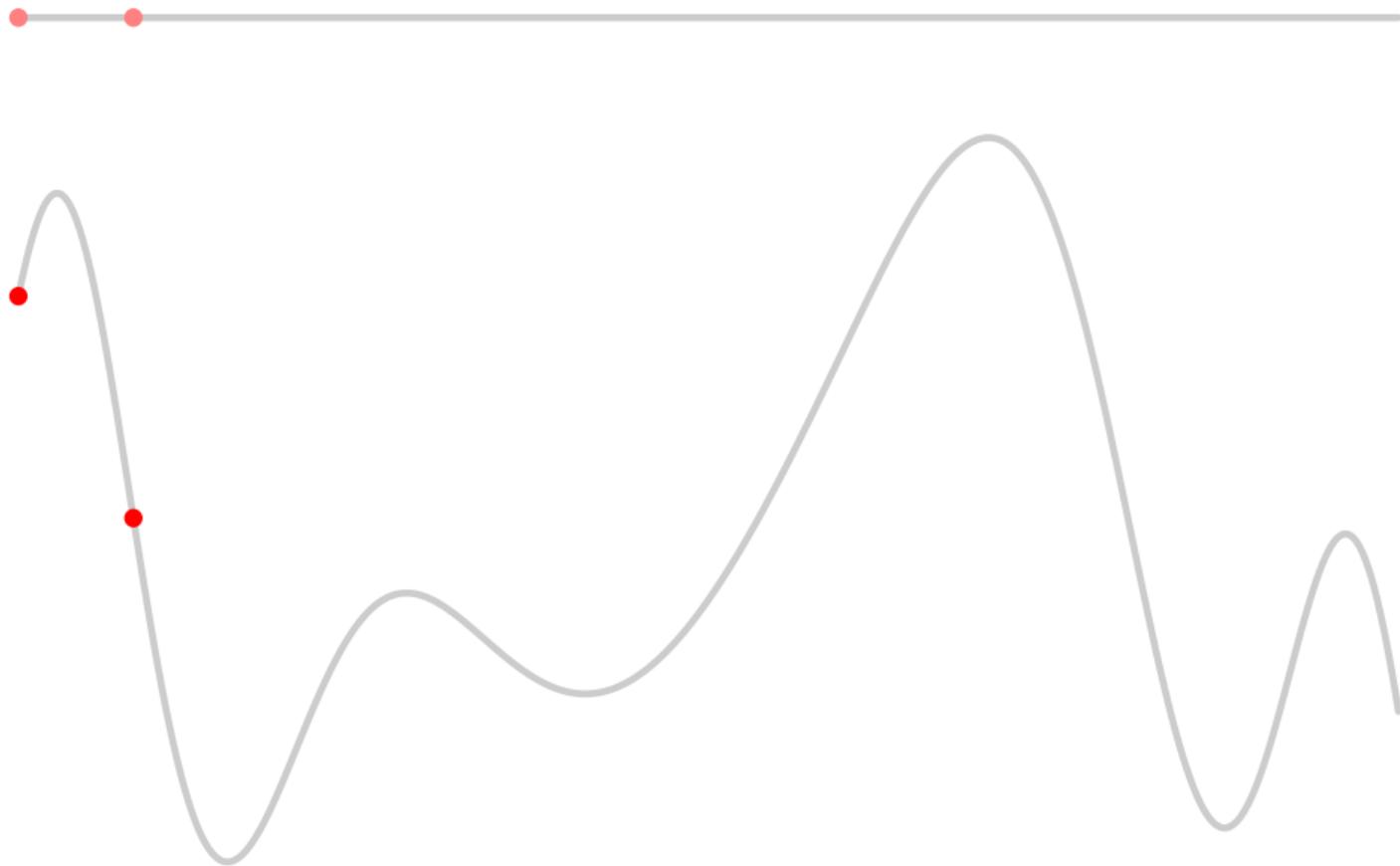
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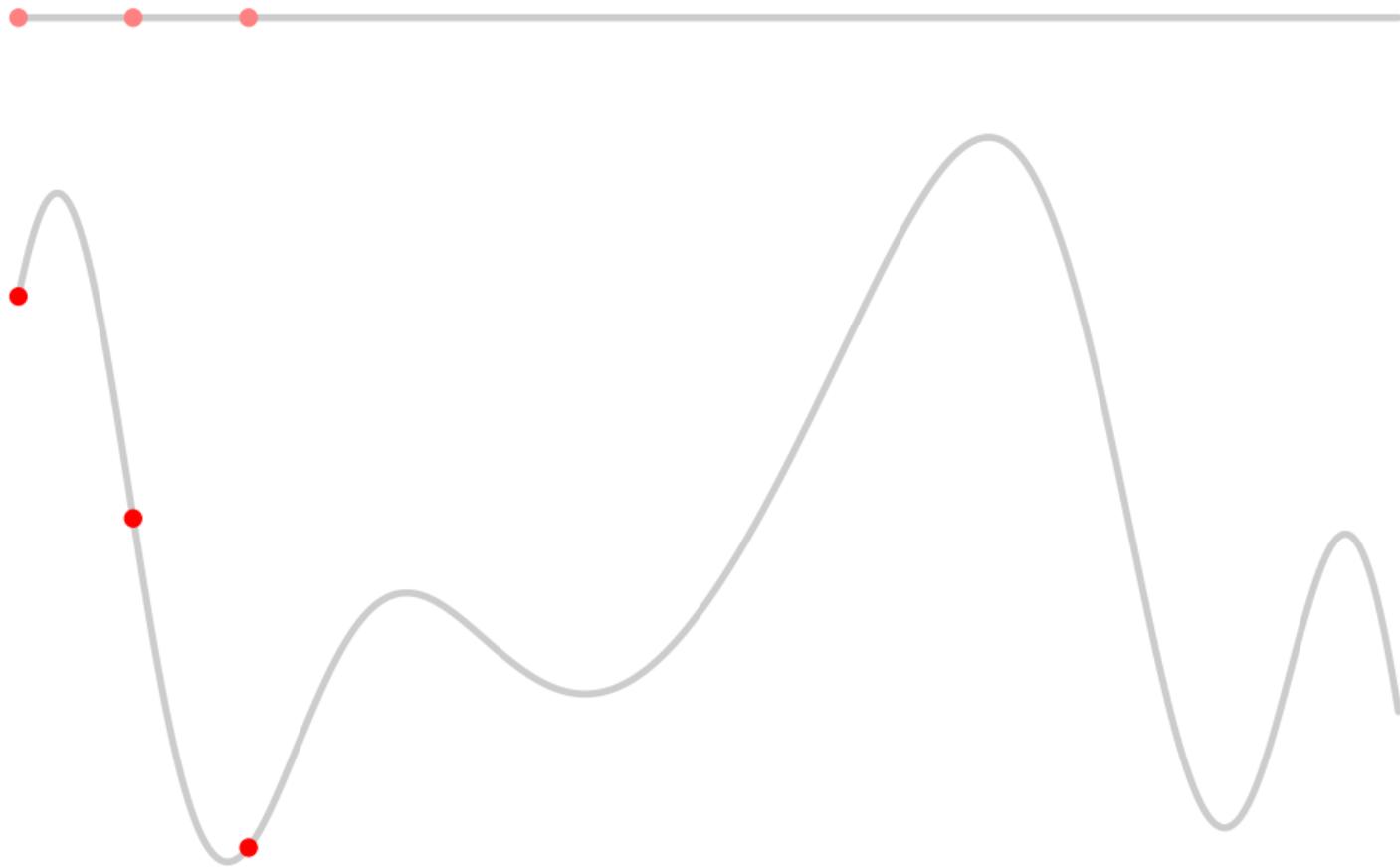
## Curva paramétrica – amostragem uniforme no domínio



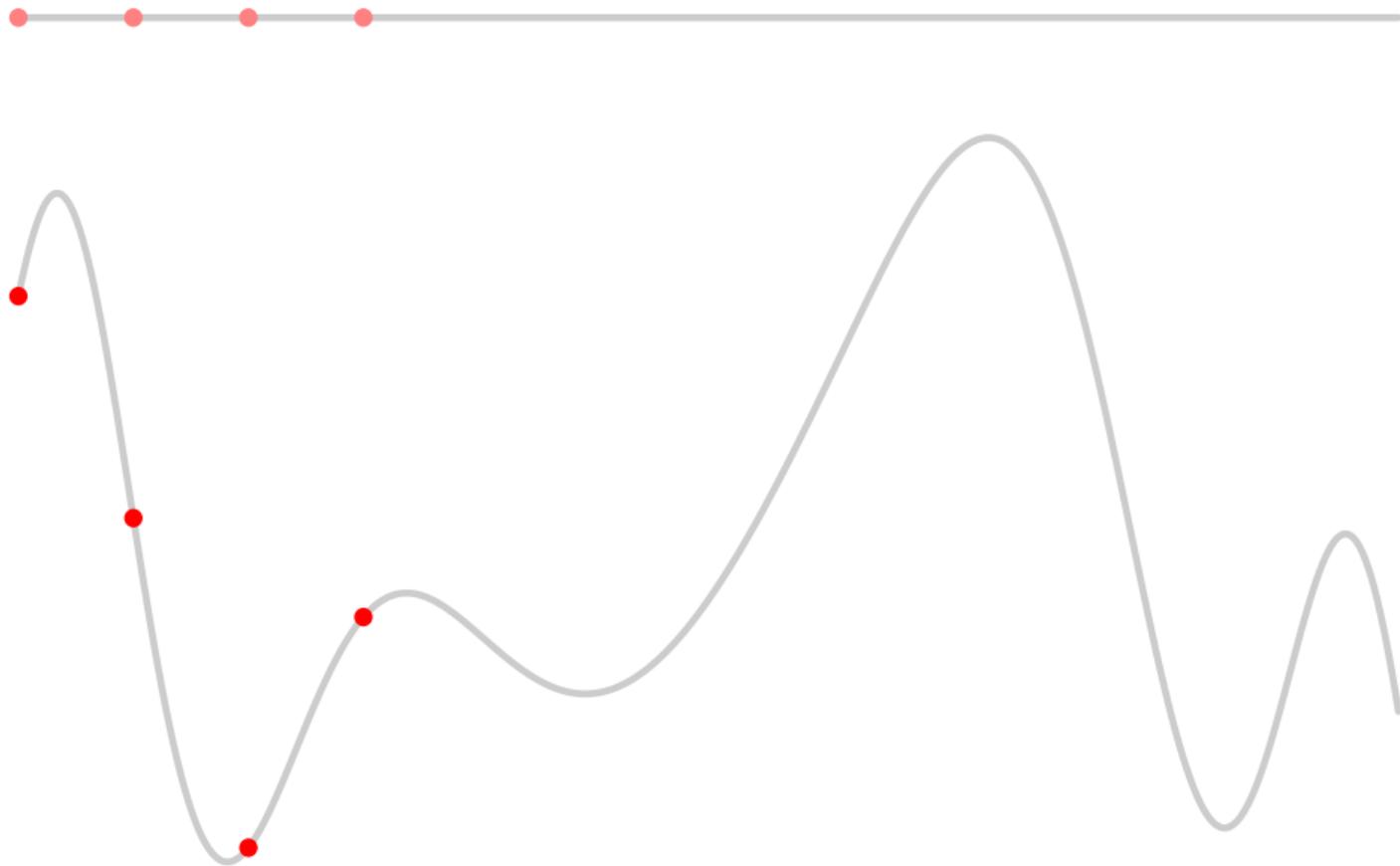
## Curva paramétrica – amostragem uniforme no domínio



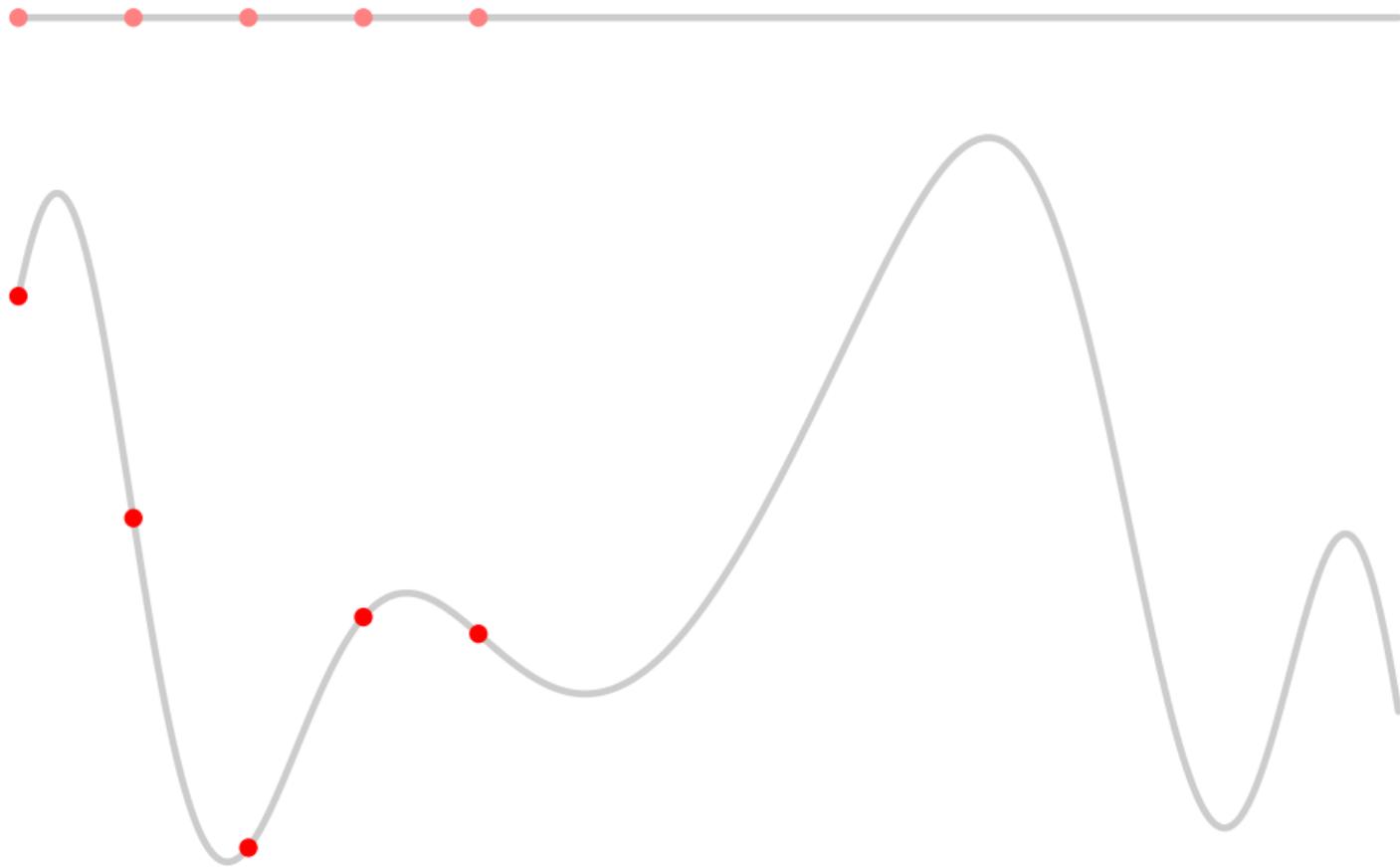
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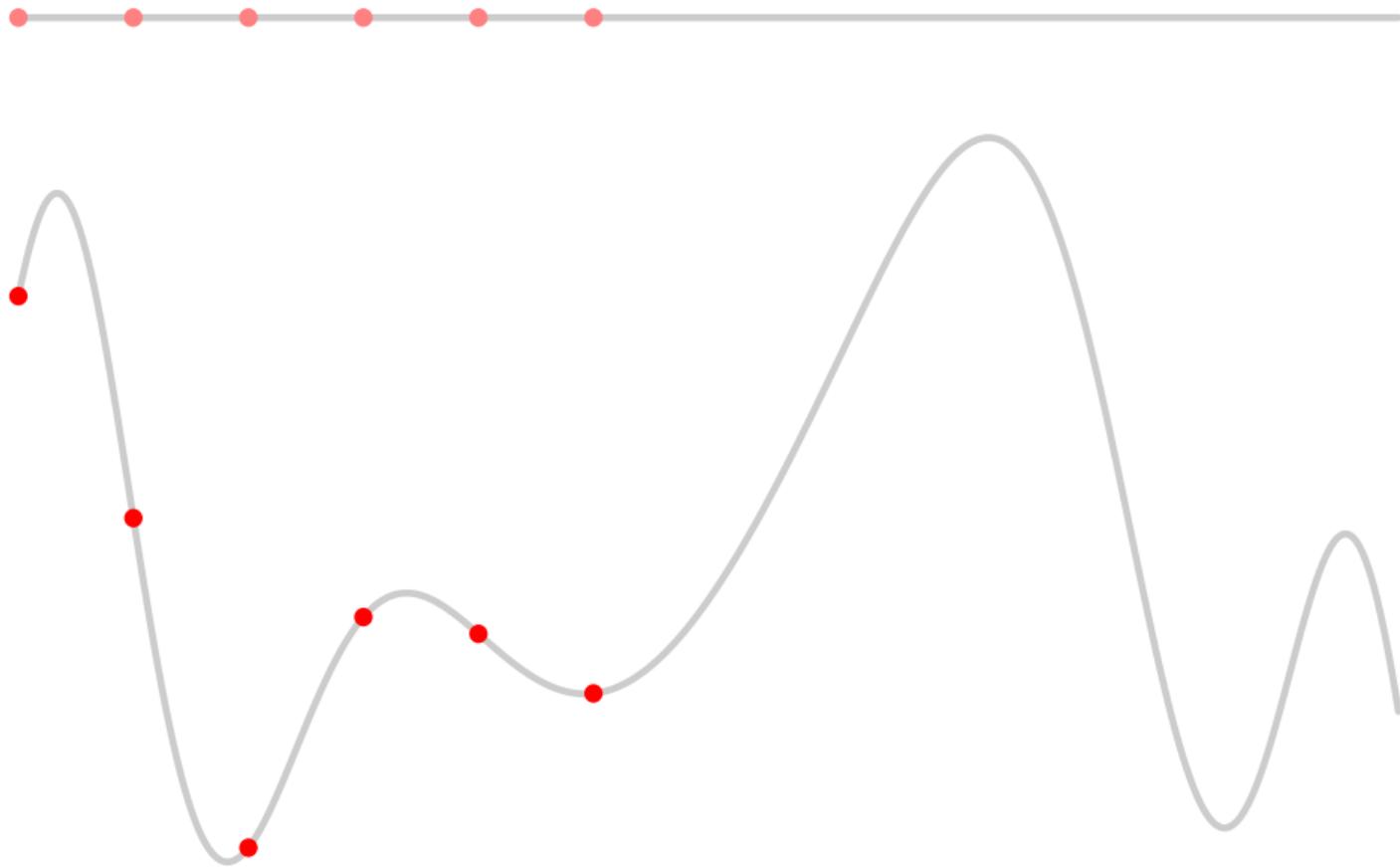
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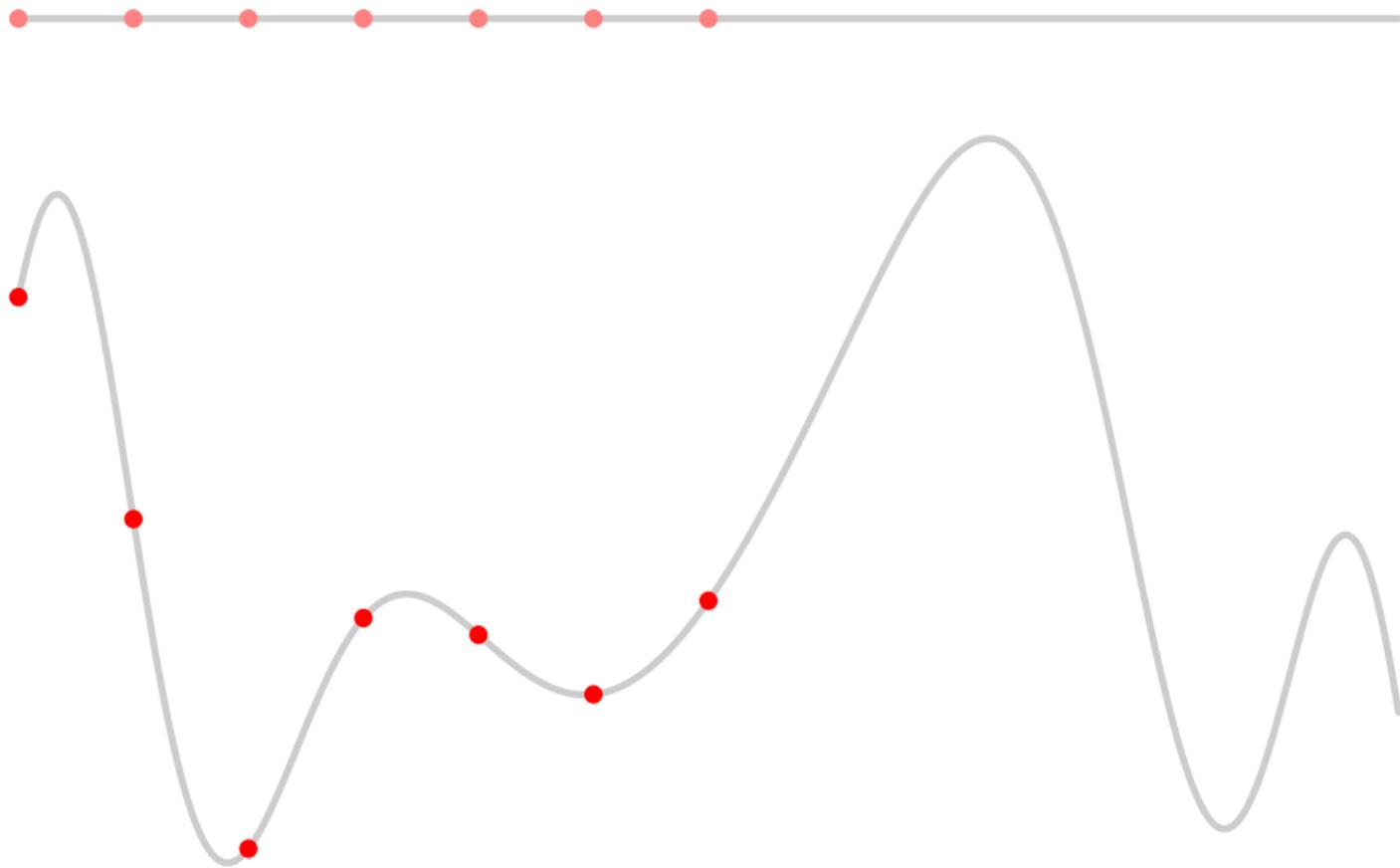
## Curva paramétrica – amostragem uniforme no domínio



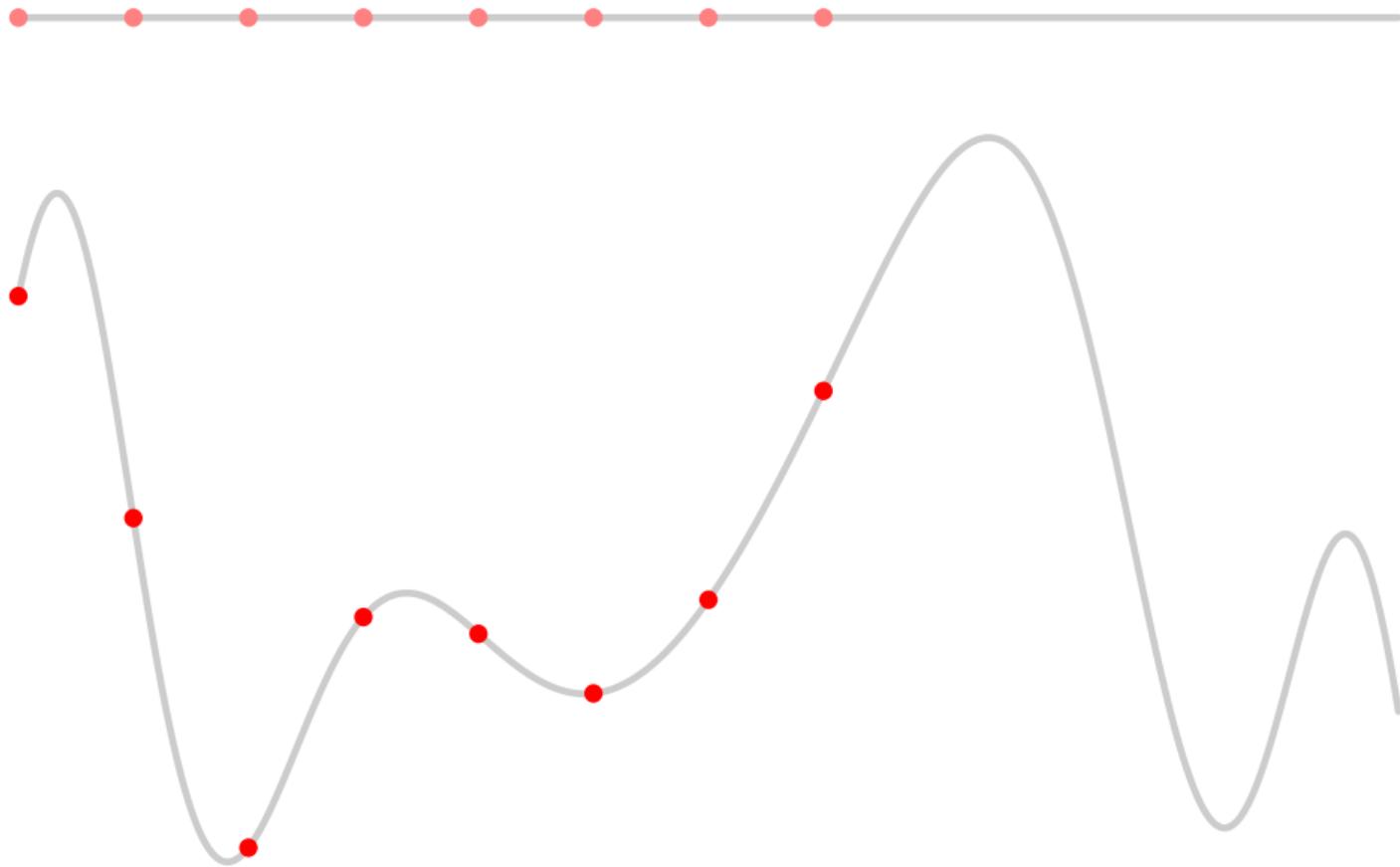
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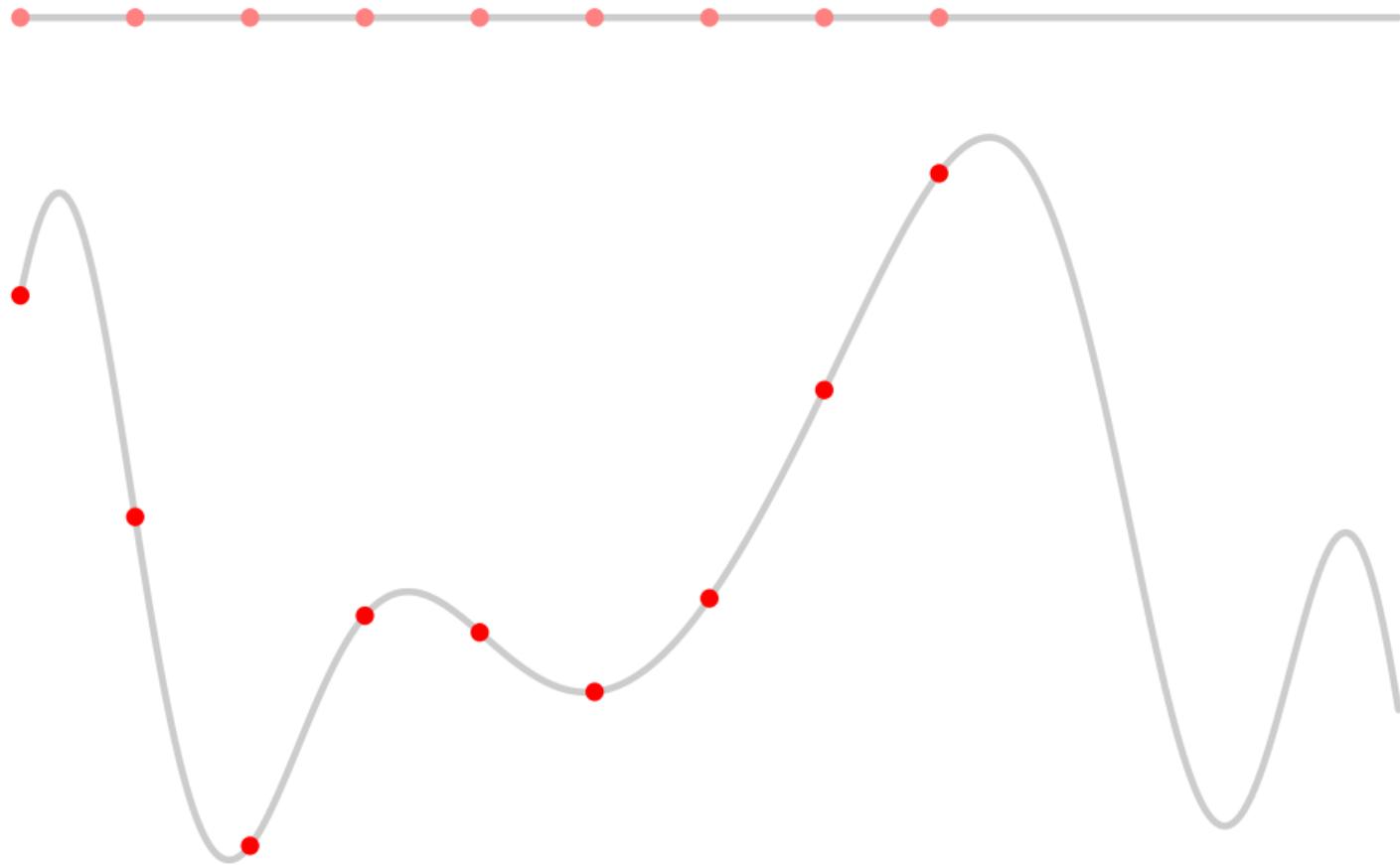
## Curva paramétrica – amostragem uniforme no domínio



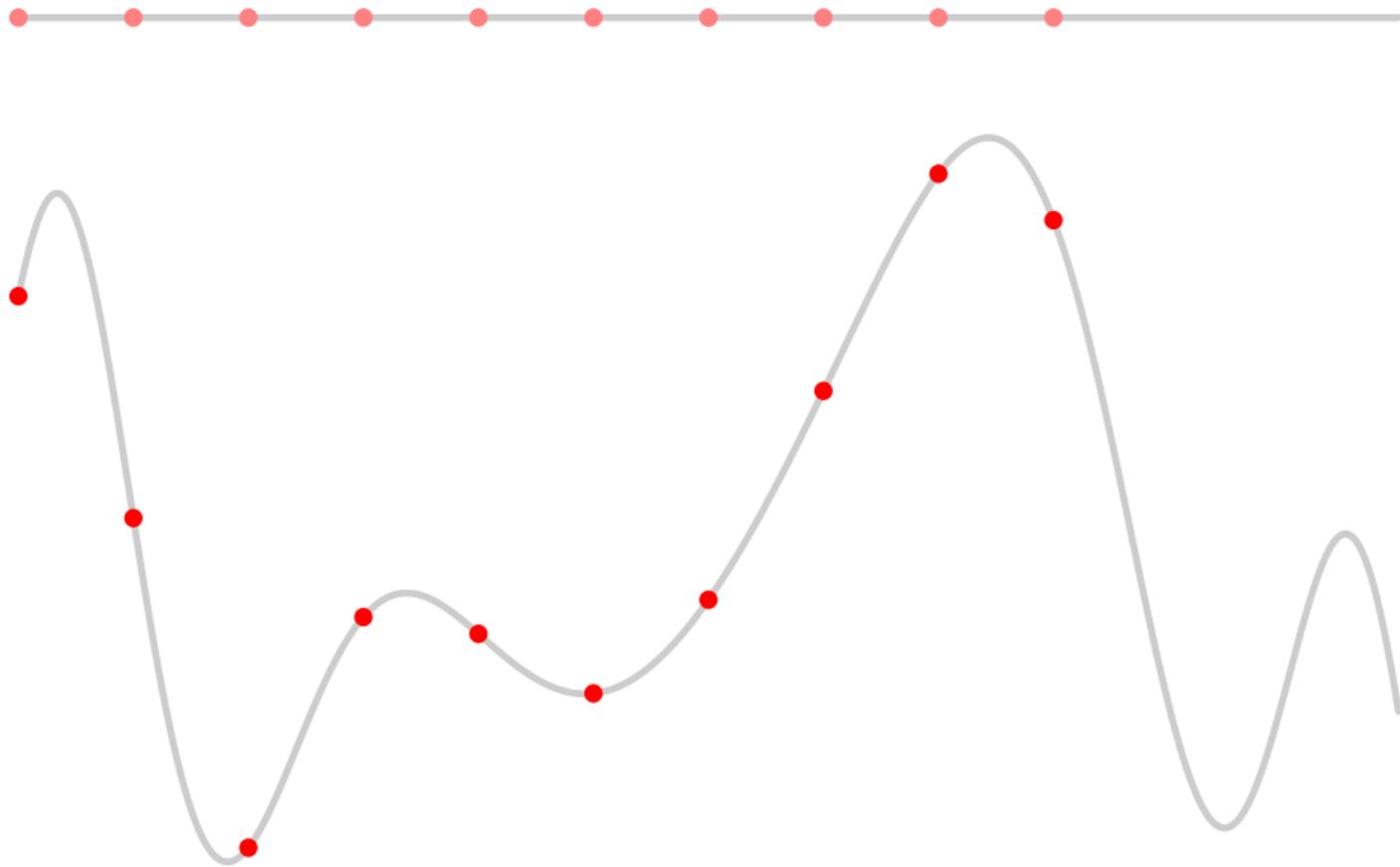
## Curva paramétrica – amostragem uniforme no domínio



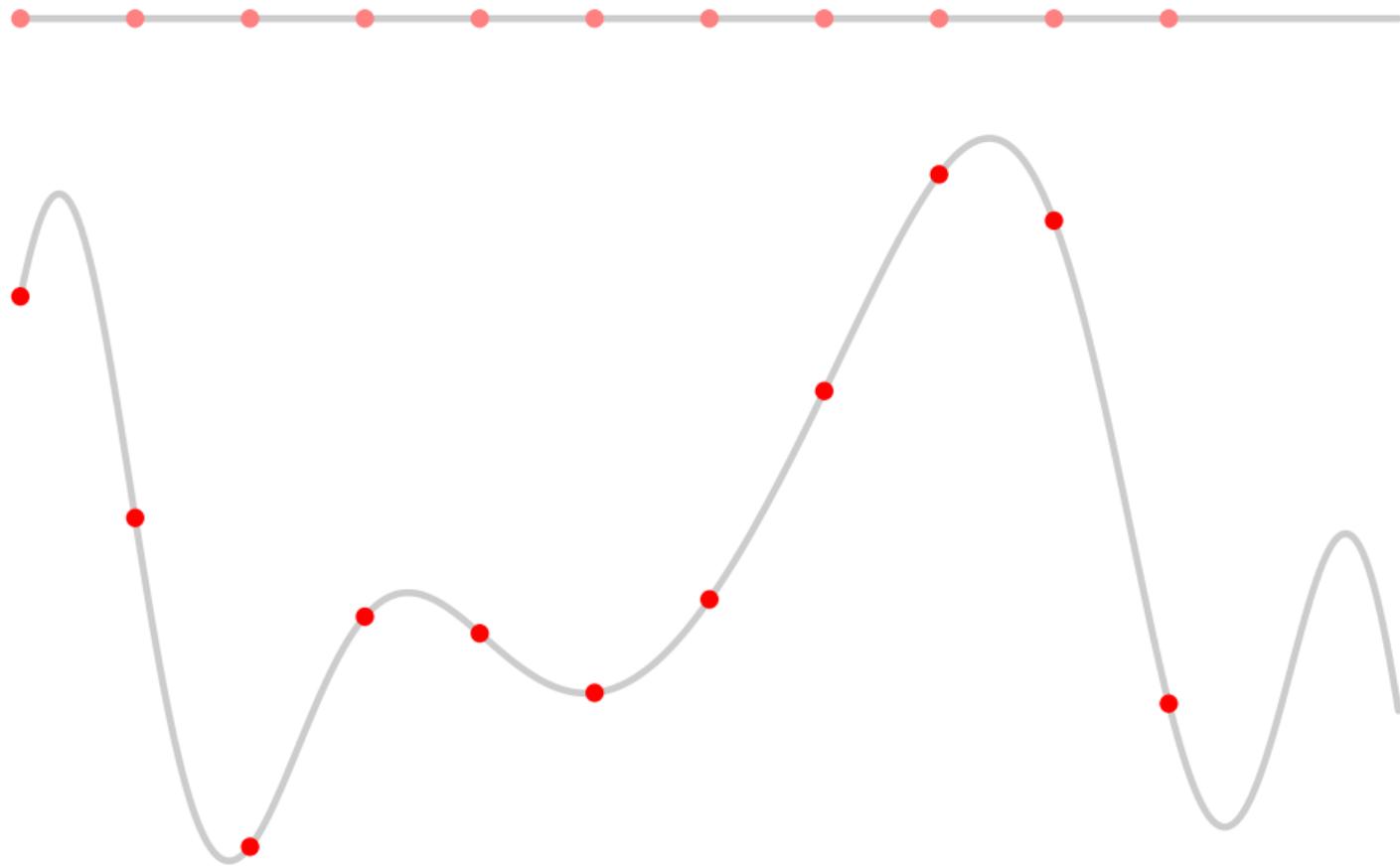
## Curva paramétrica – amostragem uniforme no domínio



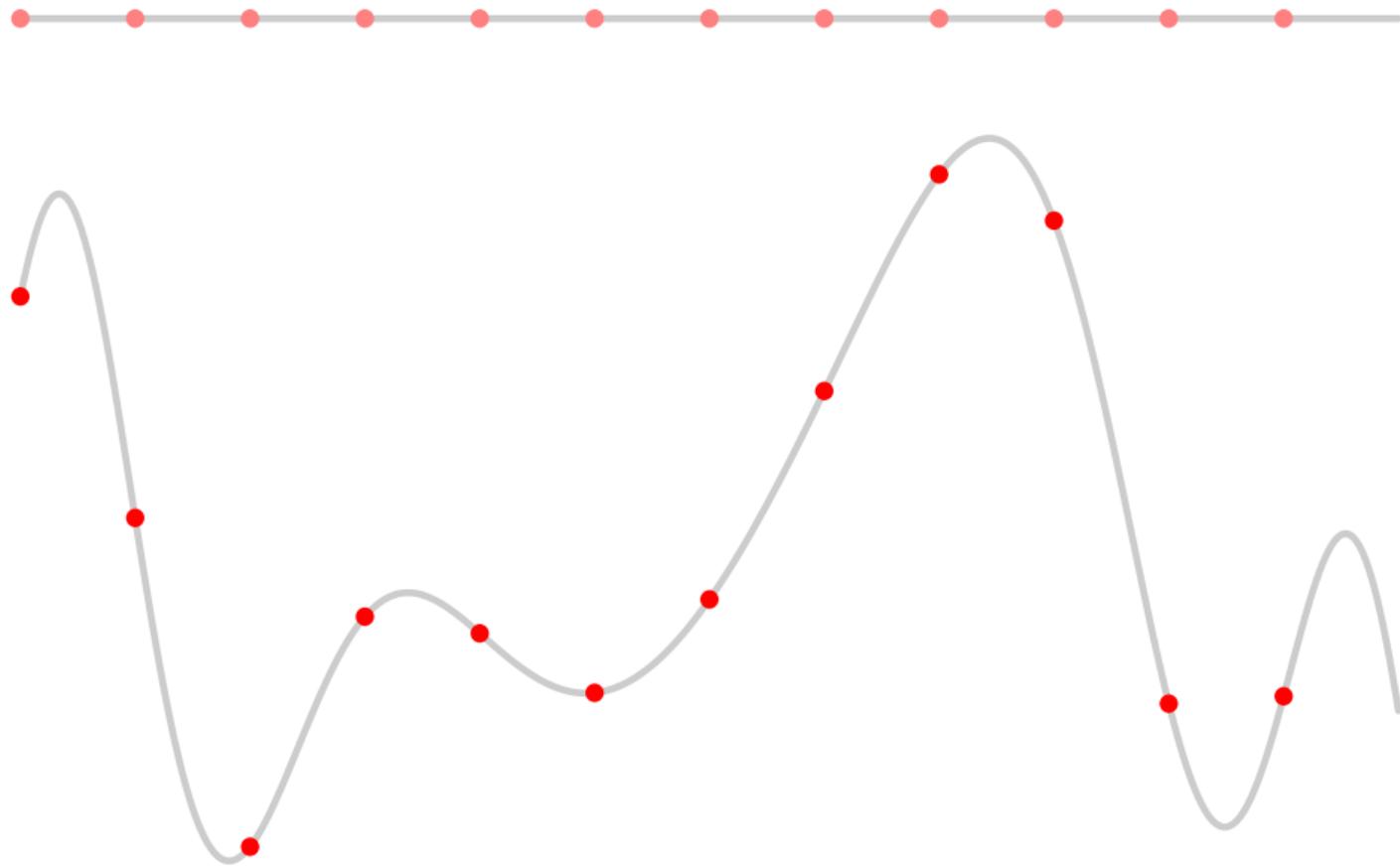
## Curva paramétrica – amostragem uniforme no domínio



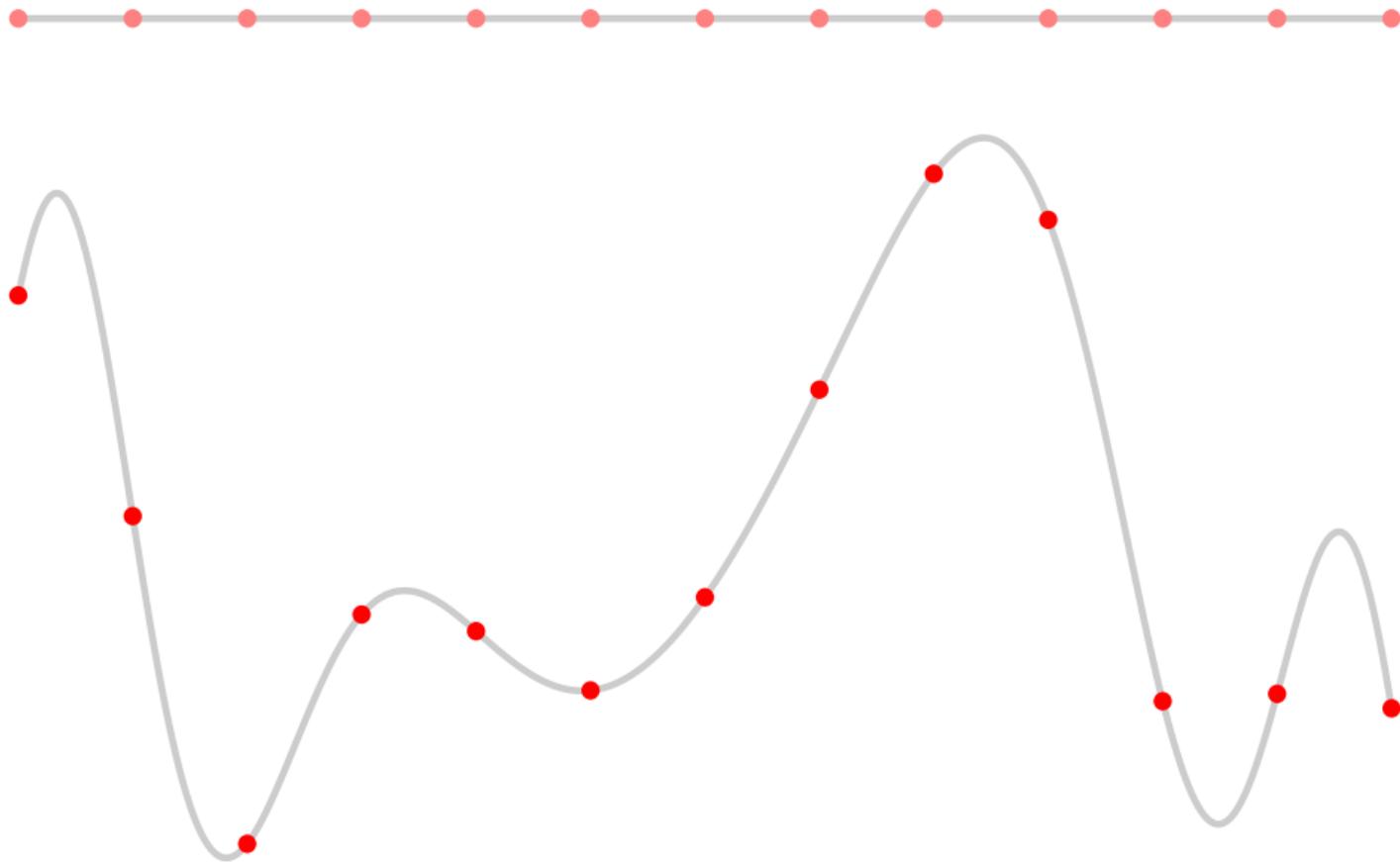
## Curva paramétrica – amostragem uniforme no domínio



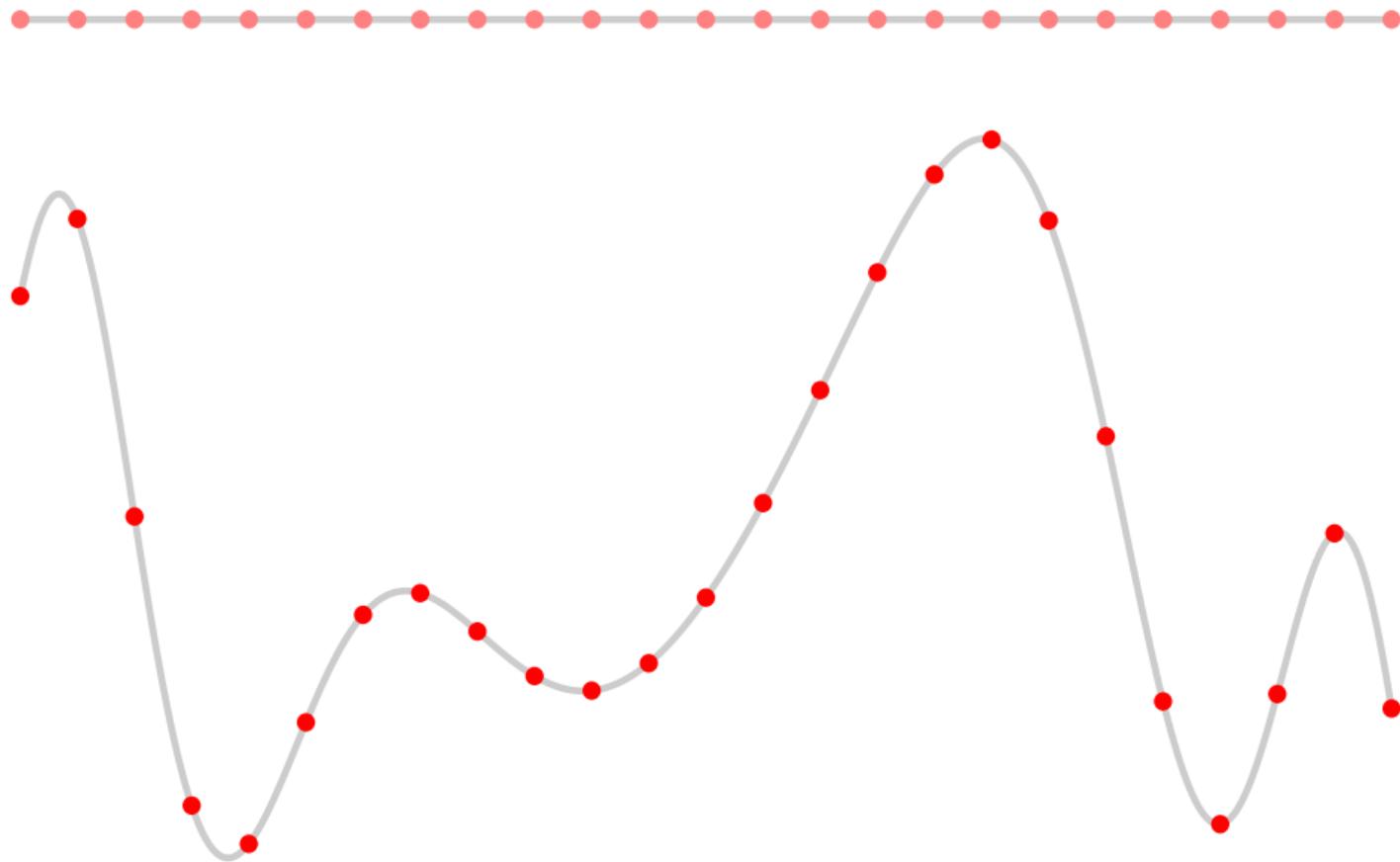
## Curva paramétrica – amostragem uniforme no domínio



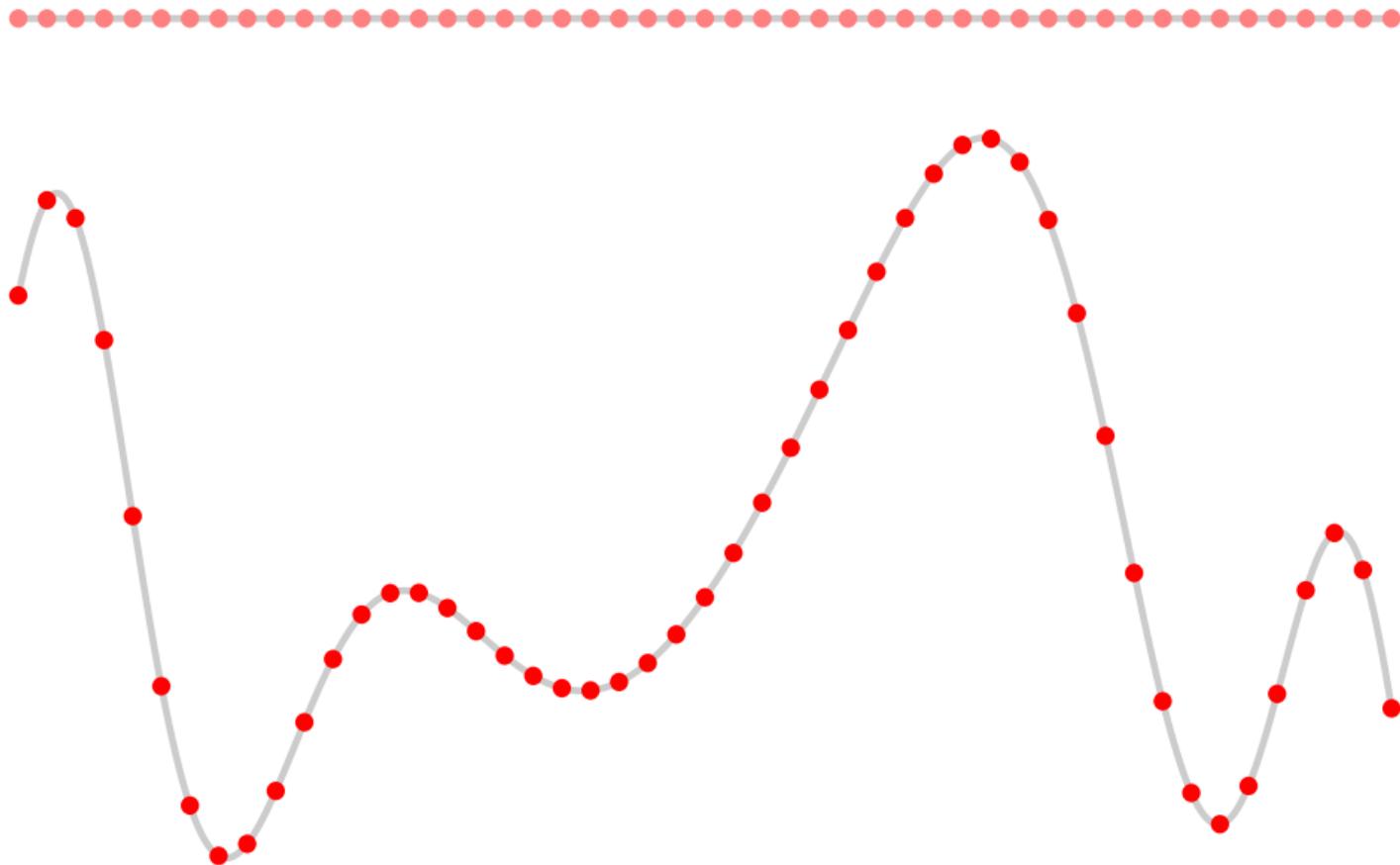
## Curva paramétrica – amostragem uniforme no domínio



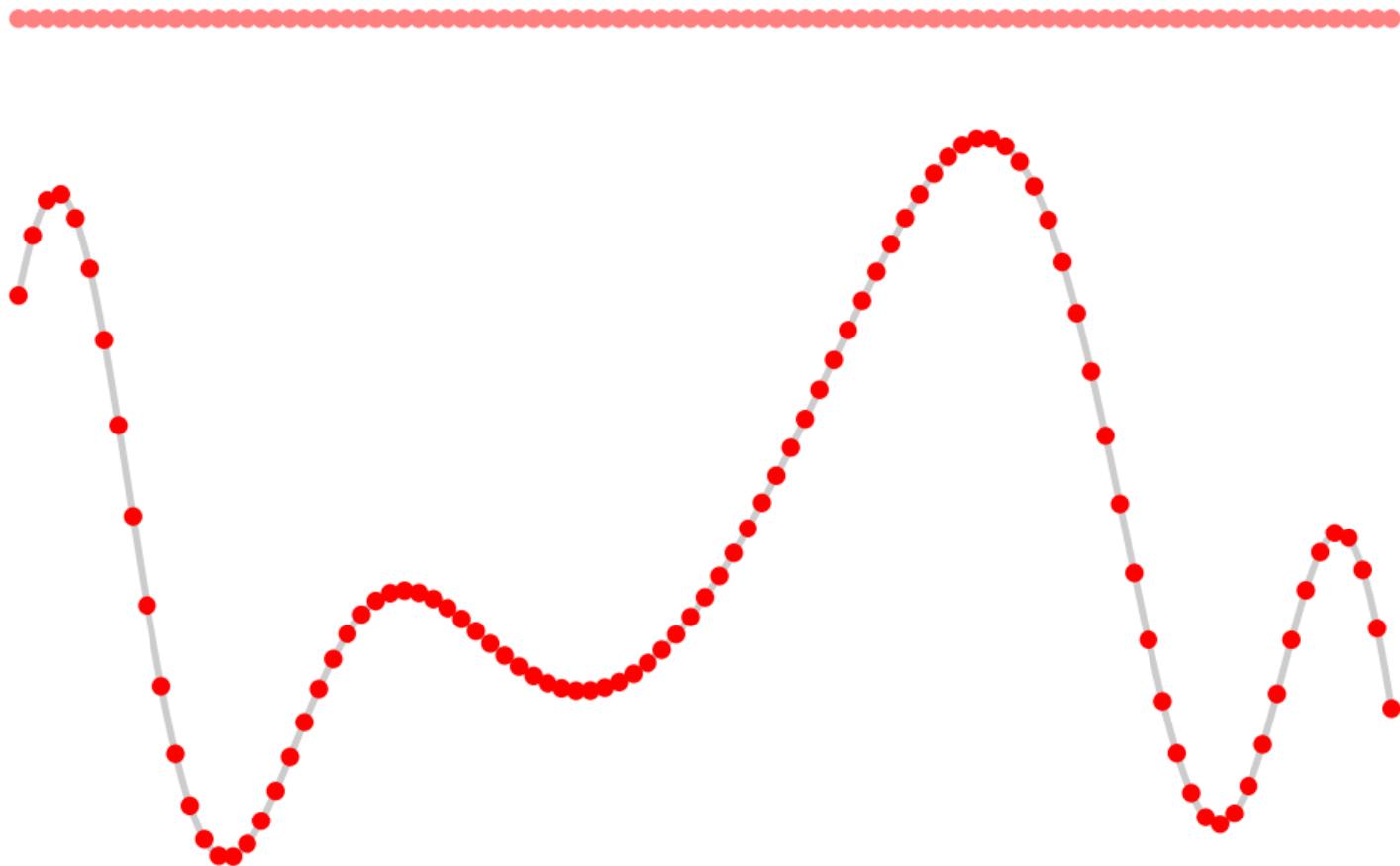
## Curva paramétrica – amostragem uniforme no domínio



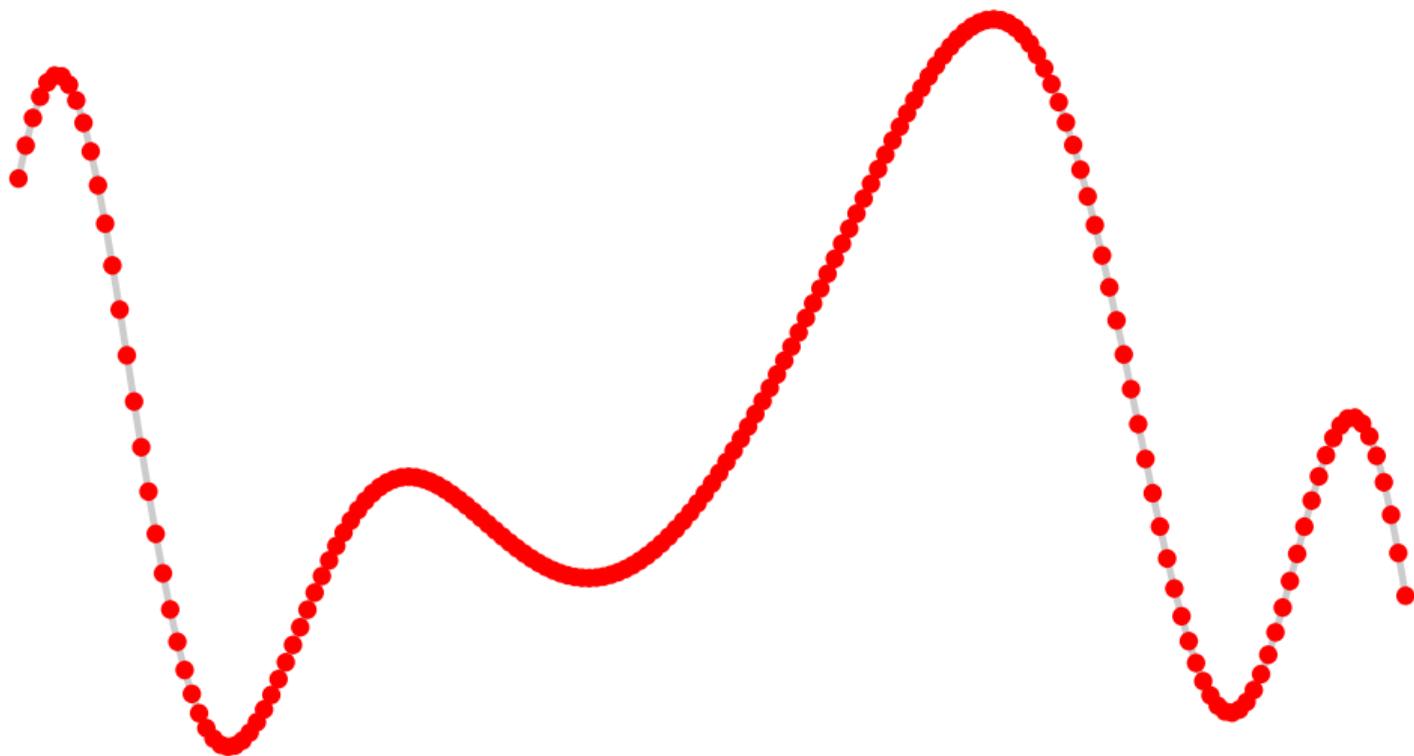
## Curva paramétrica – amostragem uniforme no domínio



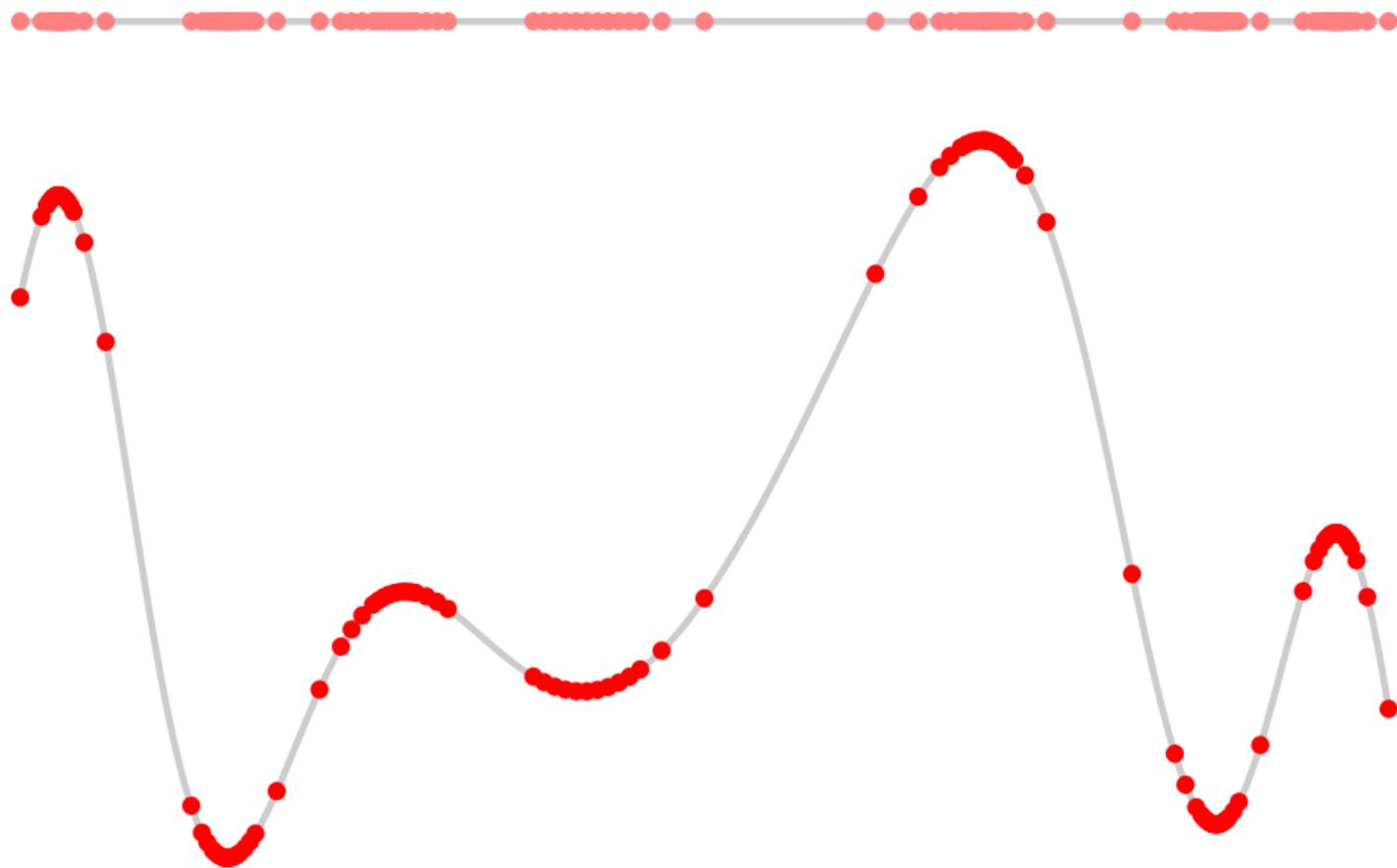
## Curva paramétrica – amostragem uniforme no domínio



## Curva paramétrica – amostragem uniforme no domínio

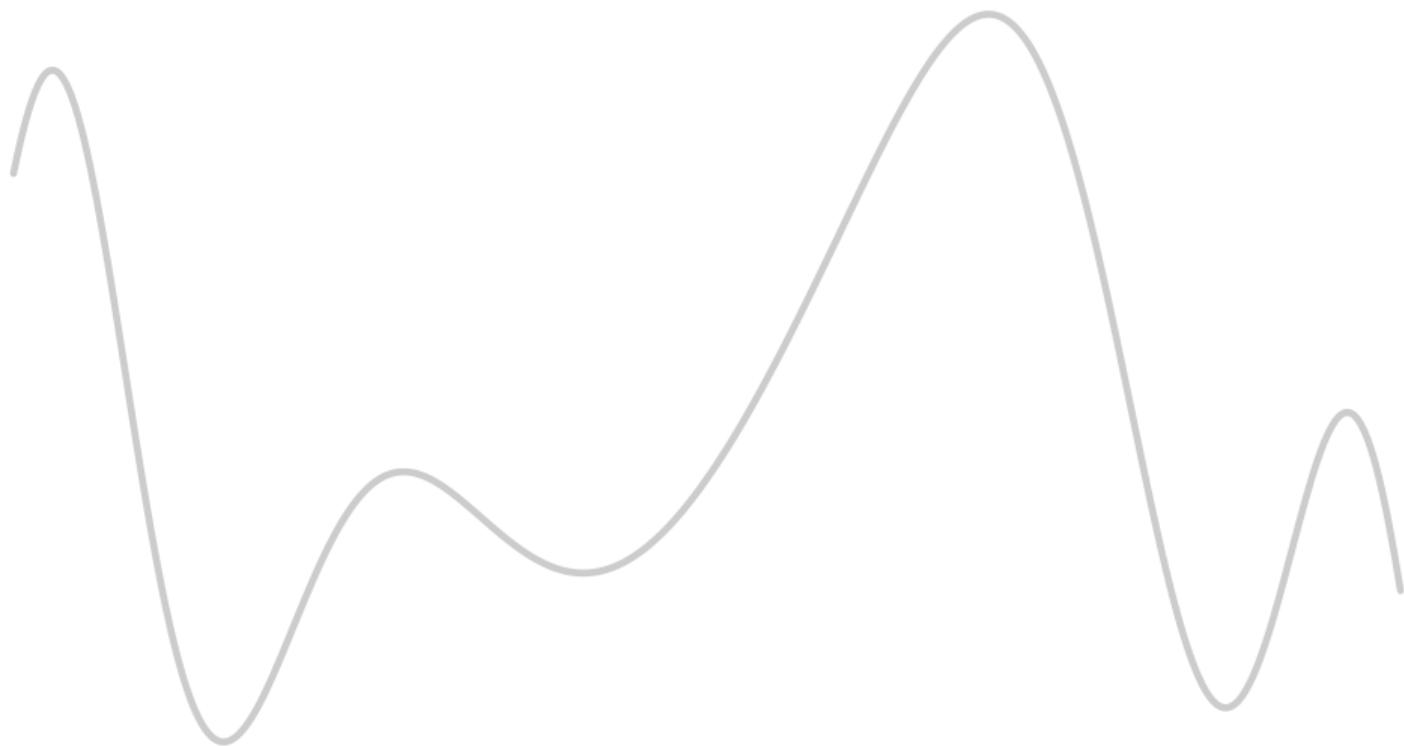


## Curva paramétrica – amostragem adaptativa

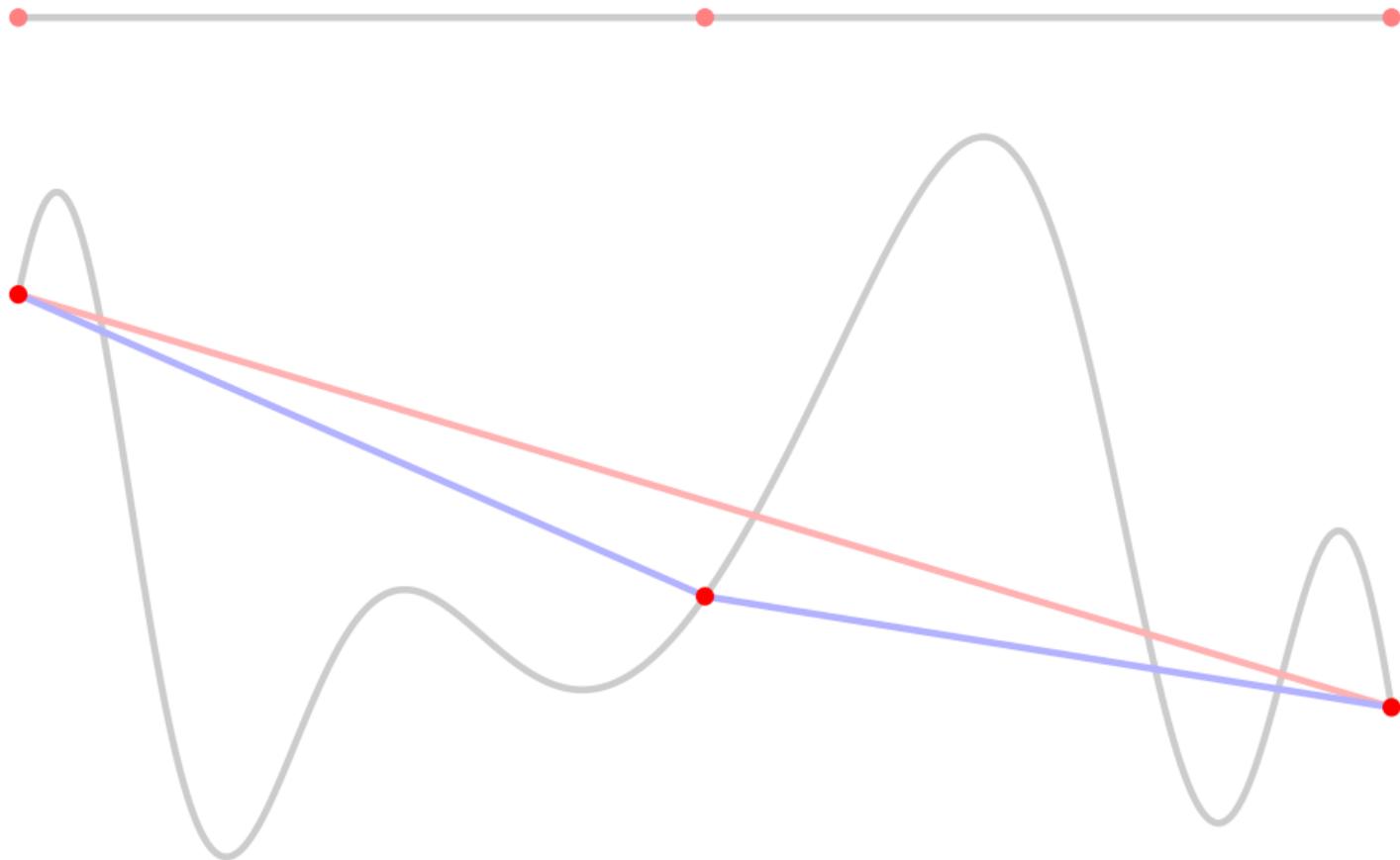


## Curva paramétrica – amostragem adaptativa

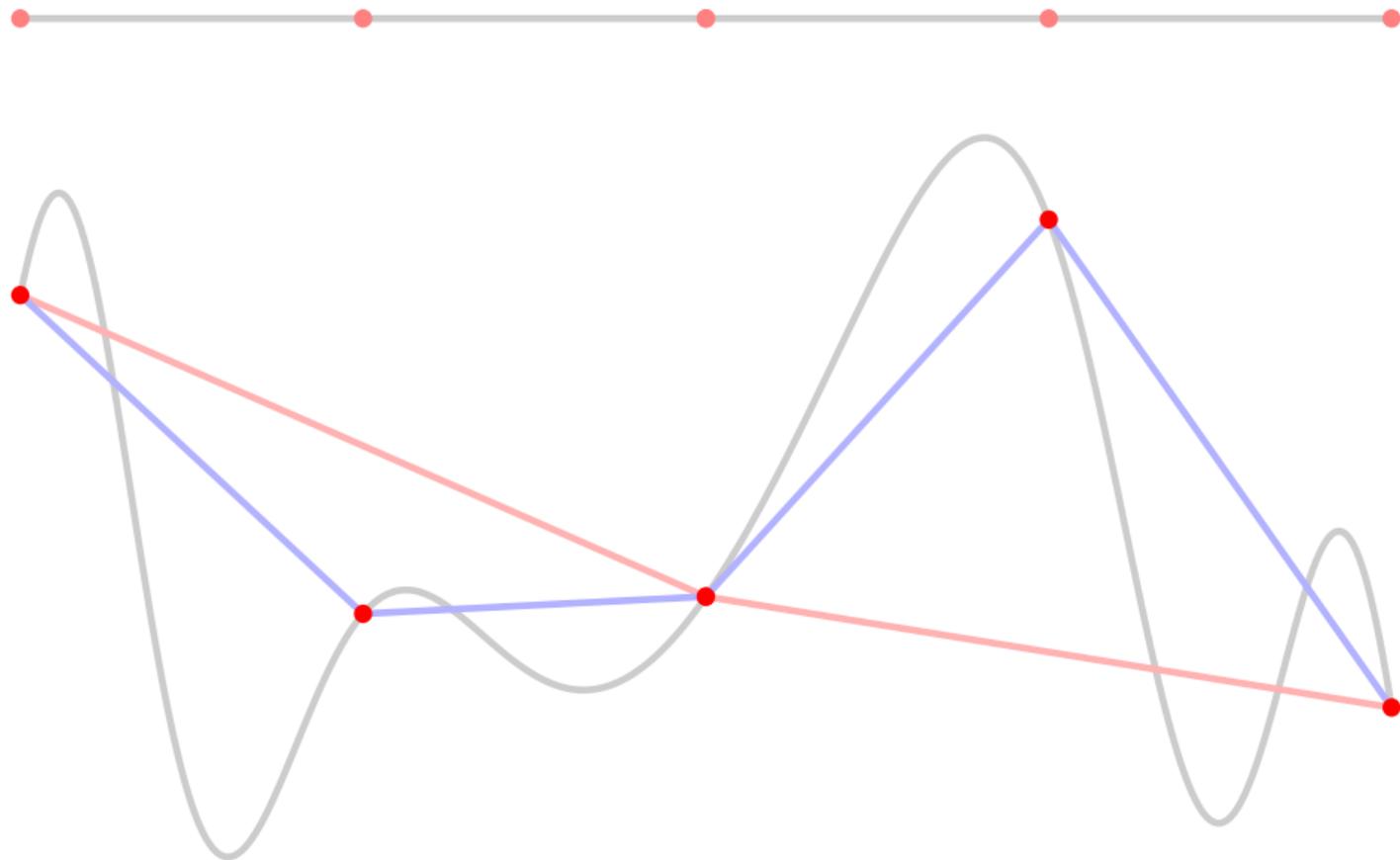
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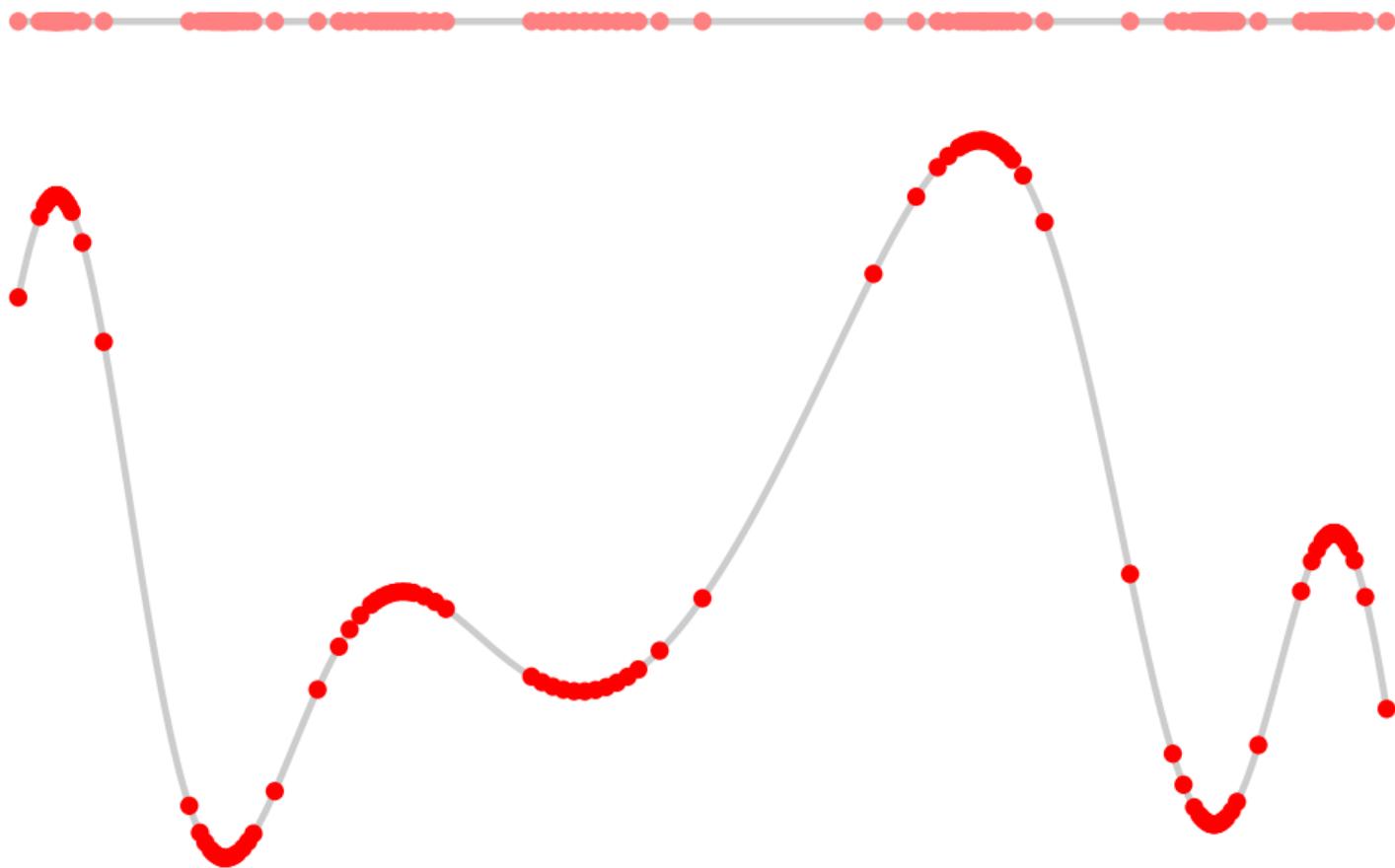
## Curva paramétrica – amostragem adaptativa



## Curva paramétrica – amostragem adaptativa



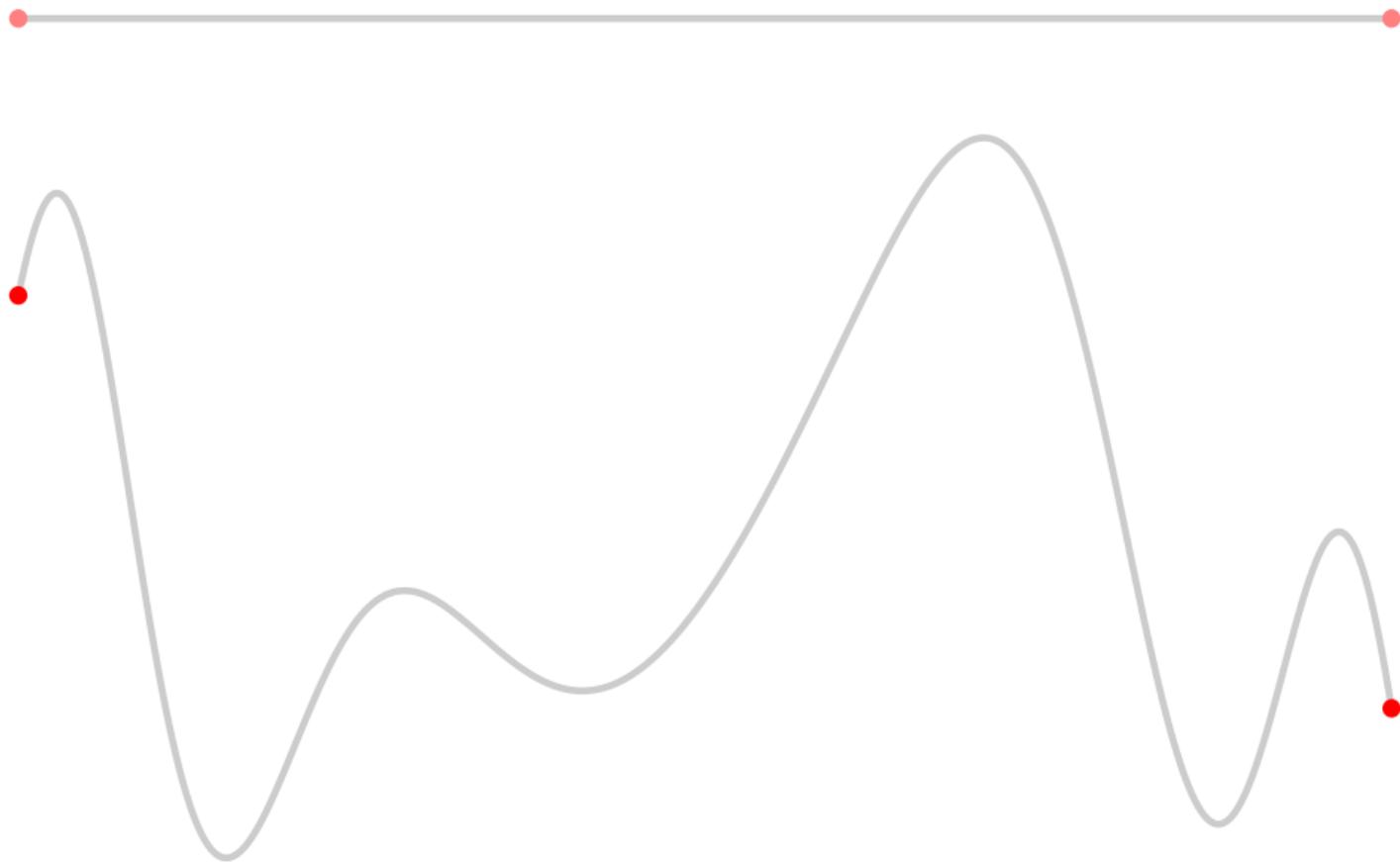
## Curva paramétrica – amostragem adaptativa



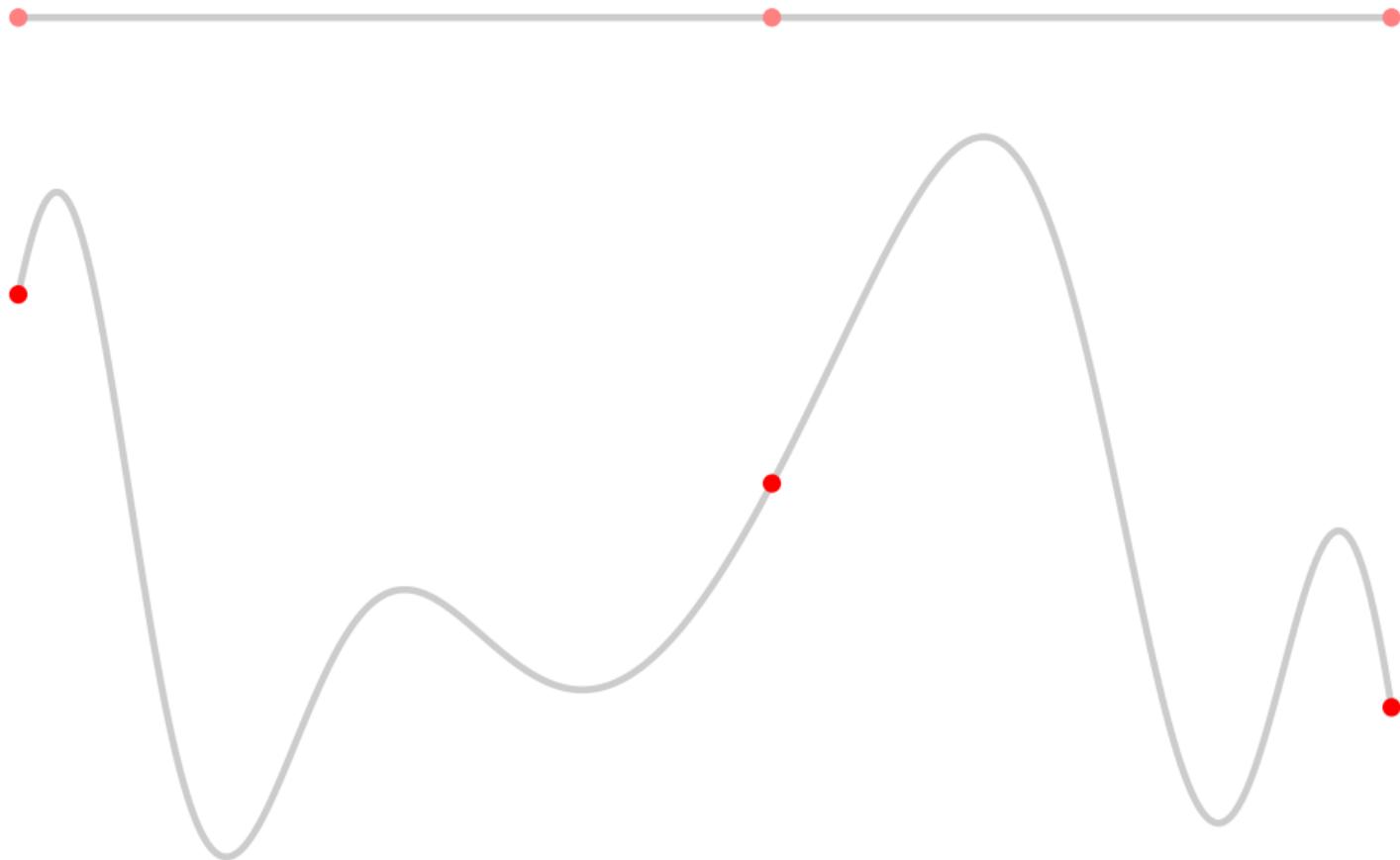
## Curva paramétrica – amostragem adaptativa

```
function sample(t0,t1)
    local tm=(t0+t1)/2
    local x0,y0=gamma(t0)
    local x1,y1=gamma(t1)
    local xm,ym=gamma(tm)
    if flat(x0,y0,x1,y1,xm,ym) then
        save(t0,x0,y0)
    else
        sample(t0,tm)
        sample(tm,t1)
    end
end
```

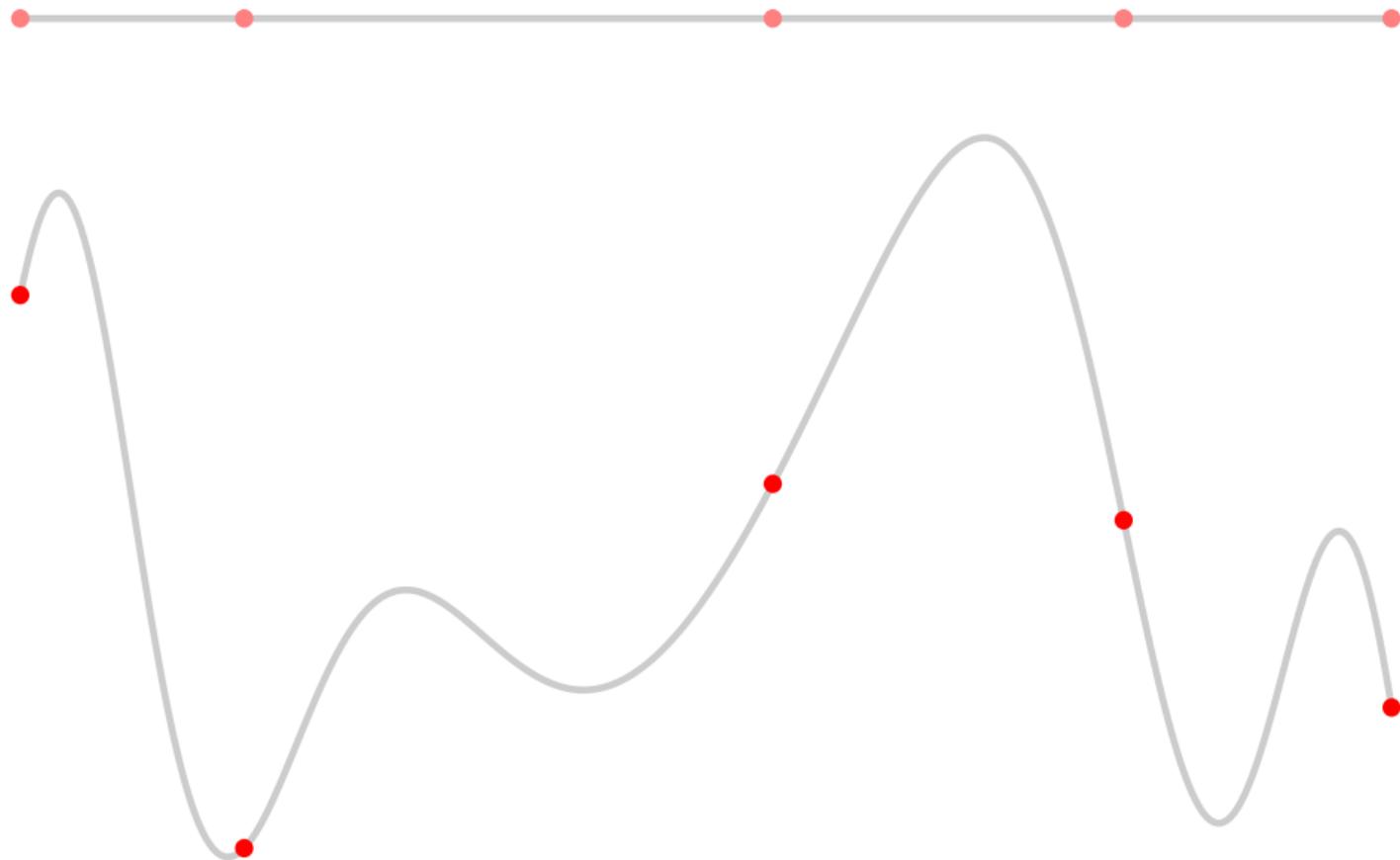
## Curva paramétrica – amostragem uniforme na curva



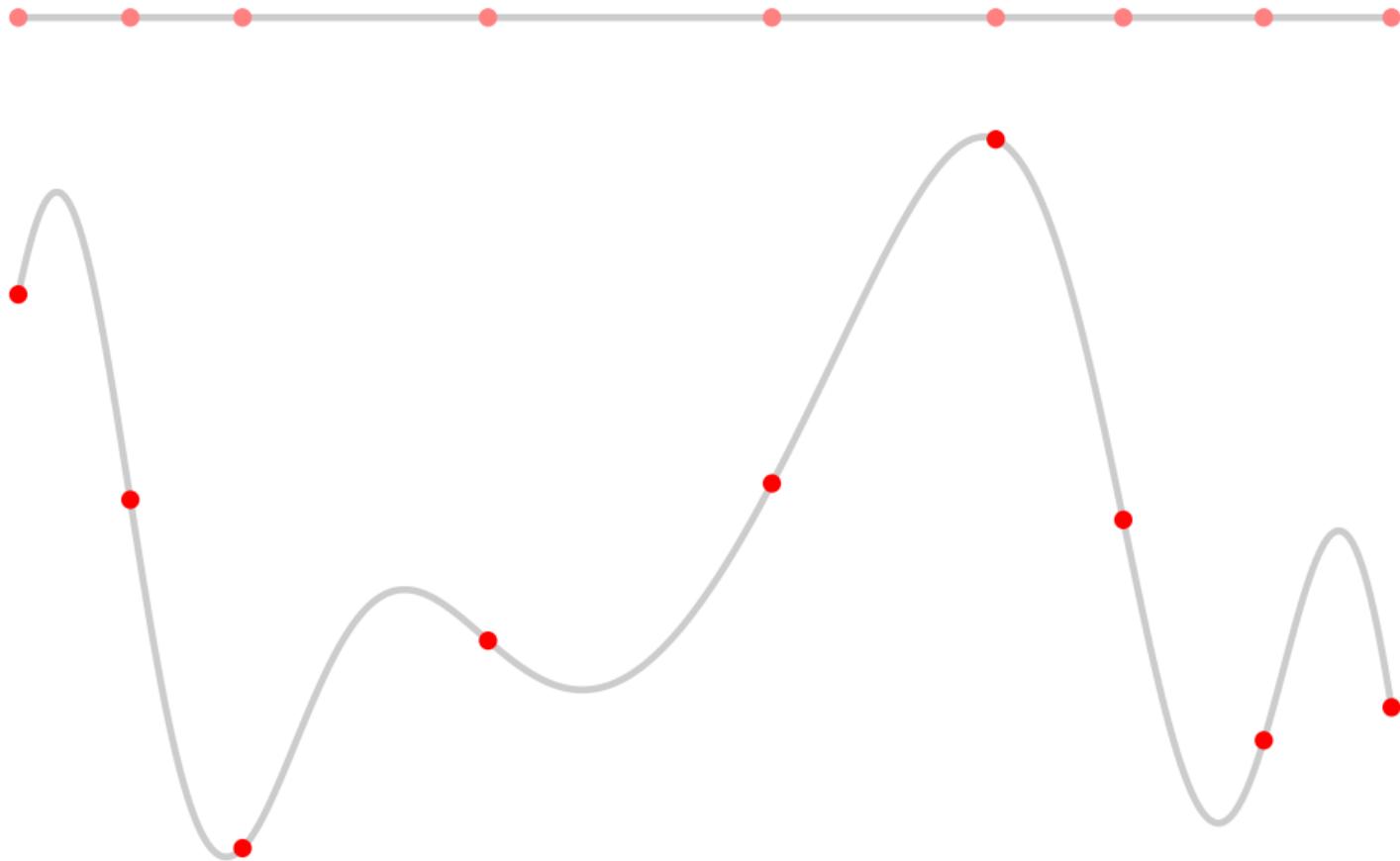
## Curva paramétrica – amostragem uniforme na curva



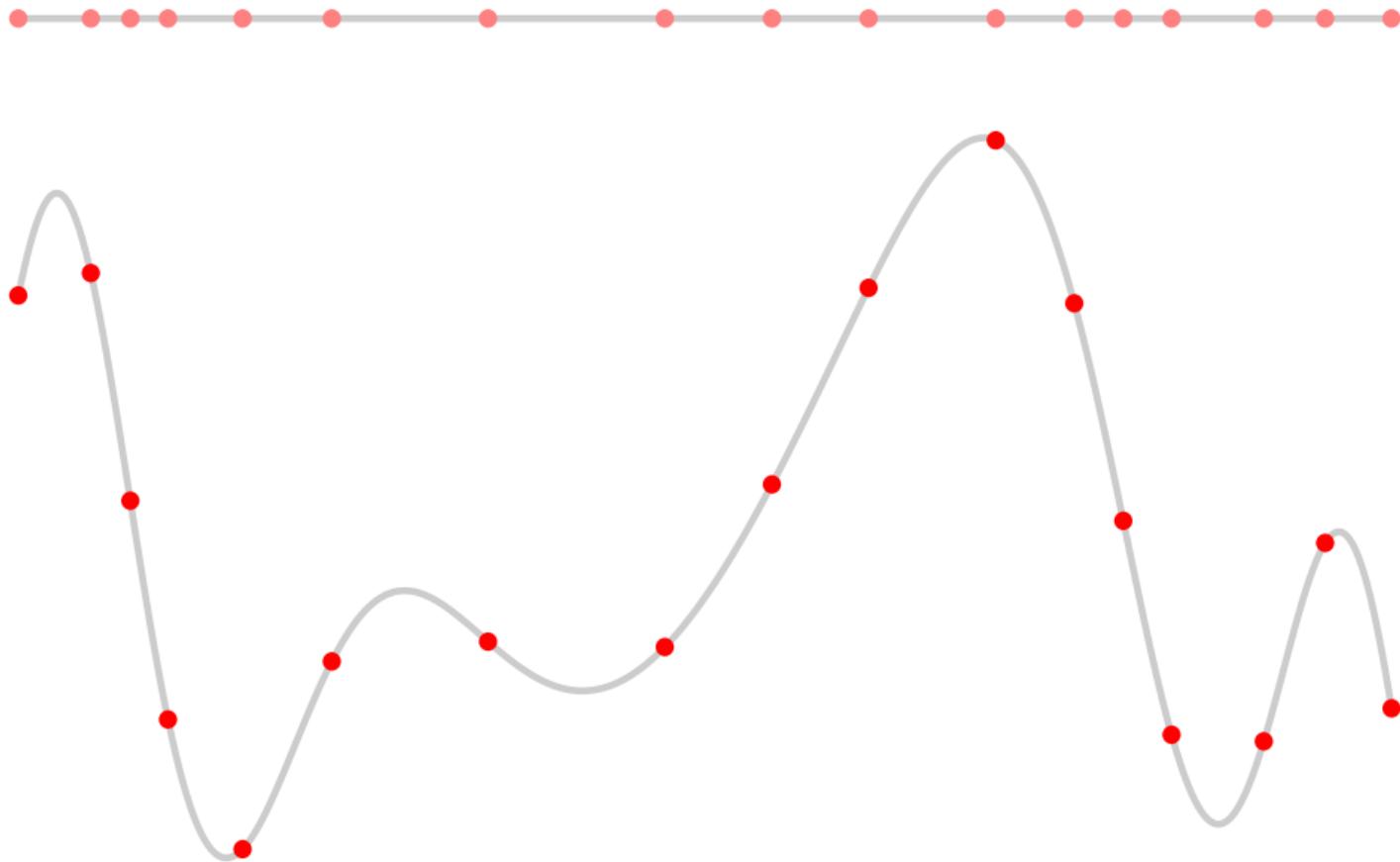
## Curva paramétrica – amostragem uniforme na curva



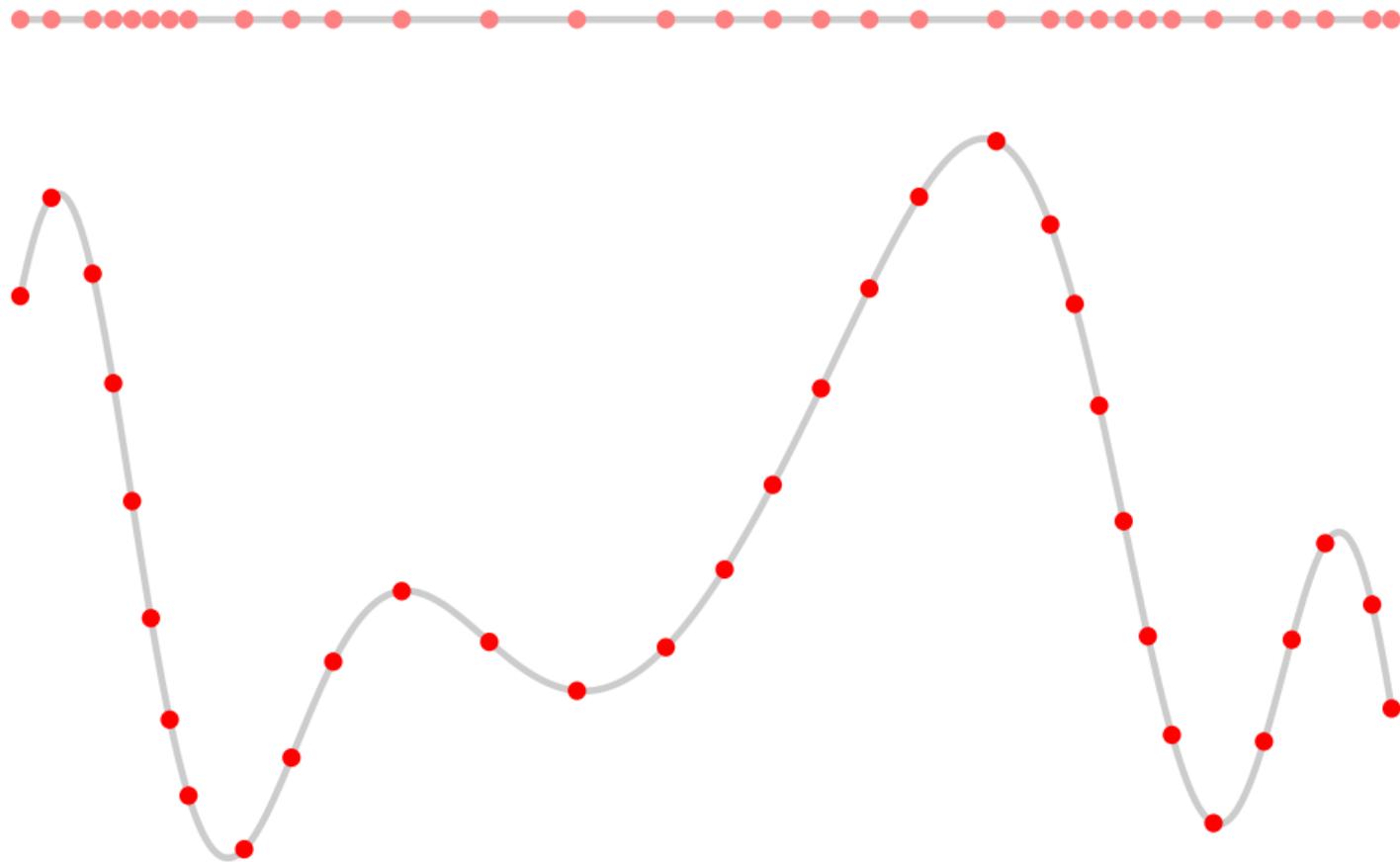
## Curva paramétrica – amostragem uniforme na curva



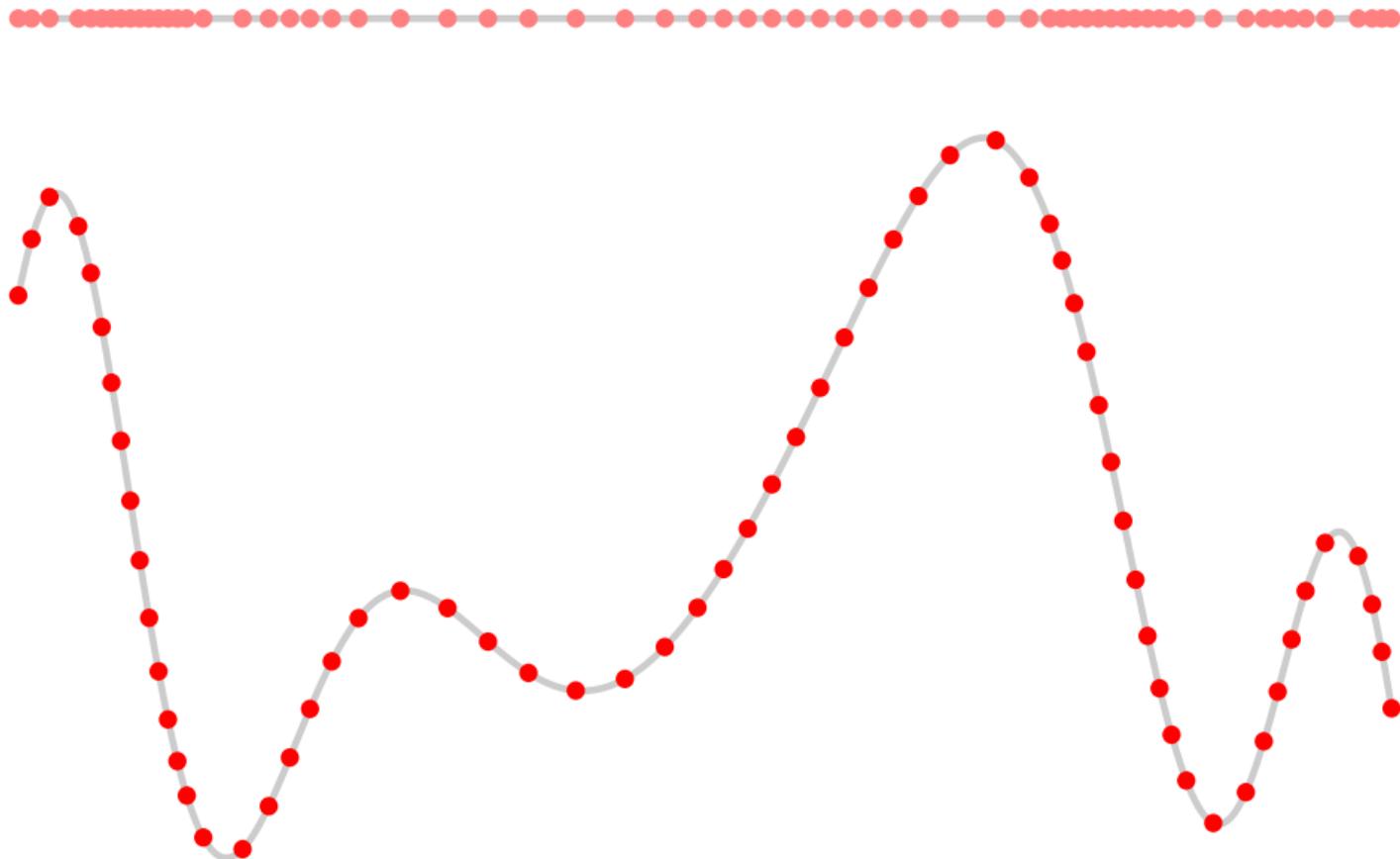
## Curva paramétrica – amostragem uniforme na curva



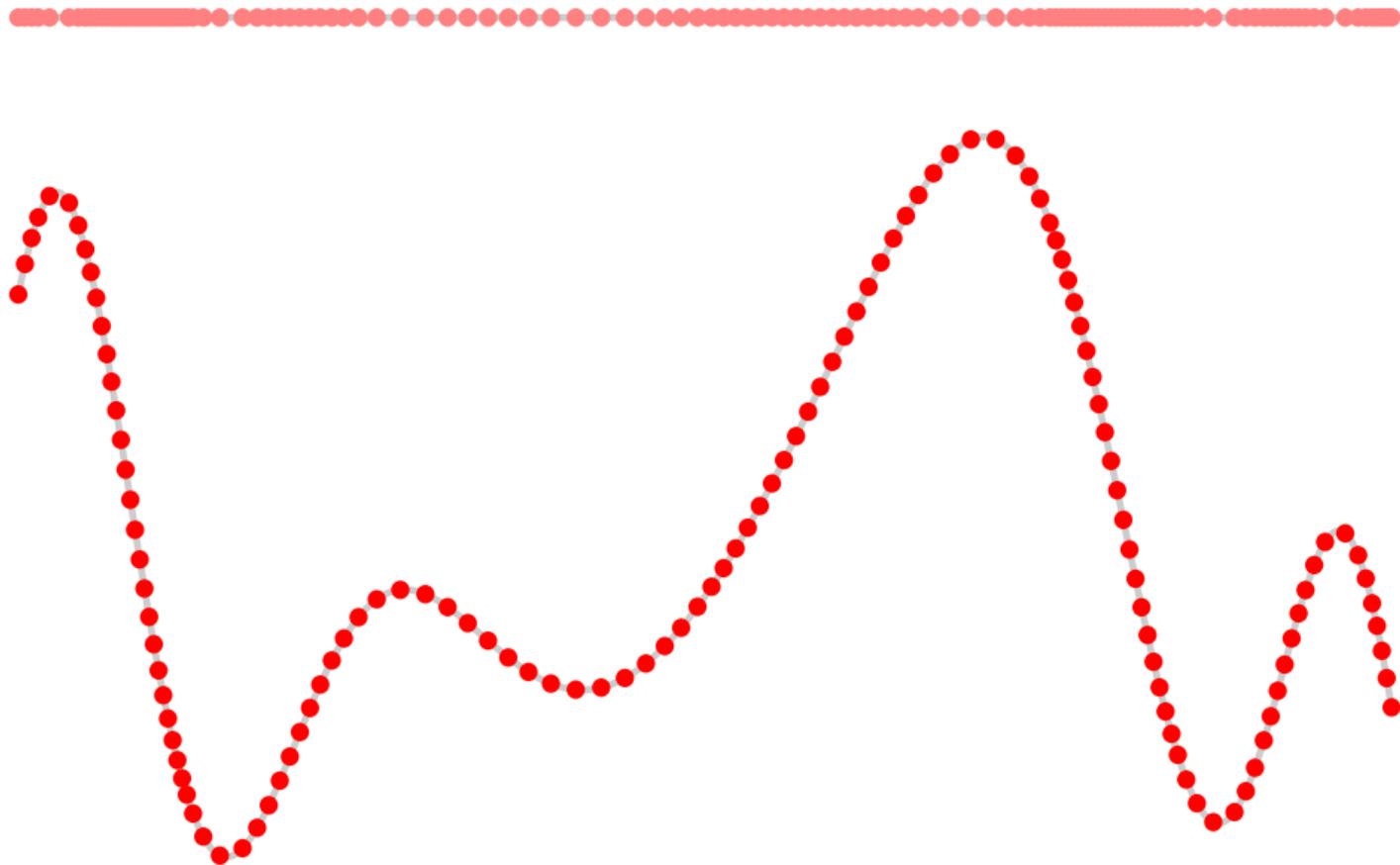
## Curva paramétrica – amostragem uniforme na curva



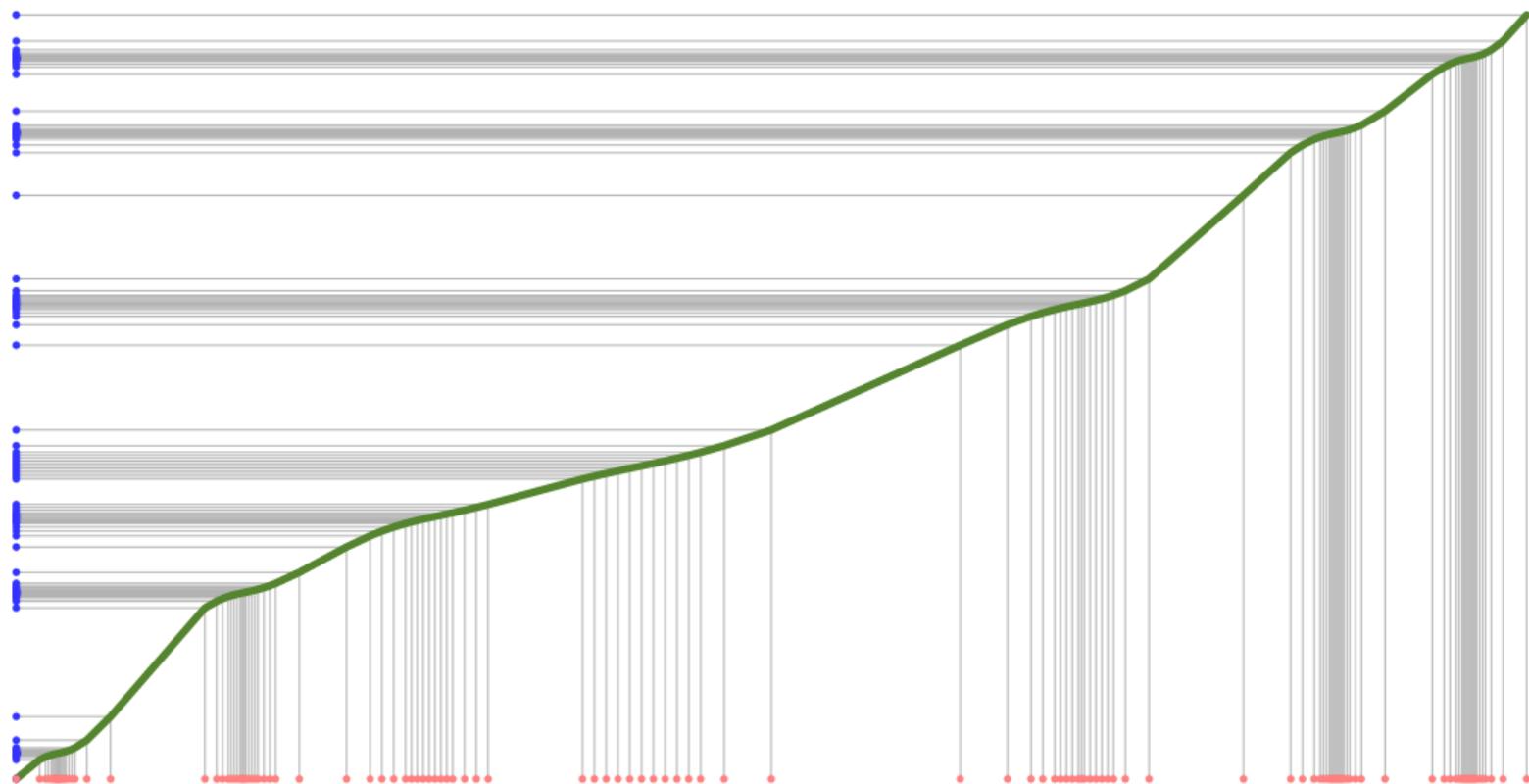
## Curva paramétrica – amostragem uniforme na curva



## Curva paramétrica – amostragem uniforme na curva



## Curva paramétrica – amostragem uniforme na curva

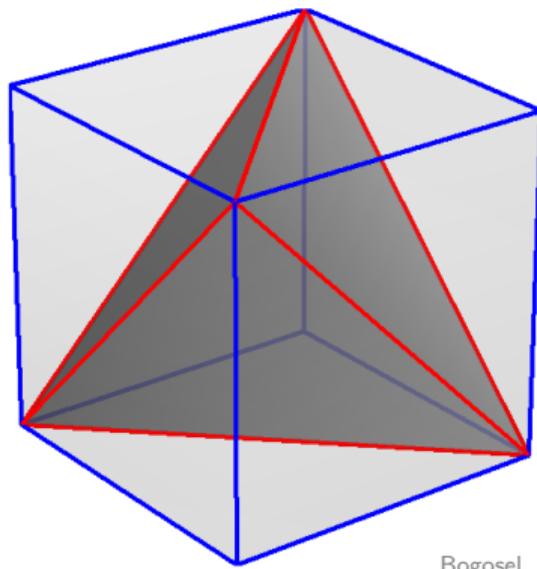


representação de malhas

# Representação de malhas: sopa de polígonos

STL

- conjunto de faces sem organização
- vértices não compartilhados
- difícil de editar
- geometria das faces



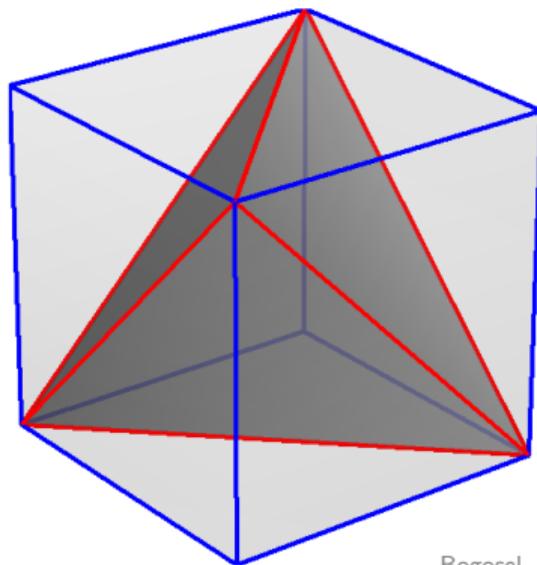
Bogoşel

```
f
-1  1 -1
 1  1  1
 1 -1 -1
f
-1 -1  1
-1  1 -1
 1 -1 -1
f
 1 -1 -1
 1  1  1
-1 -1  1
f
-1 -1  1
 1  1  1
-1  1 -1
```

# Representação de malhas: geometria + topologia

OBJ

- conjunto de faces sem organização
- vértices compartilhados
- mais fácil de editar
- geometria dos vértices
- topologia das faces



geometria

v	1	1	1
v	-1	1	-1
v	1	-1	-1
v	-1	-1	1

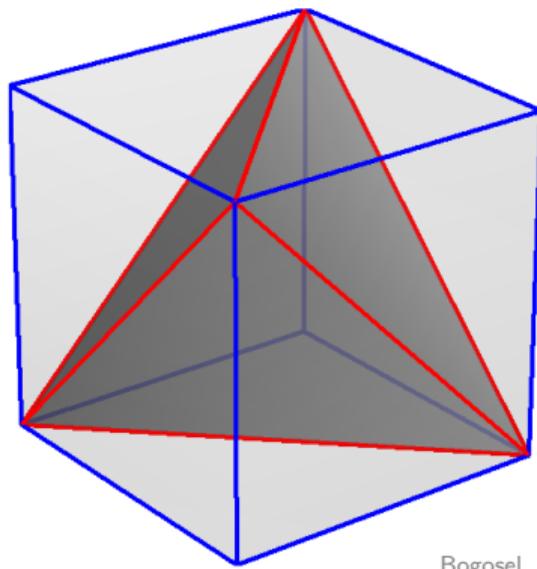
topologia

f	1	2	3
f	2	4	3
f	1	3	4
f	1	4	2

Bogoşel

## Representação de malhas: geometria + topologia + adjacências

- conjunto de faces com organização
- vértices compartilhados
- menos fácil de editar
- geometria dos vértices
- topologia das faces



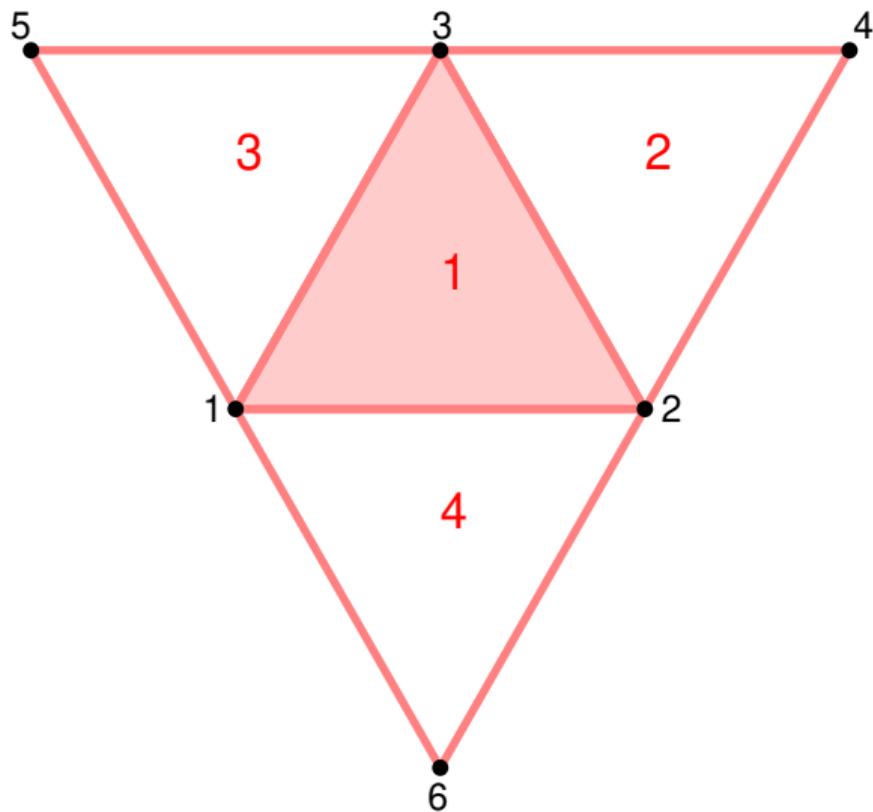
geometria

v	1	1	1
v	-1	1	-1
v	1	-1	-1
v	-1	-1	1

topologia

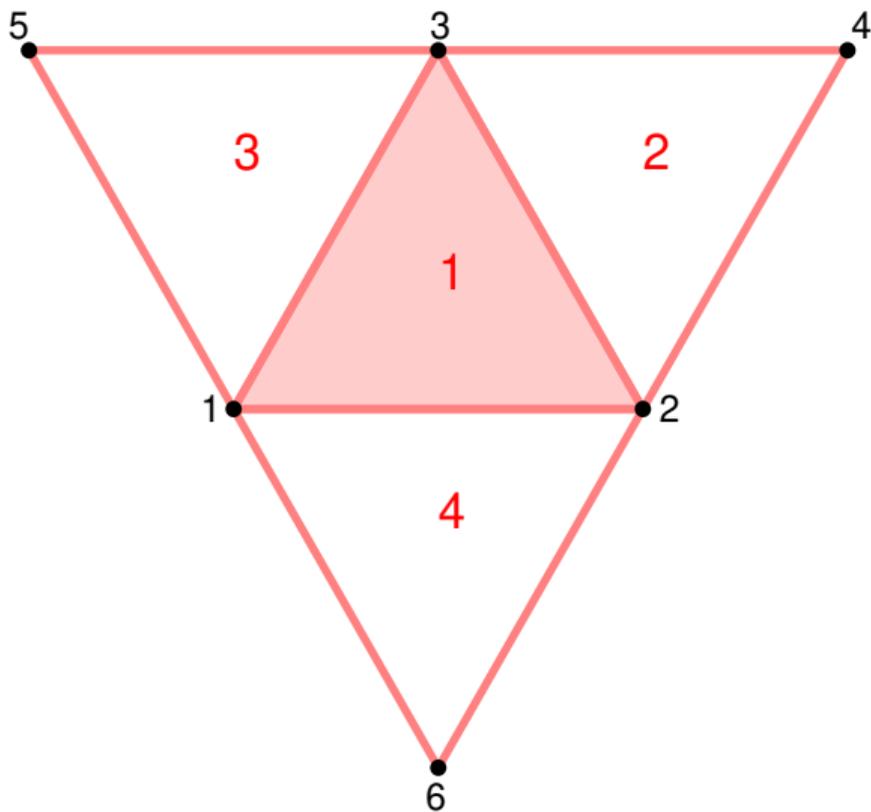
f	1	2	3
	2	3	4
f	2	4	3
	3	1	4
f	1	3	4
	2	4	1
f	1	4	2
	2	1	3

## Representação de malhas de triângulos



1	1	2	3
	2	3	4

## Representação de malhas de triângulos



1	1	2	3
	2	3	4
2	2	4	3
	•	1	•
3	1	3	5
	•	•	1
4	1	6	2
	•	1	•

superfícies paramétricas

## Superfície paramétrica

superfície = fronteira da região

$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

## Superfície paramétrica

superfície = fronteira da região (retalho)

$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

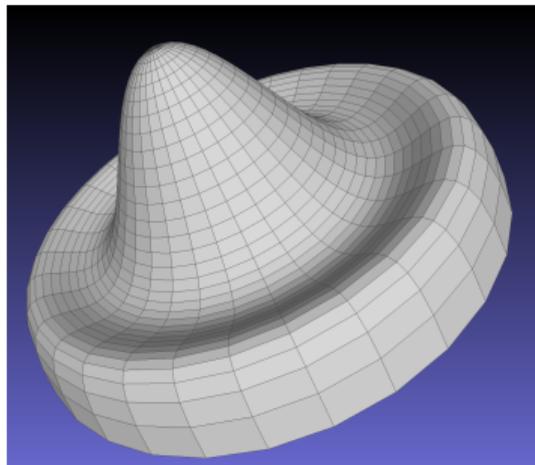
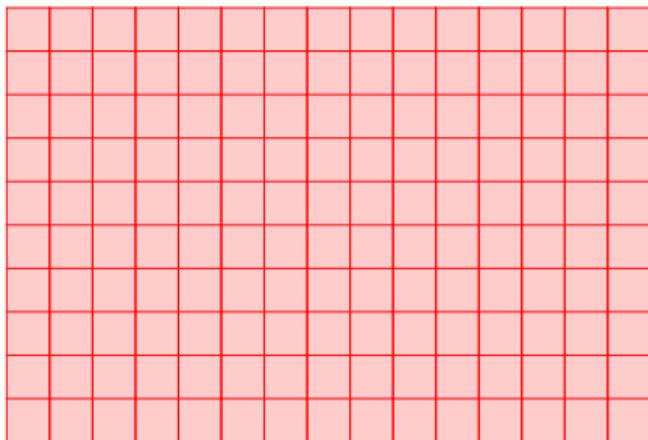
## Superfície paramétrica

superfície = fronteira da região (retalho)

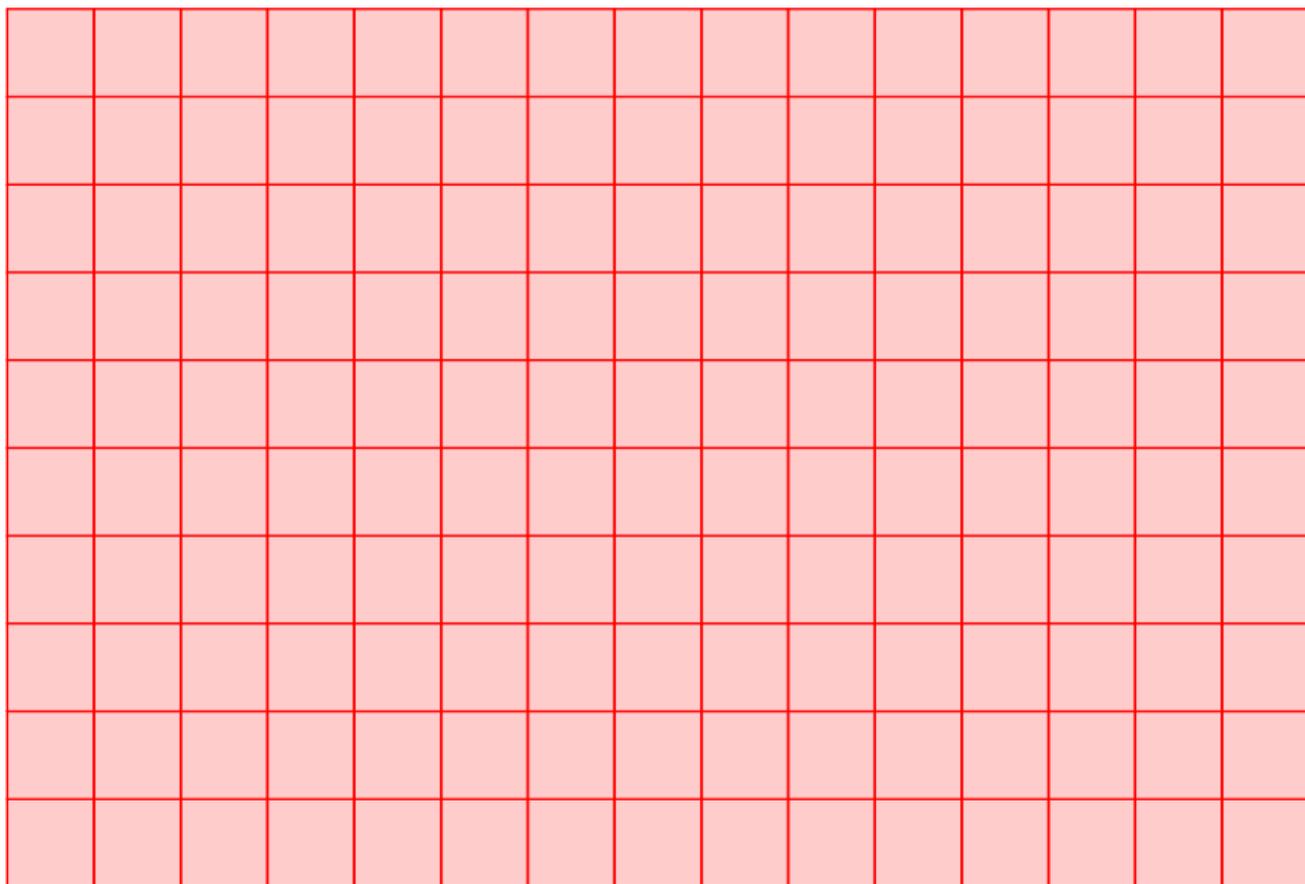
$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

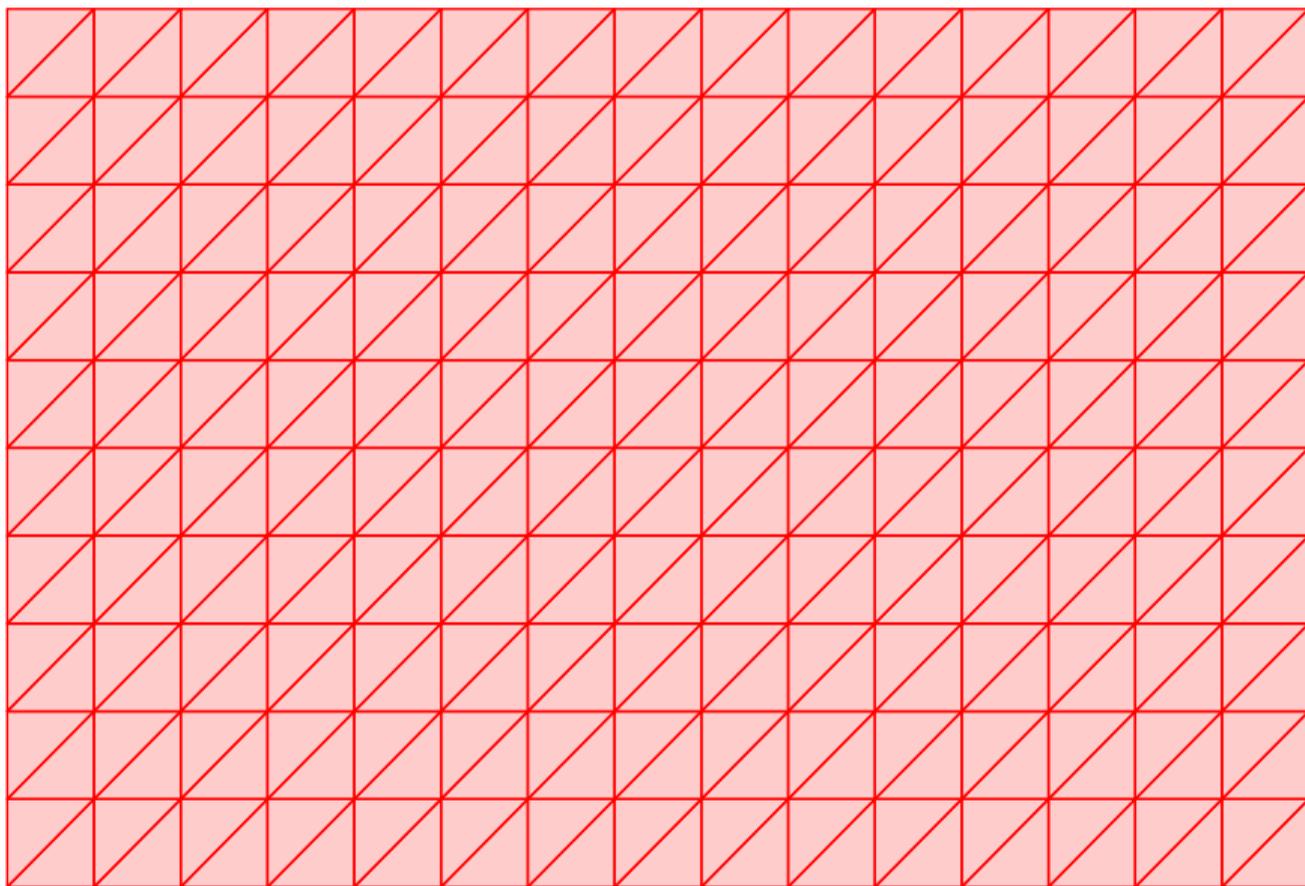
discretização da superfície = discretização do domínio



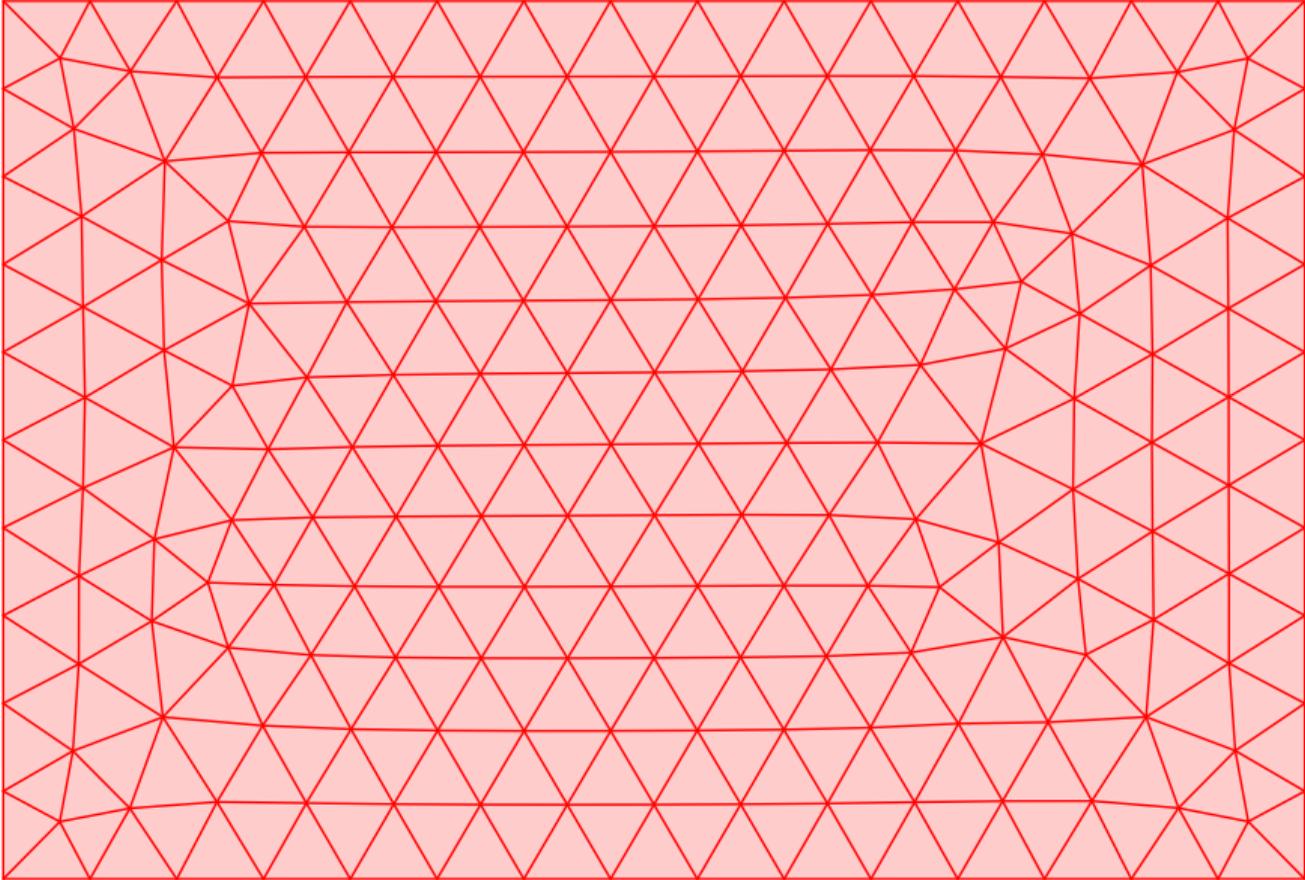
## Superfície paramétrica – discretização do domínio



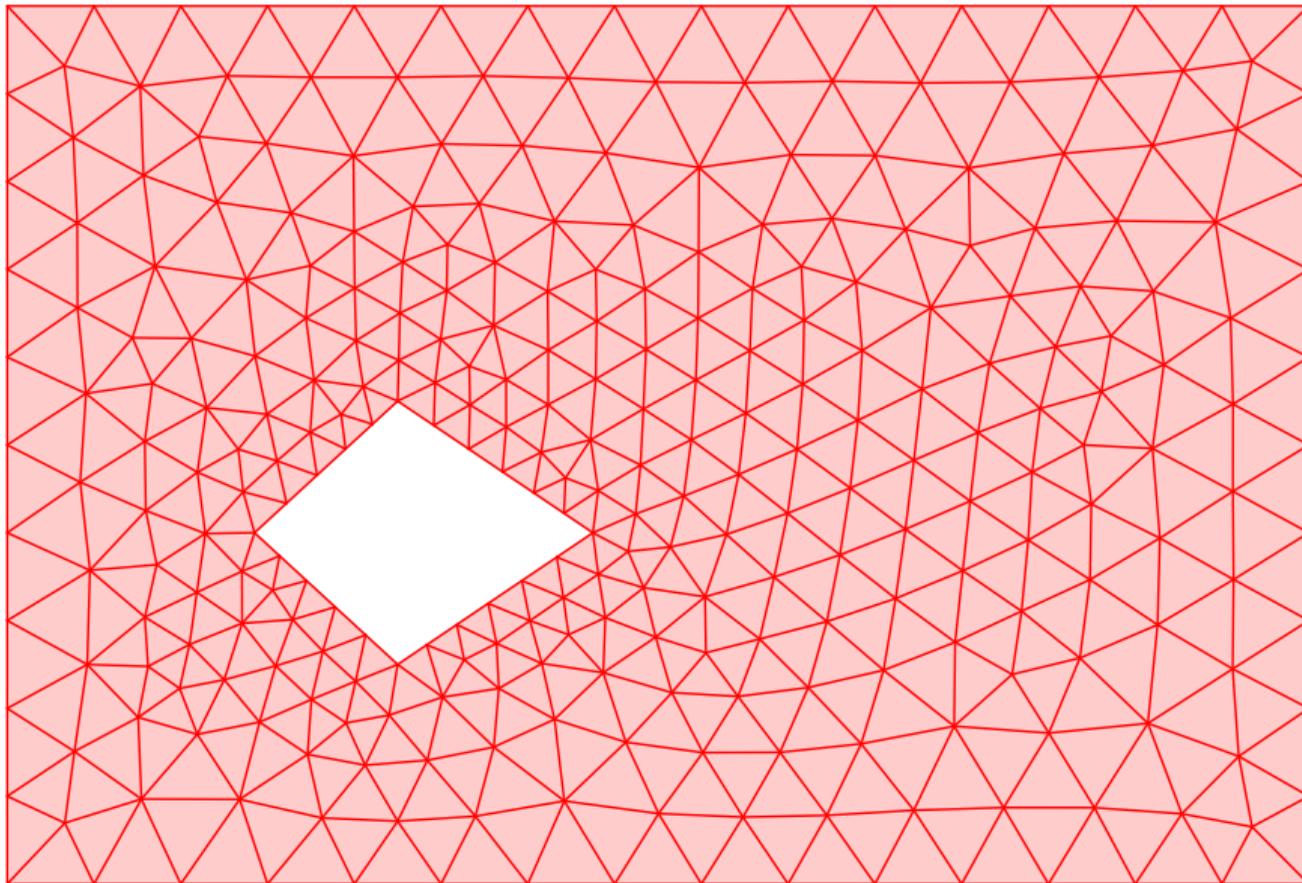
## Superfície paramétrica – discretização do domínio



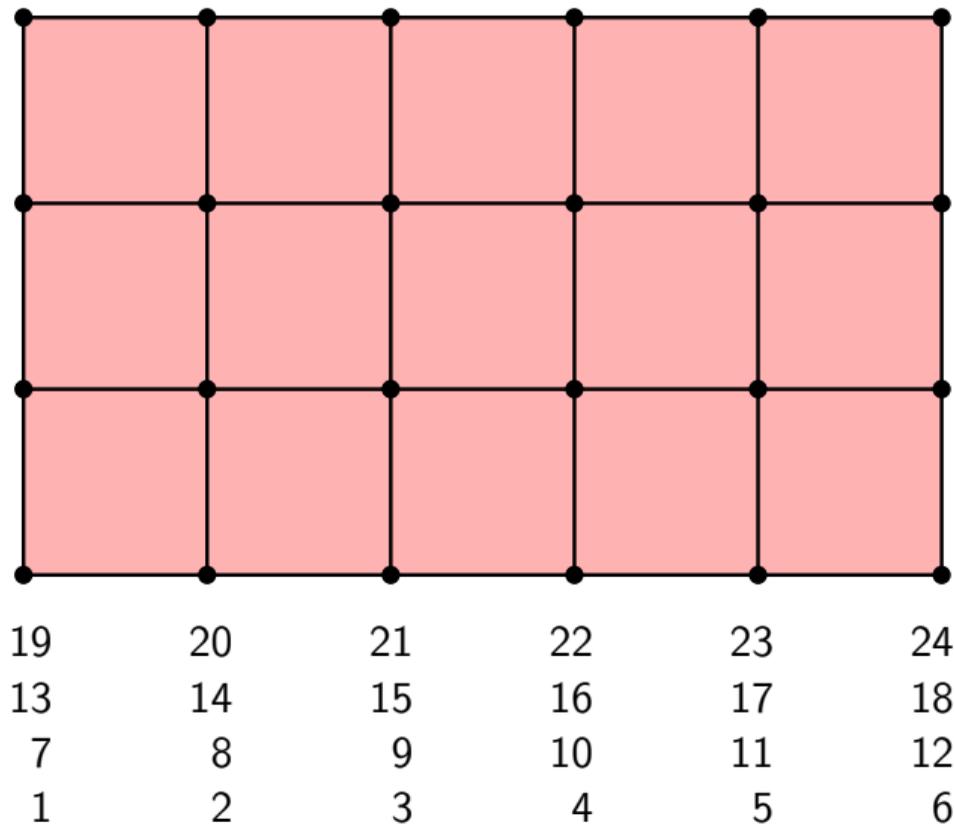
# Superfície paramétrica – discretização do domínio



## Superfície paramétrica – discretização do domínio

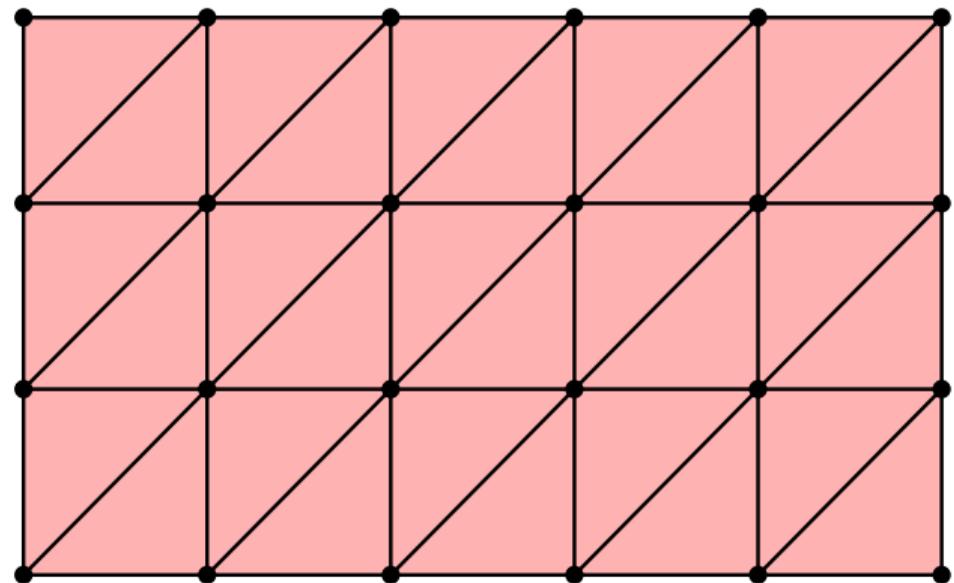


## Superfície paramétrica – representação



f	1	2	8	7
f	2	3	9	8
f	3	4	10	9
f	4	5	11	10
f	5	6	12	11
f	7	8	14	13
f	8	9	15	14
f	9	10	16	15
f	10	11	17	16
f	11	12	18	17
f	13	14	20	19
f	14	15	21	20
f	15	16	22	21
f	16	17	23	22
f	17	18	24	23

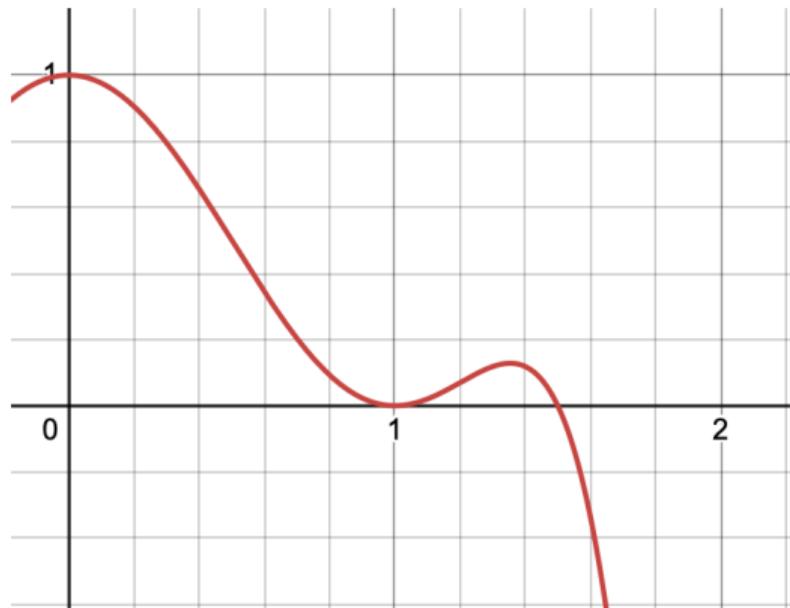
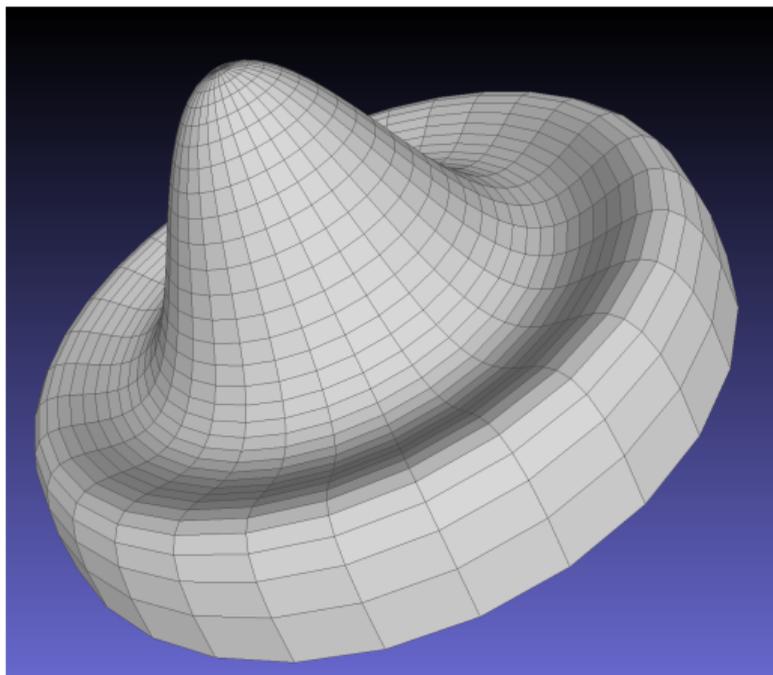
## Superfície paramétrica – representação



19	20	21	22	23	24
13	14	15	16	17	18
7	8	9	10	11	12
1	2	3	4	5	6

f	1	2	8
f	1	8	7
f	2	3	9
f	2	9	8
f	3	4	10
f	3	10	9
f	4	5	11
f	4	11	10
f	5	6	12
f	5	12	11
f	7	8	14
f	7	14	13
f	8	9	15
f	8	15	14
f	...		

## Superfície de revolução

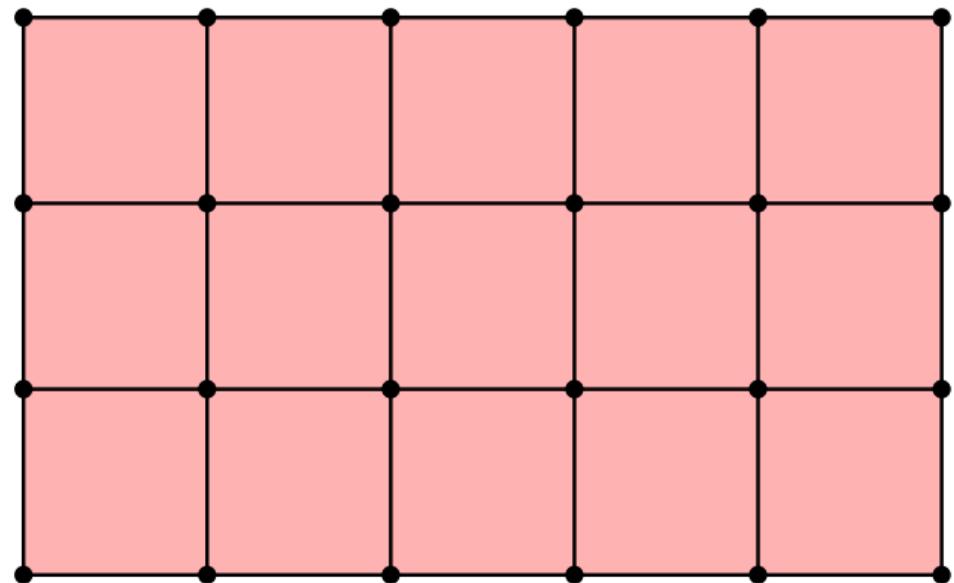


$$f(u, v) = (v \cos(u), v \sin(u), g(v))$$

$$u \in [0, 2\pi], v \in [0, 1.6]$$

$$g(x) = -\frac{4}{9}x^6 + \frac{17}{9}x^4 - \frac{22}{9}x^2 + 1$$

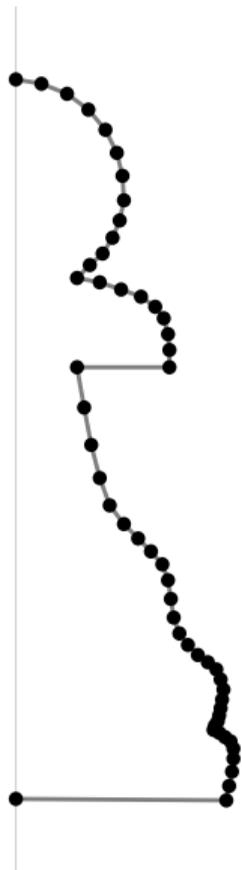
## Superfície de revolução – representação



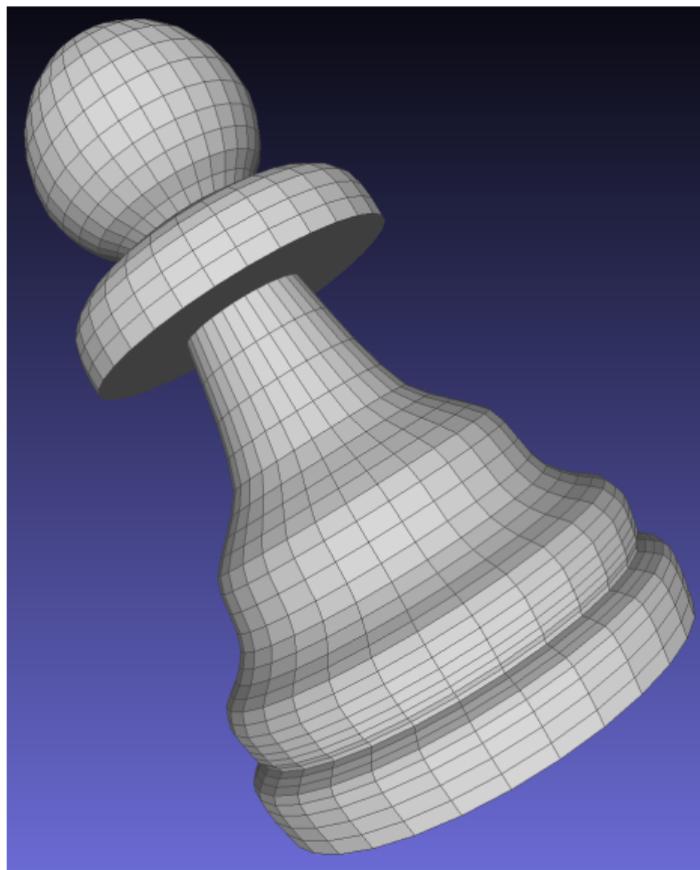
16	17	18	19	20	16
11	12	13	14	15	11
6	7	8	9	10	6
1	2	3	4	5	1

f	1	2	7	6
f	2	3	8	7
f	3	4	9	8
f	4	5	10	9
f	5	1	6	10
f	6	7	12	11
f	7	8	13	12
f	8	9	14	13
f	9	10	15	14
f	10	6	11	15
f	11	12	17	16
f	12	13	18	17
f	13	14	19	18
f	14	15	20	19
f	15	11	16	20

# Superfície de revolução



Warren–Weimer



## Retalho bilinear

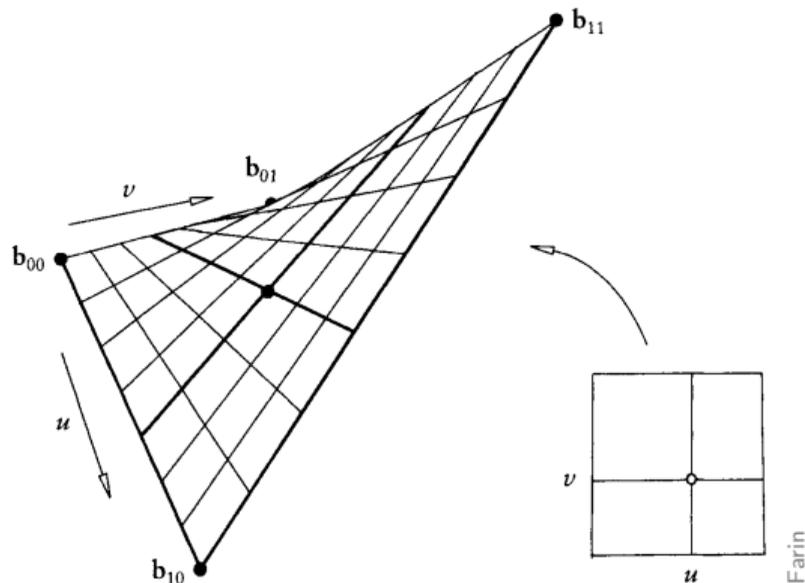
- vértices do bordo  
 $p_{00}, p_{10}, p_{01}, p_{11}$

- interpolação bilinear

$$f(u, v) = (1 - u)(1 - v) p_{00} + u(1 - v) p_{10} + (1 - u)v p_{01} + uv p_{11}$$

$$u, v \in [0, 1]$$

- parabolóide hiperbólico



- curvas de bordo

$$\alpha_0, \alpha_1, \beta_0, \beta_1: [0, 1] \rightarrow \mathbf{R}^3$$

- vértices do bordo

$$p_{00}, p_{10}, p_{01}, p_{11}$$

- superfícies regradas

$$A(u, v) = (1 - v)\alpha_0(u) + v\alpha_1(u)$$

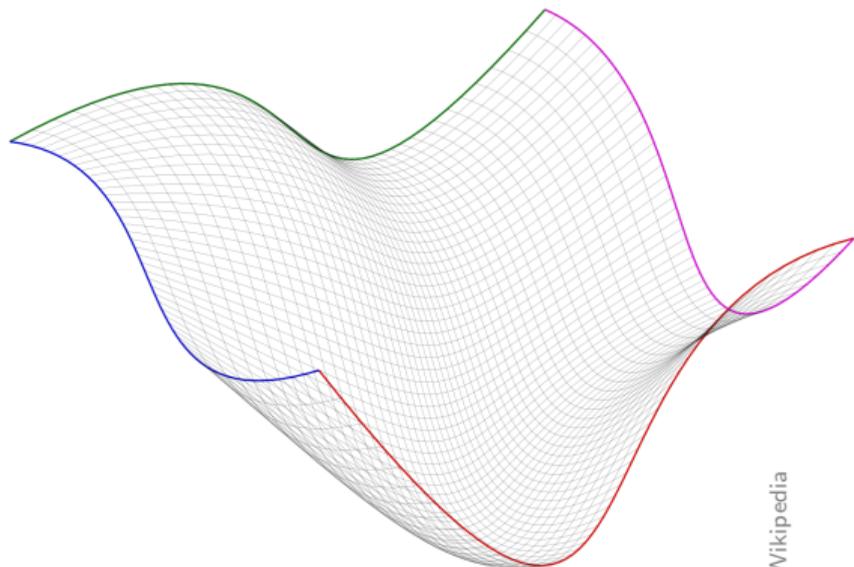
$$B(u, v) = (1 - u)\beta_0(v) + u\beta_1(v)$$

- interpolação bilinear

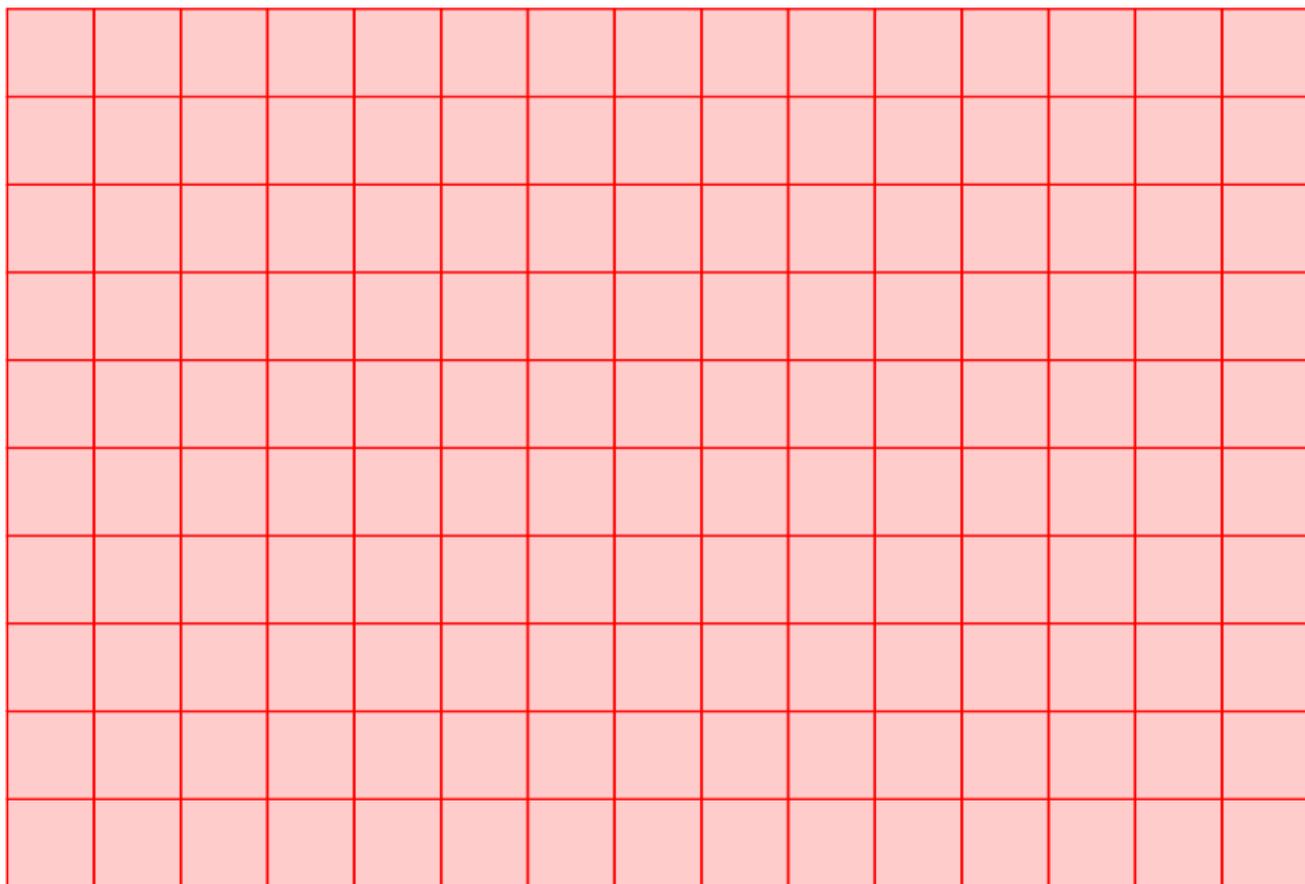
$$C(u, v) = (1 - u)(1 - v)p_{00} + u(1 - v)p_{10} + (1 - u)v p_{01} + uv p_{11}$$

- $f(u, v) = A(u, v) + B(u, v) - C(u, v)$

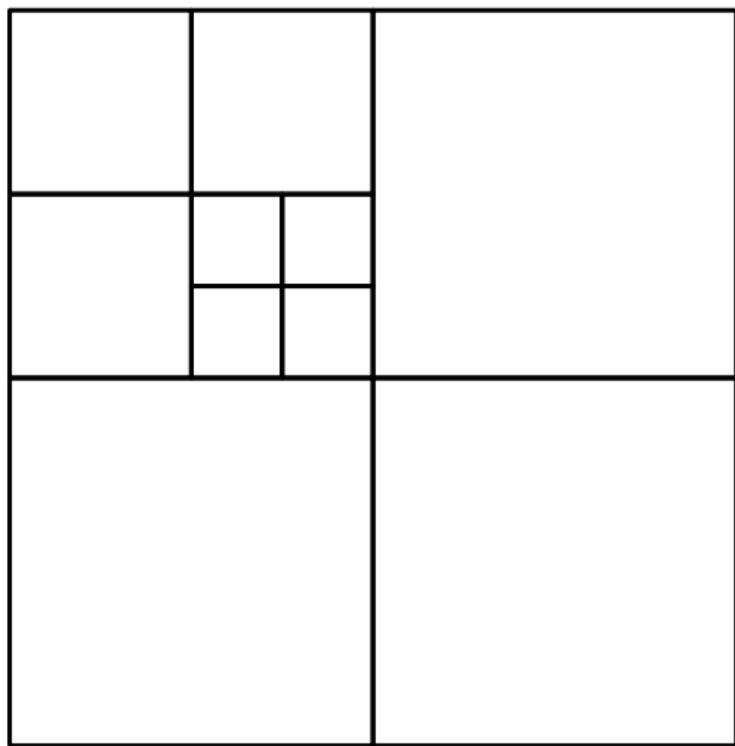
$$u, v \in [0, 1]$$



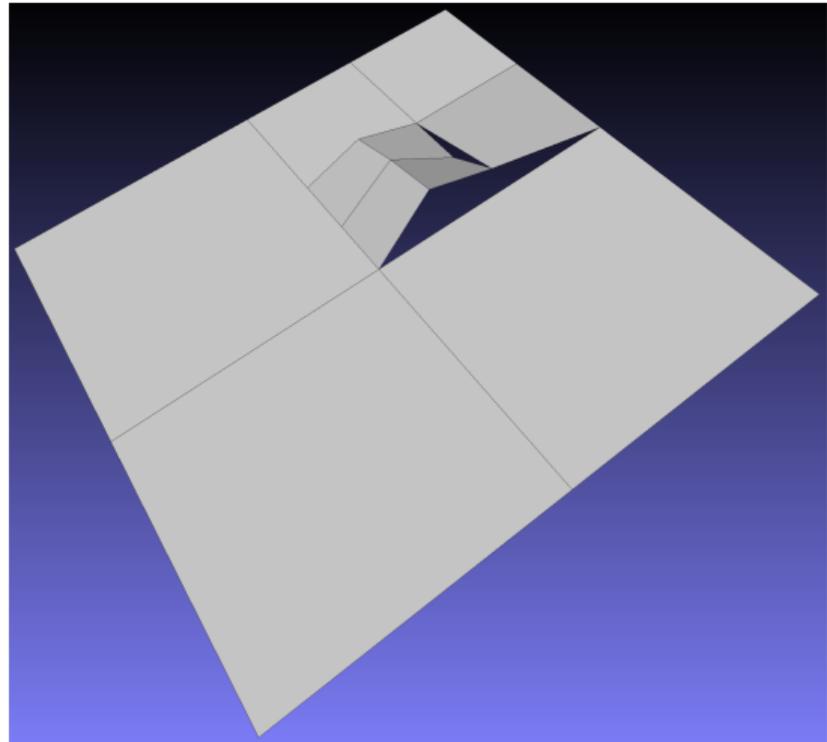
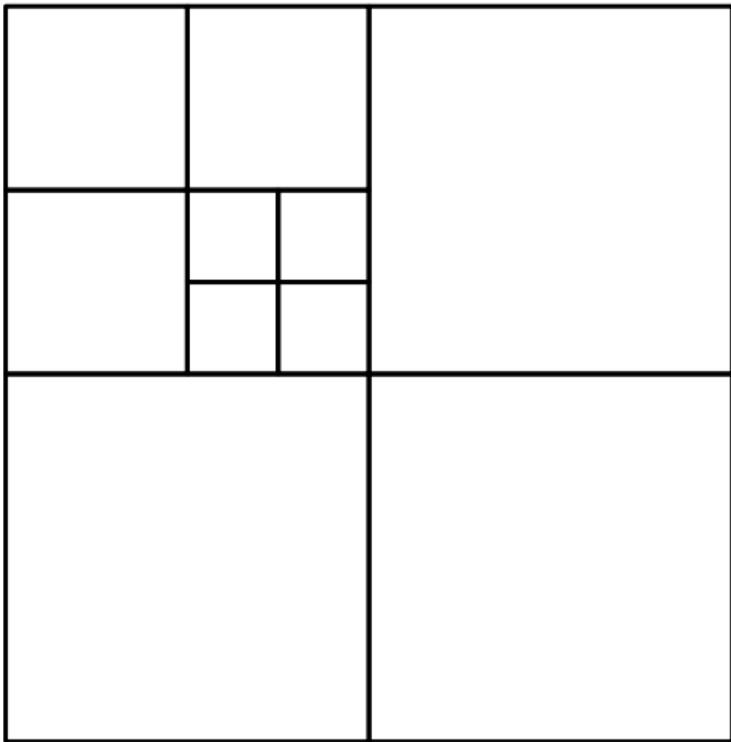
## Superfície paramétrica – discretização do domínio



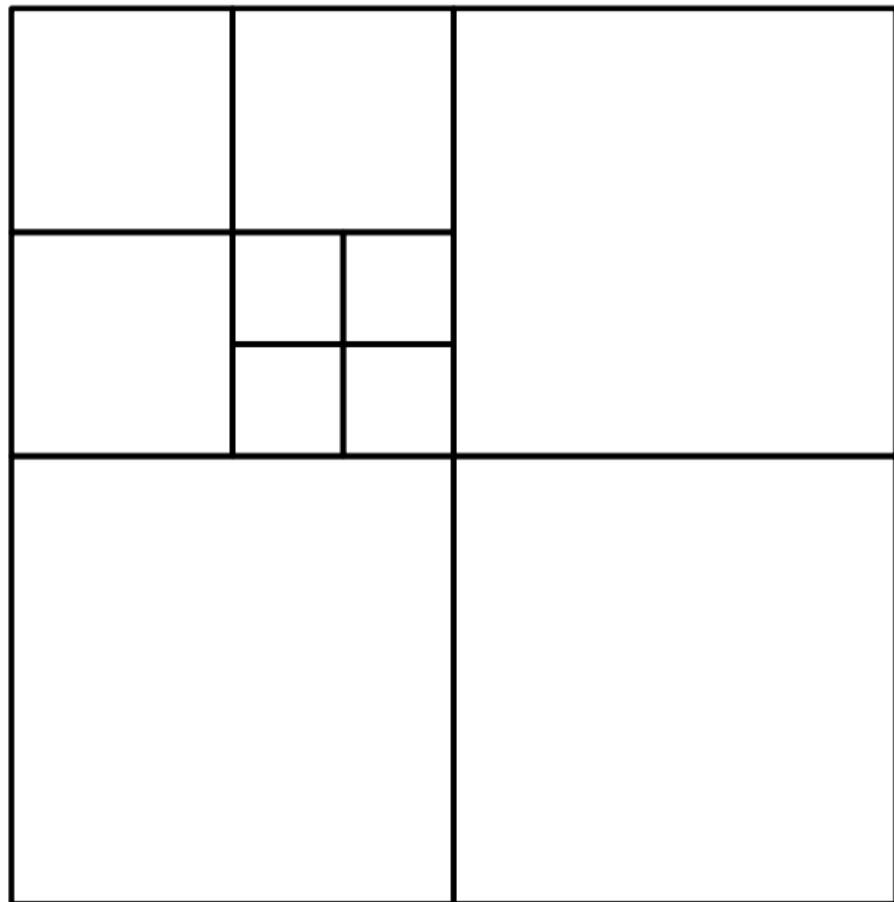
## Superfície paramétrica – discretização do domínio: quadtree



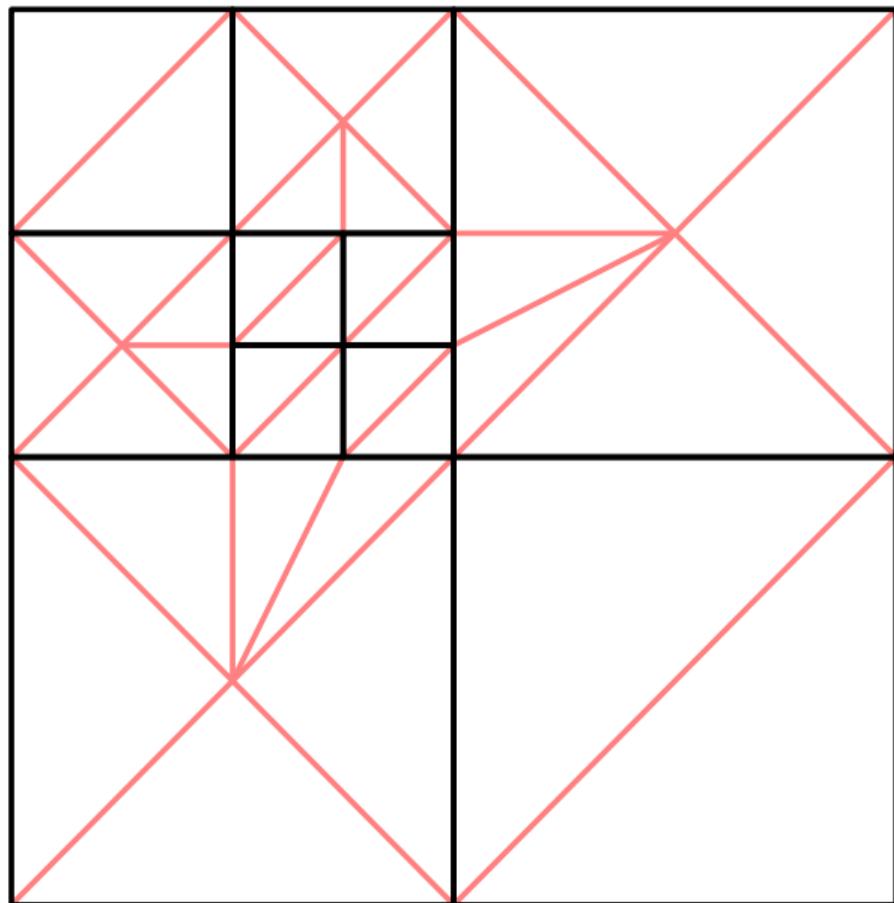
## Superfície paramétrica – discretização do domínio: quadtree



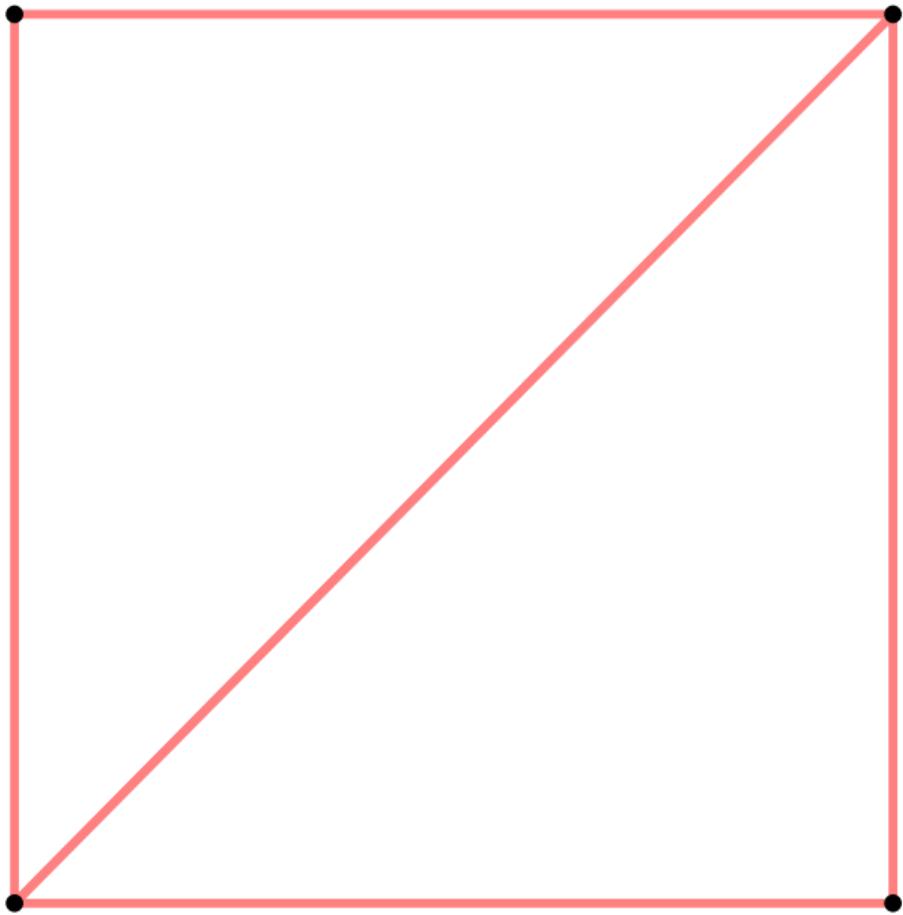
Superfície paramétrica – discretização do domínio: quadtree



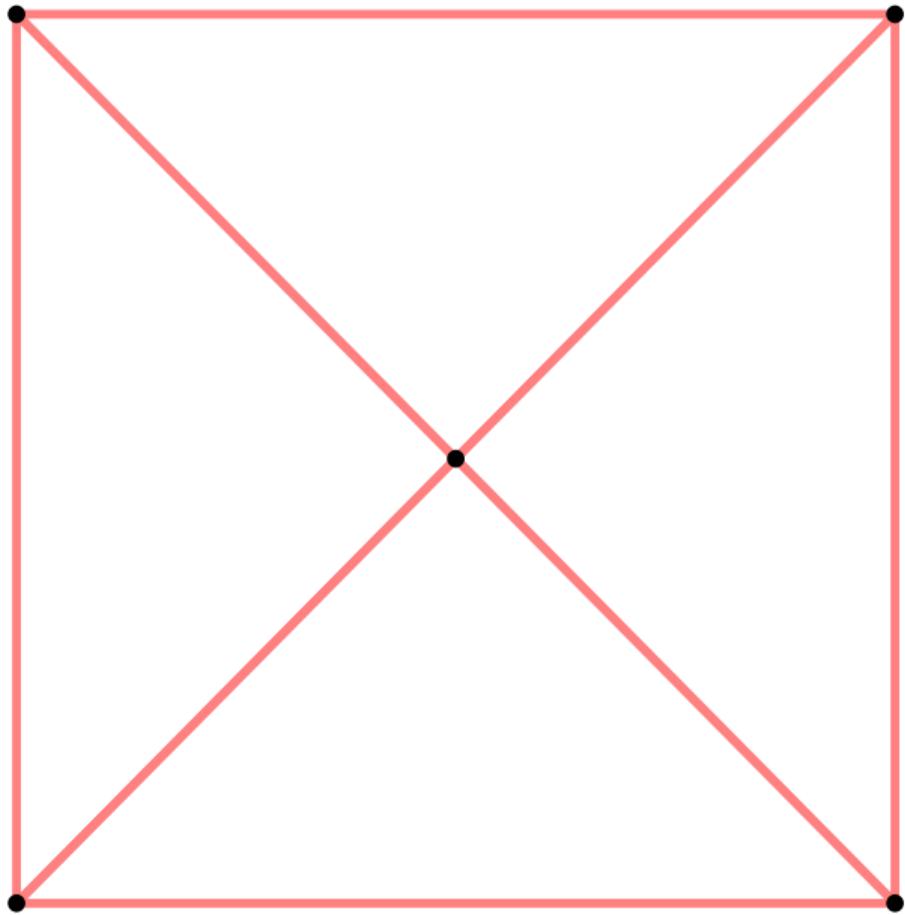
Superfície paramétrica – discretização do domínio: quadtree triangulada



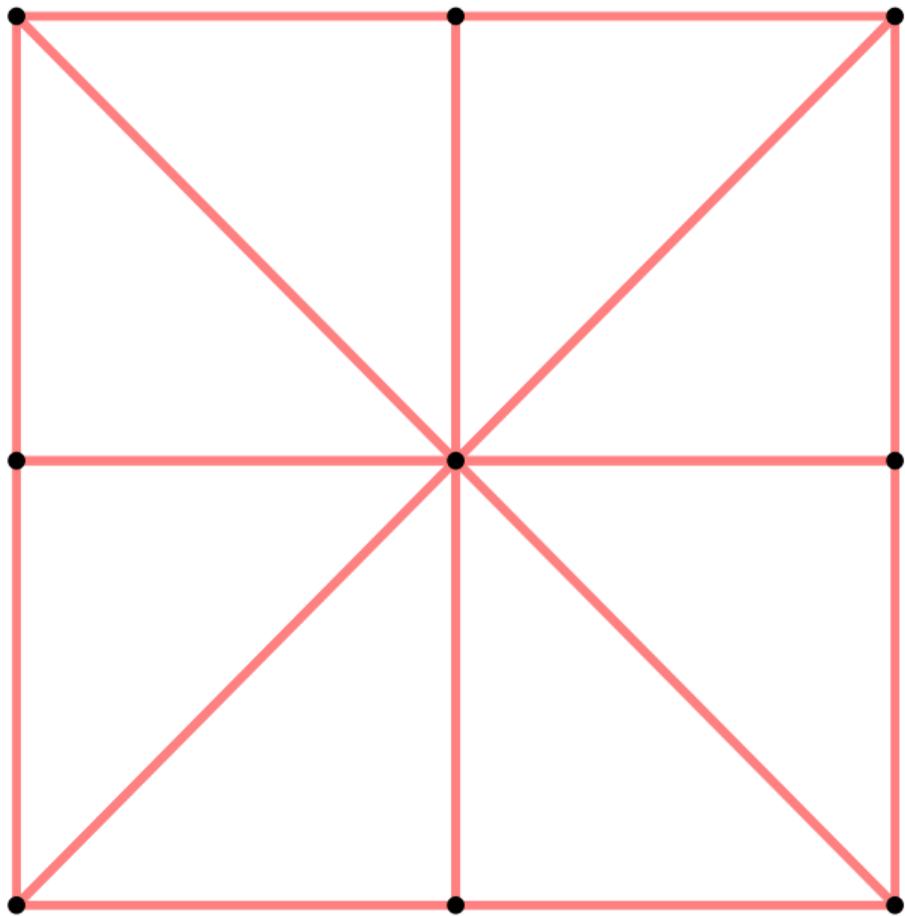
Malha 4-8 uniforme



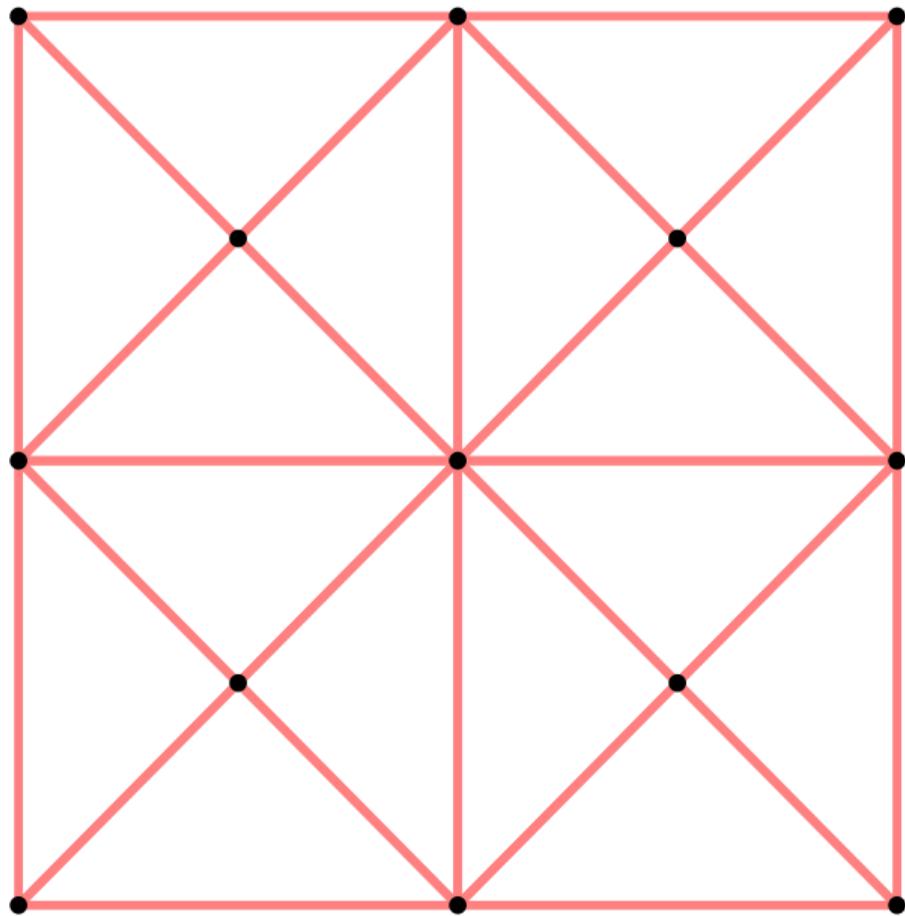
Malha 4-8 uniforme



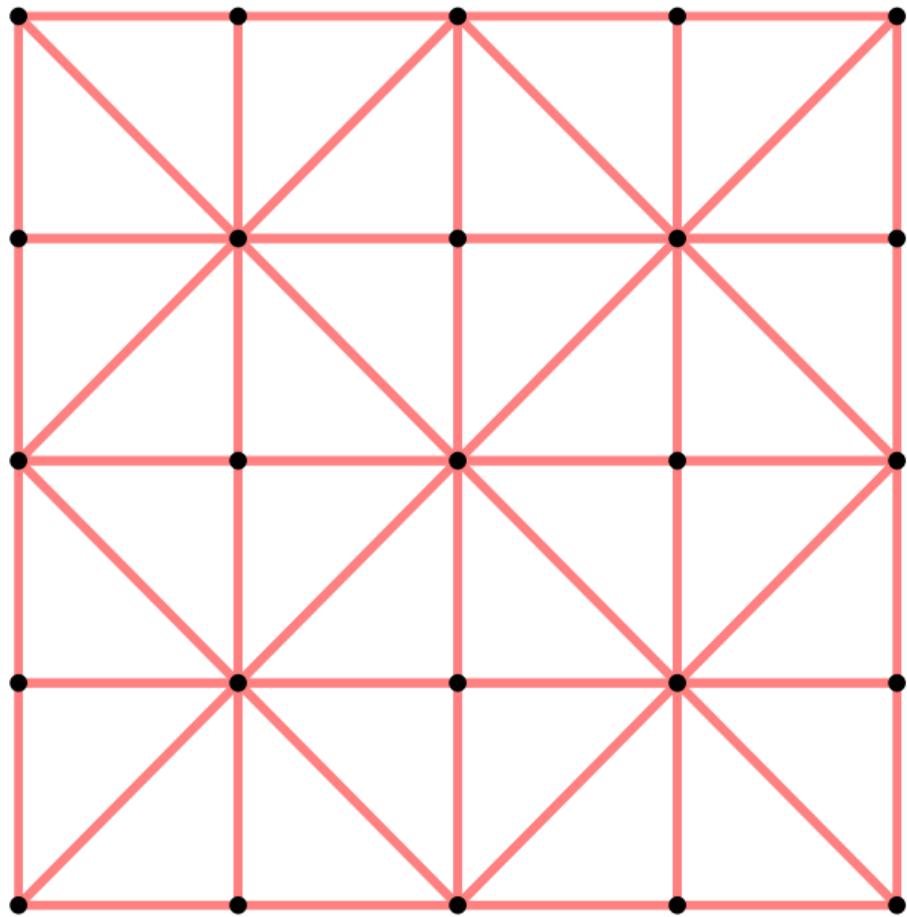
Malha 4-8 uniforme



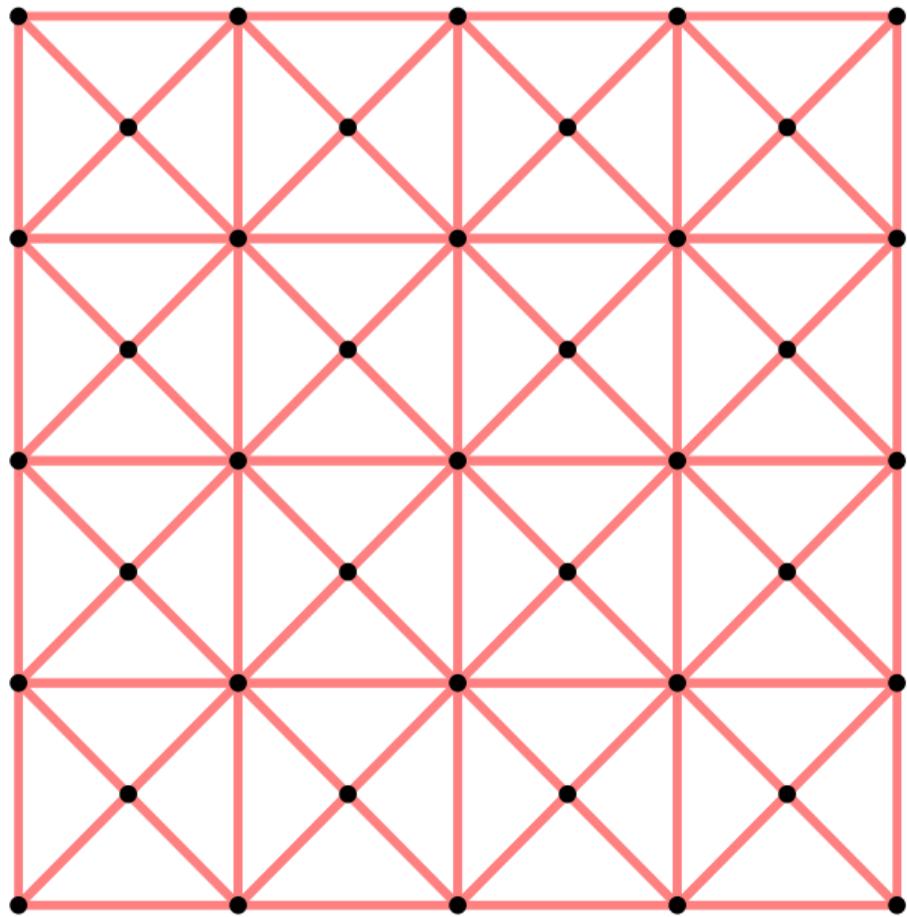
Malha 4-8 uniforme



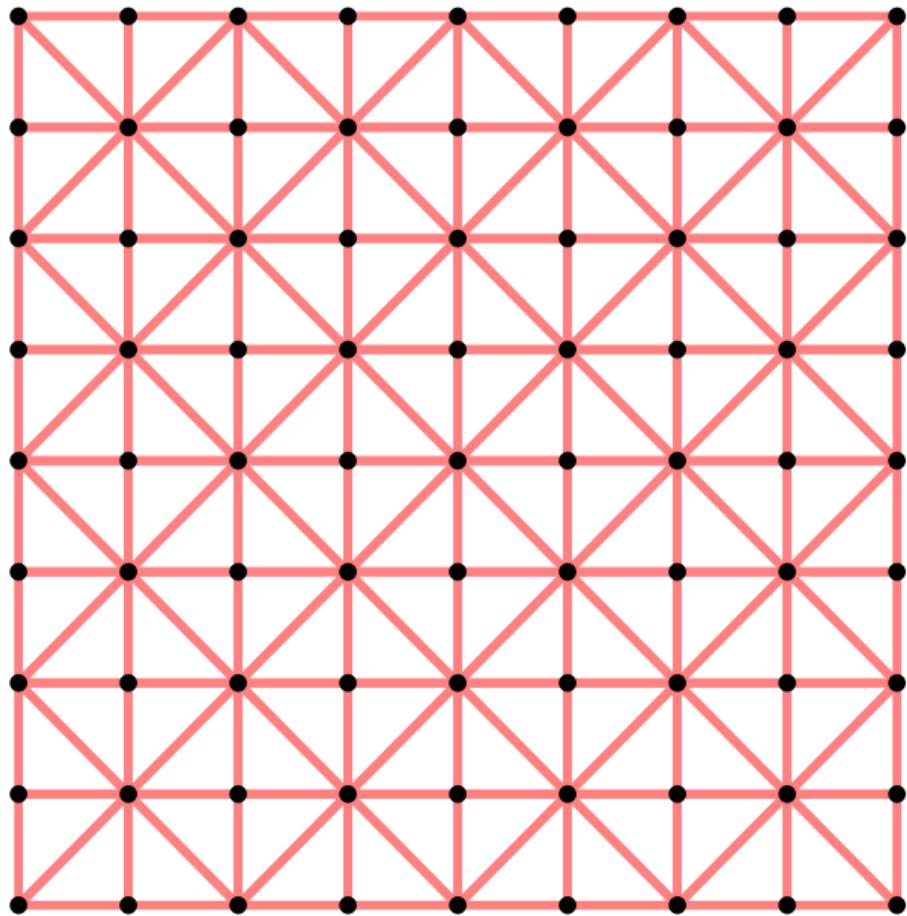
Malha 4-8 uniforme



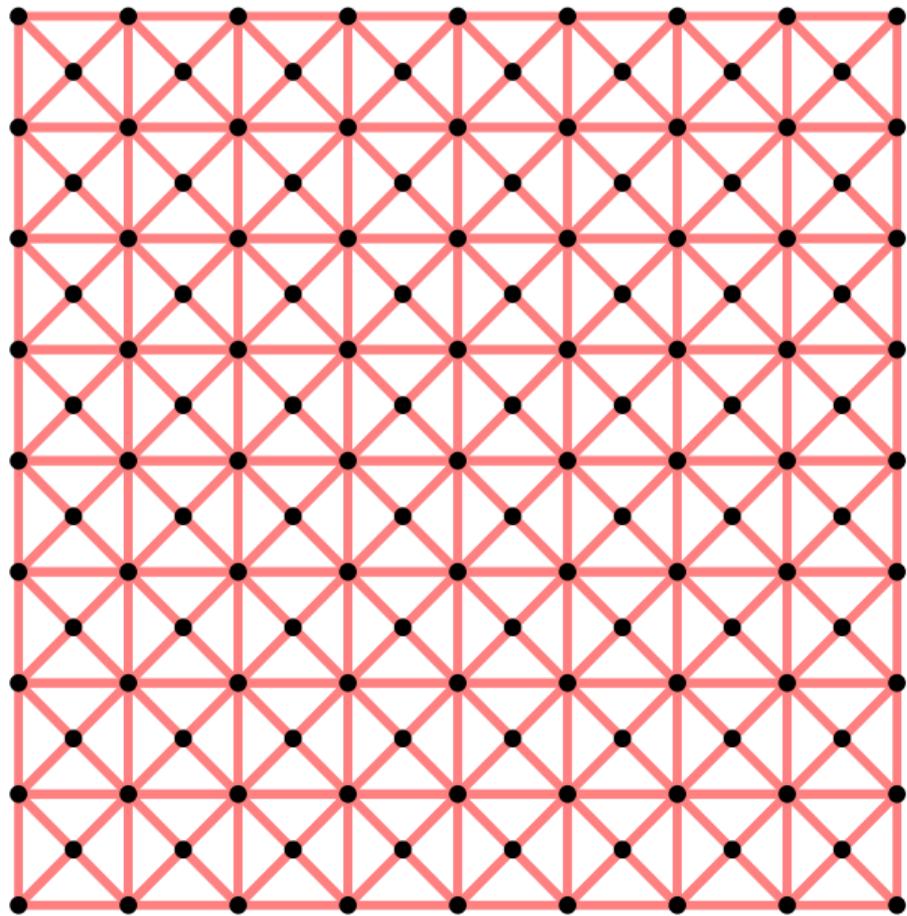
Malha 4-8 uniforme



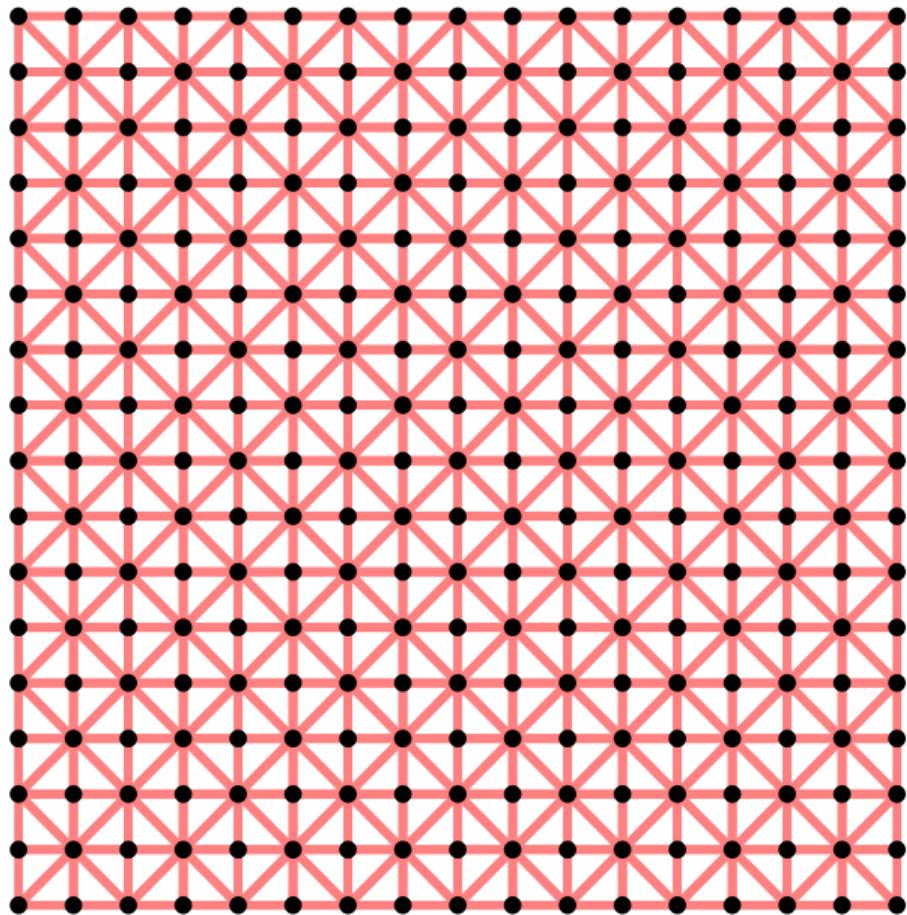
Malha 4-8 uniforme



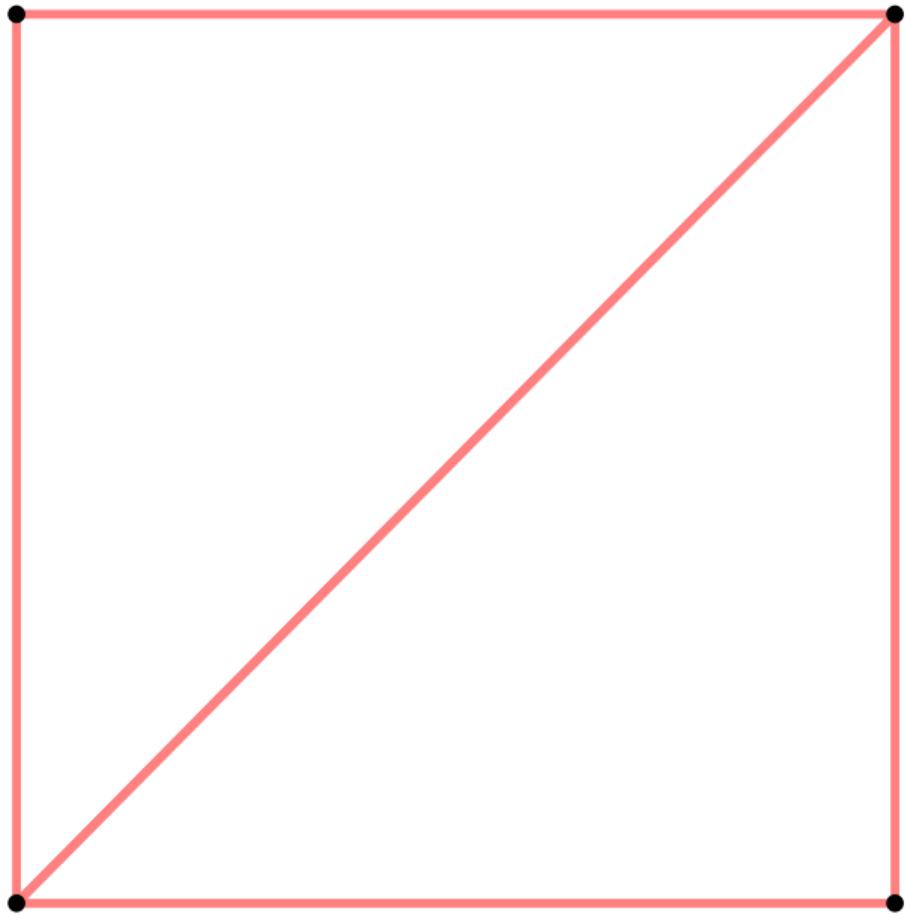
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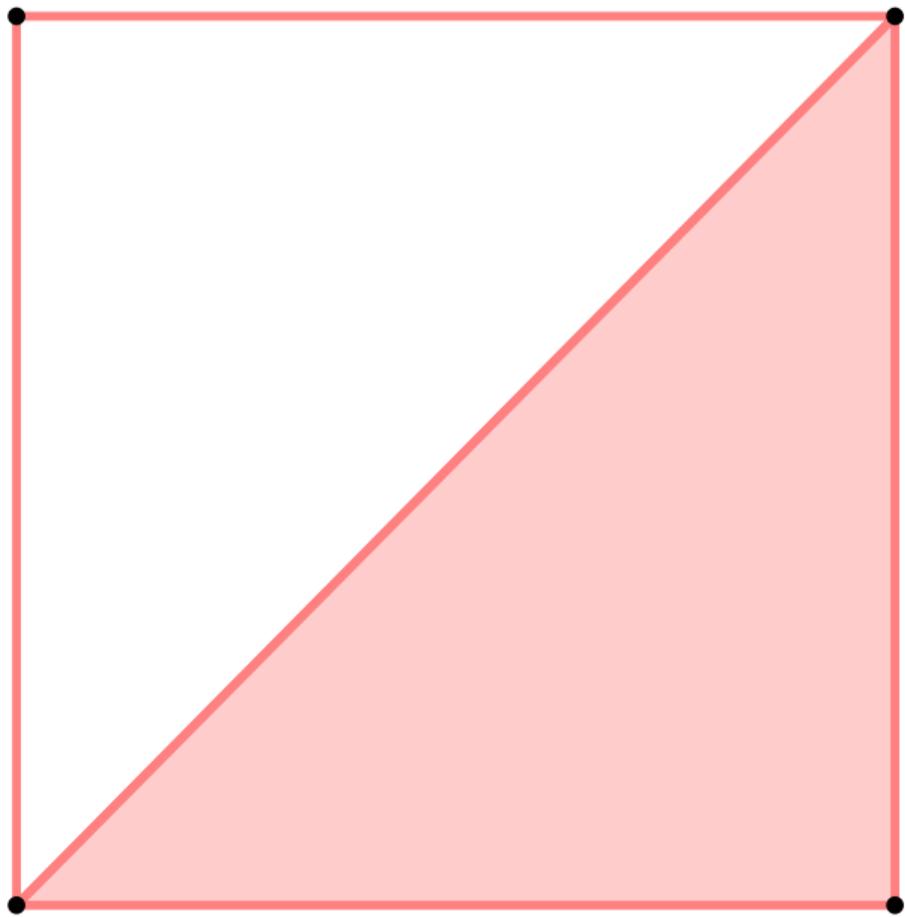
Malha 4-8 uniforme



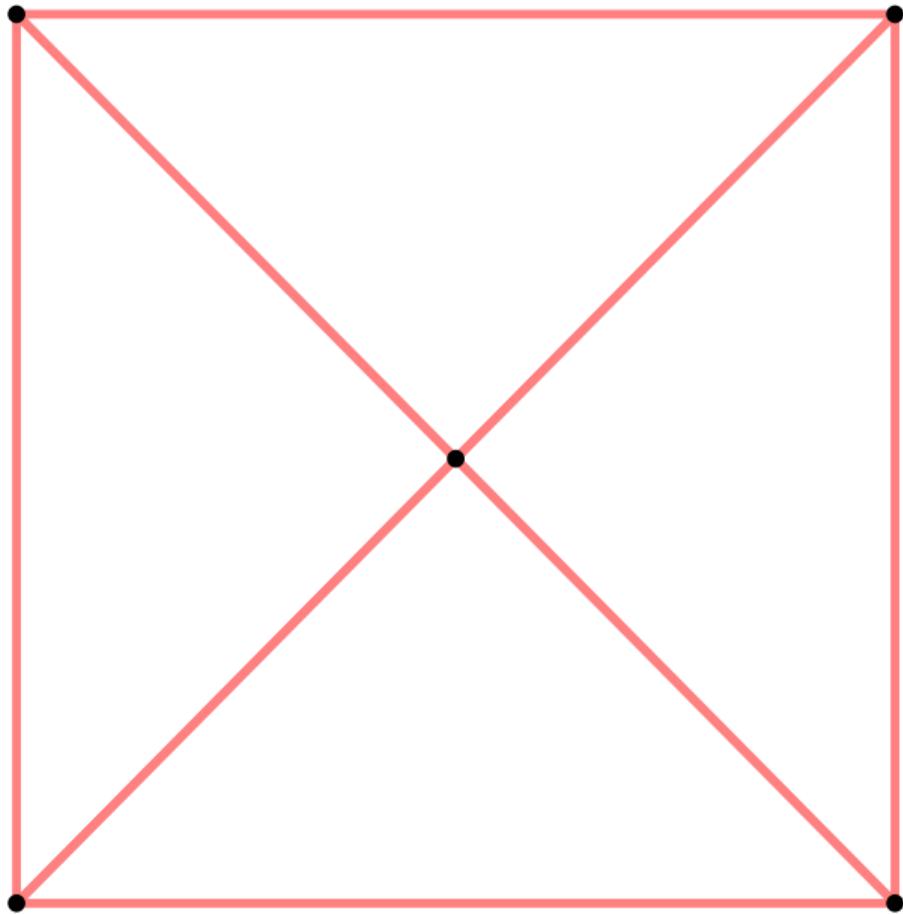
Malha 4-8 adaptativa



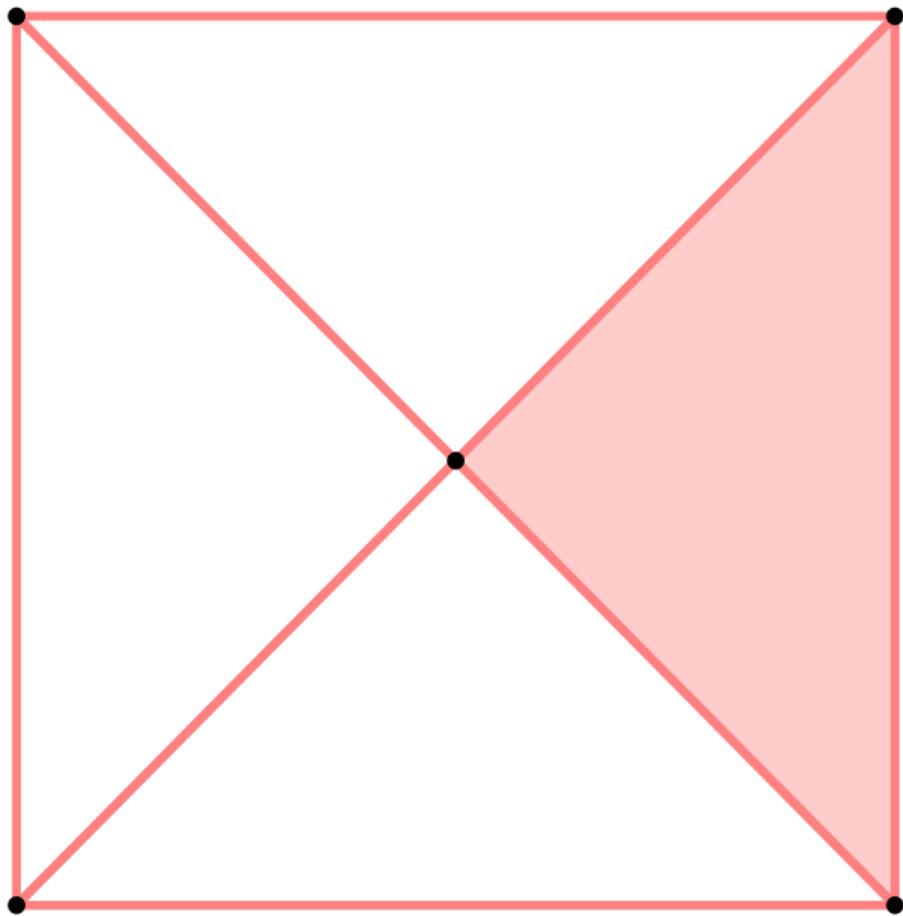
Malha 4-8 adaptativa



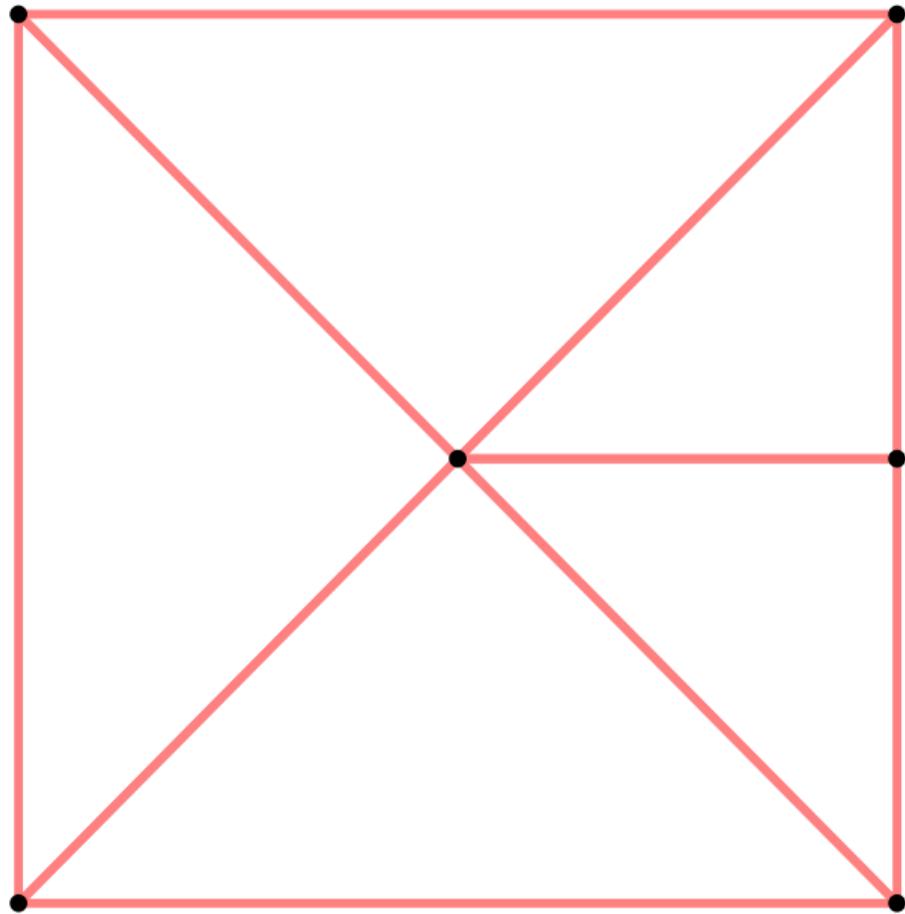
Malha 4-8 adaptativa



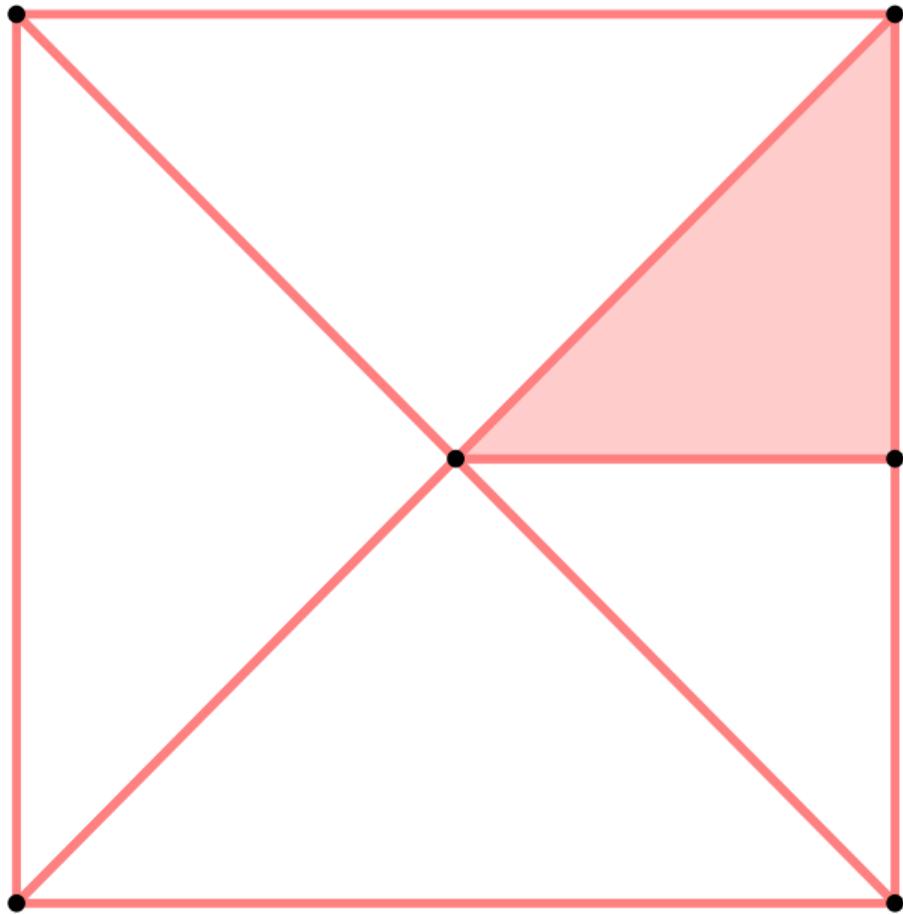
Malha 4-8 adaptativa



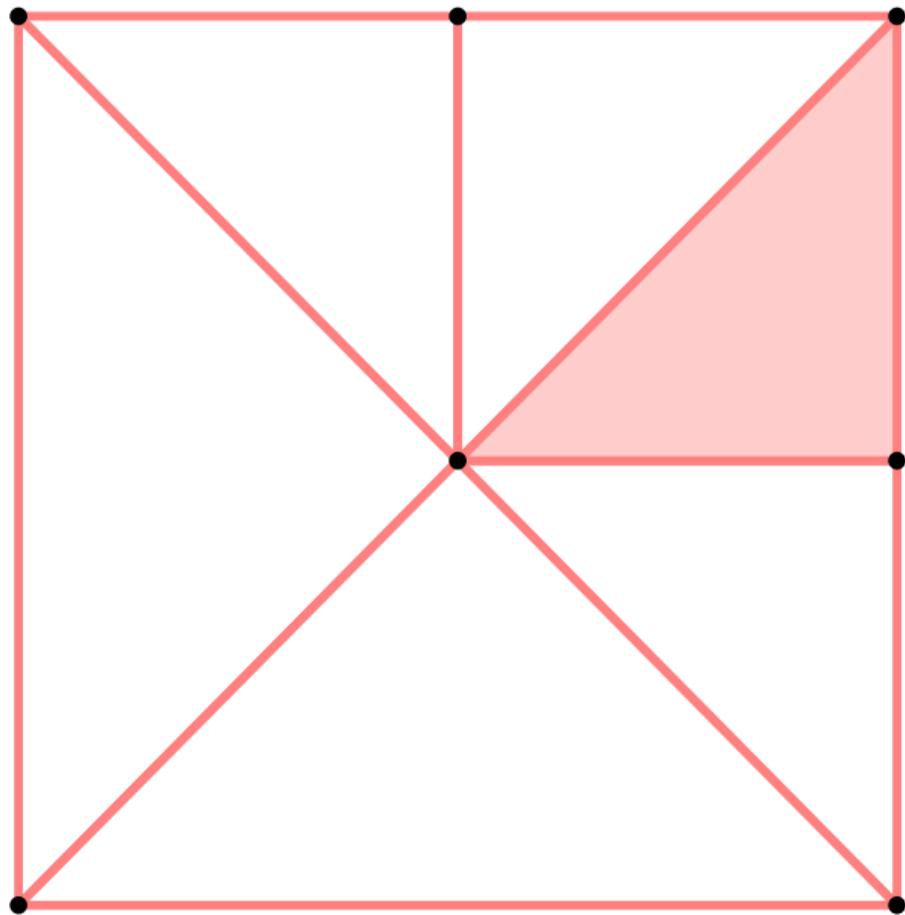
Malha 4-8 adaptativa



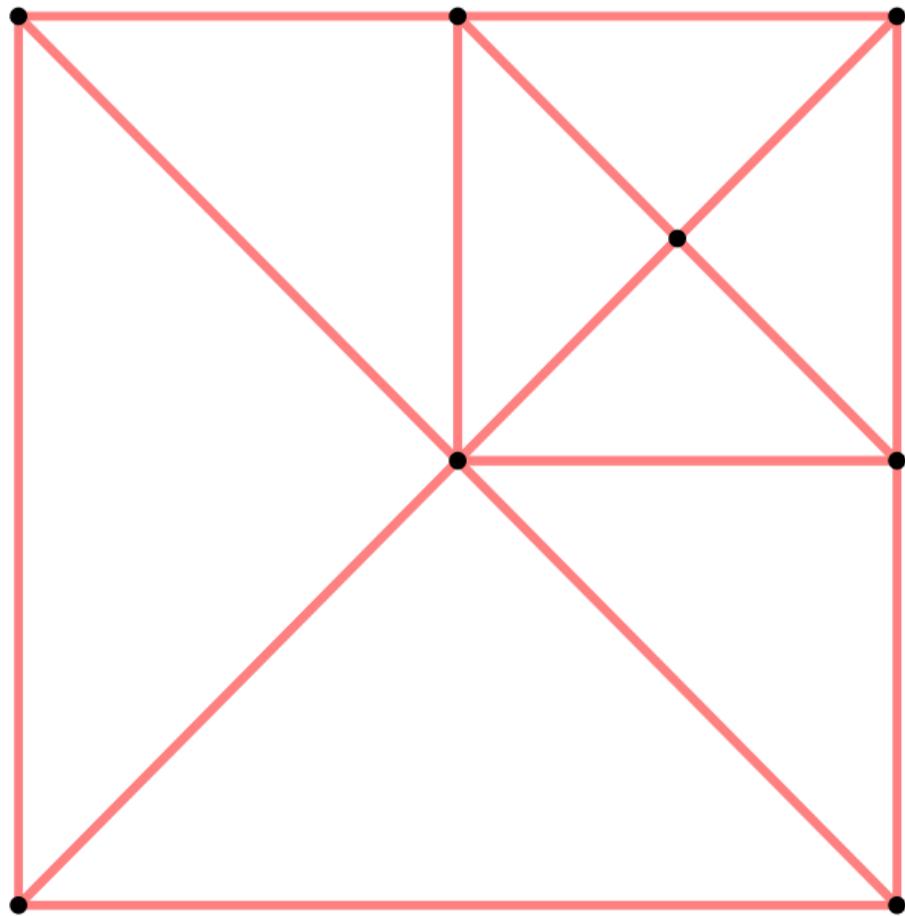
Malha 4-8 adaptativa



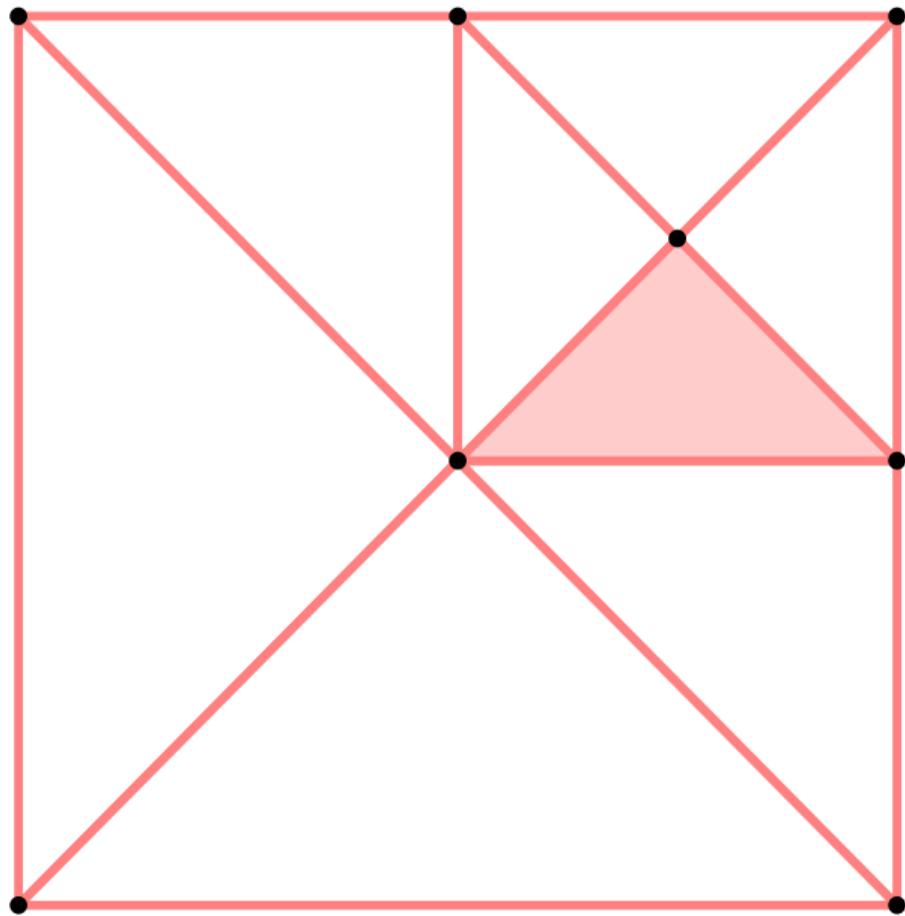
Malha 4-8 adaptativa



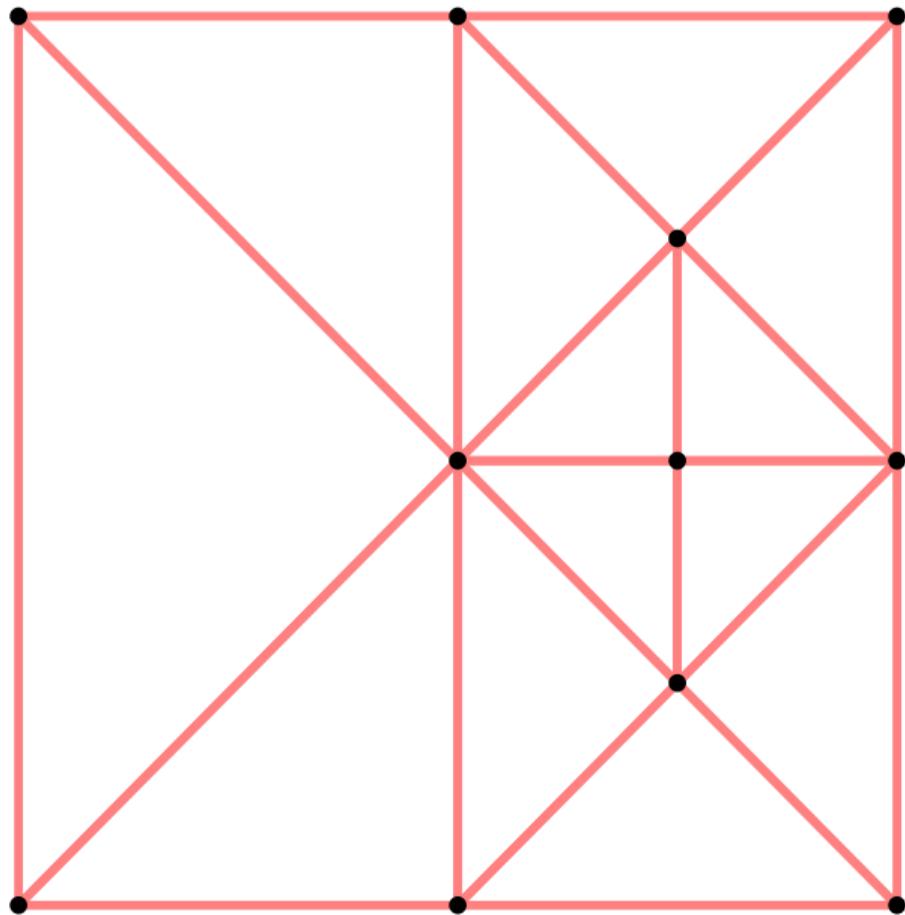
Malha 4-8 adaptativa



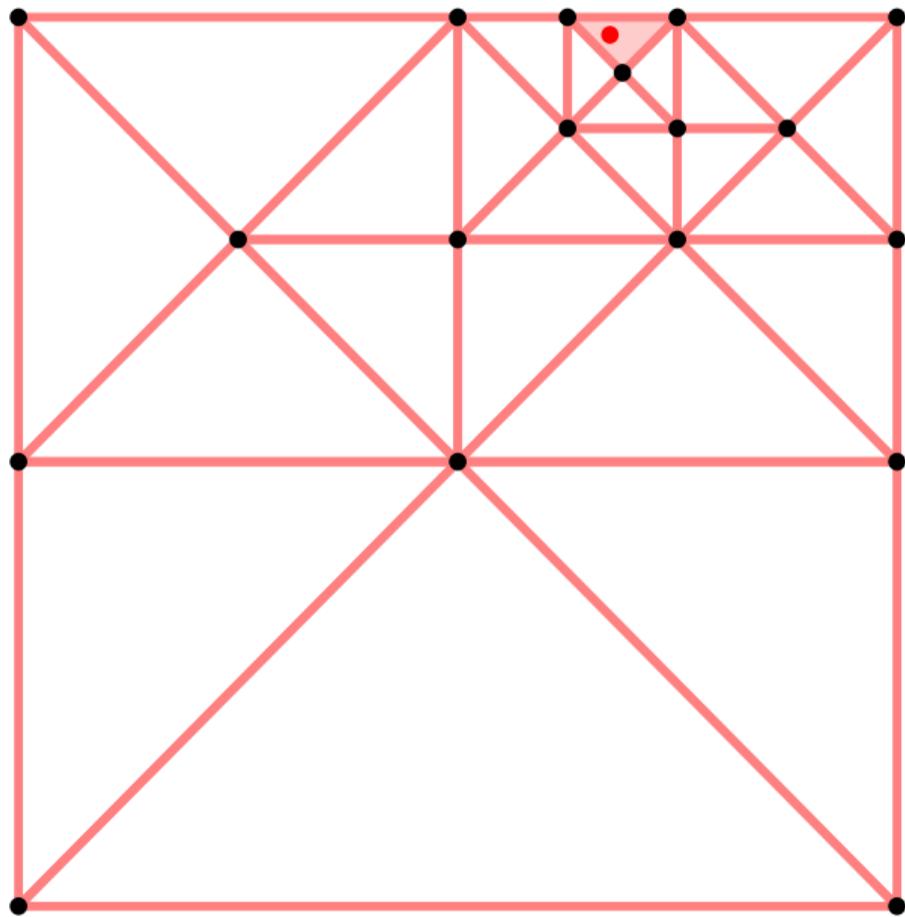
Malha 4-8 adaptativa



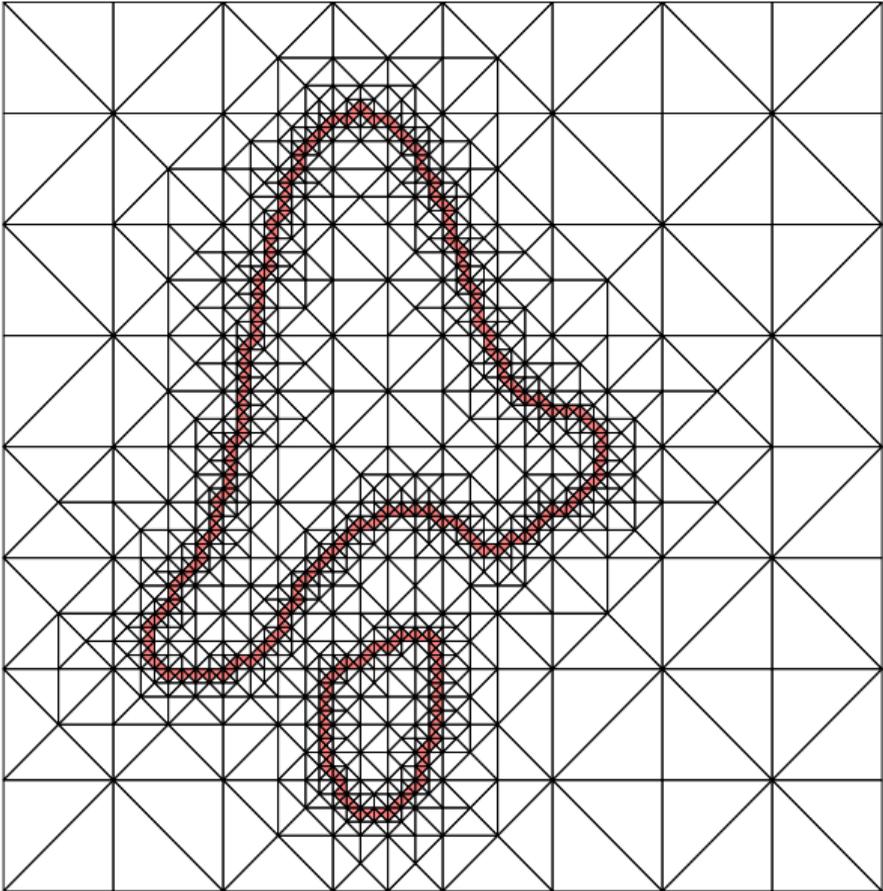
Malha 4-8 adaptativa



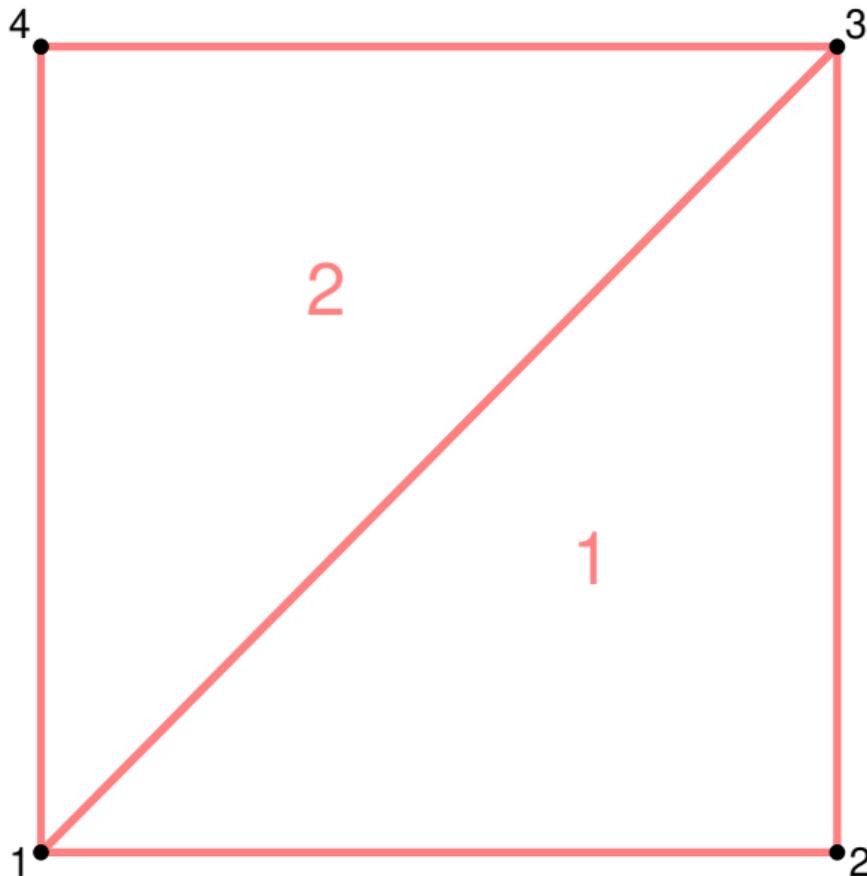
# Malha 4-8 adaptativa



# Malha 4-8 adaptativa – curva implícita



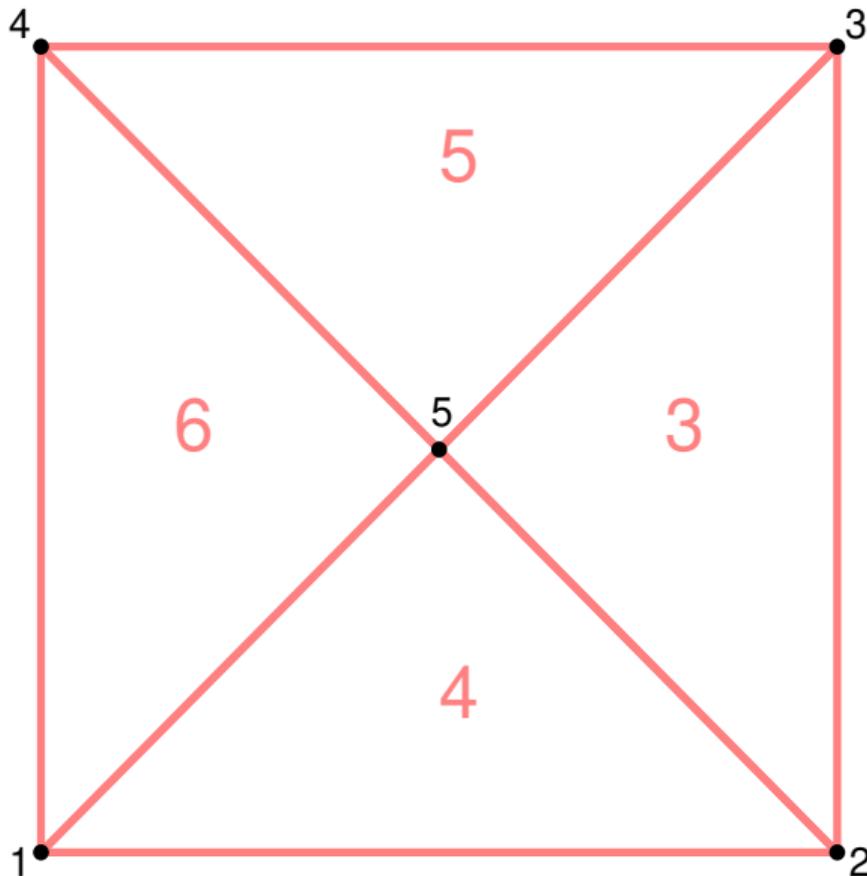
## Malha 4-8 – representação



v	1	0	0	0
v	2	1	0	0
v	3	1	1	0
v	4	0	1	0

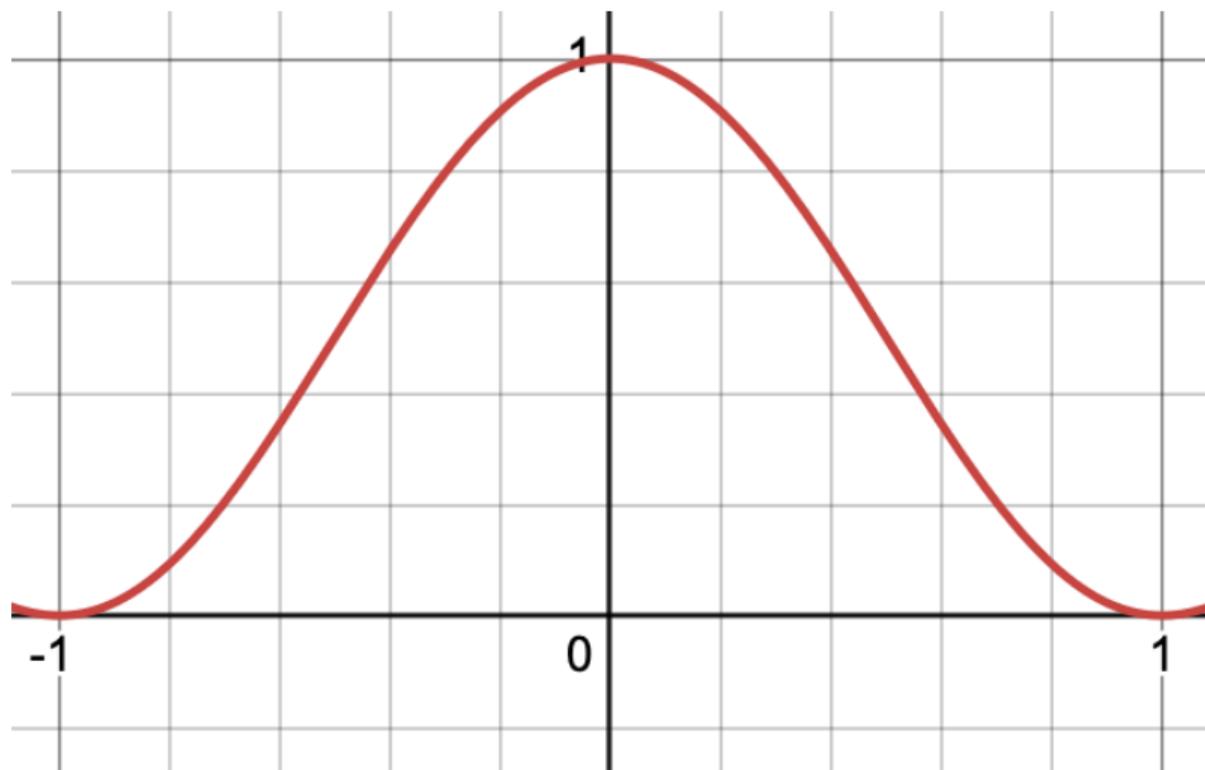
f	1	2	3	1
		2	-	-
f	2	4	1	3
		1	-	-

# Malha 4-8 – representação



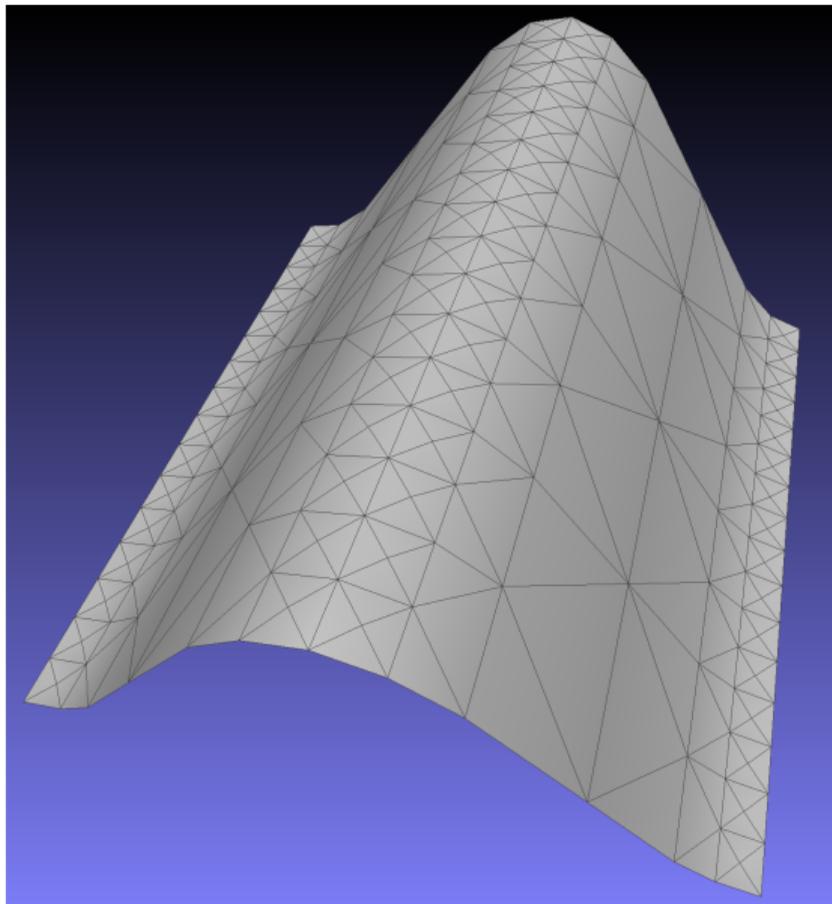
v	1	0	0	0	
v	2	1	0	0	
v	3	1	1	0	
v	4	0	1	0	
v	5	0.5	0.5	0	
f	1	2	3	1	×
		2	-	-	
f	2	4	1	3	×
		1	-	-	
f	3	5	2	3	
		-	5	4	
f	4	5	1	2	
		-	3	6	
f	5	5	3	4	
		-	6	3	
f	6	5	4	1	
		-	4	5	

## Malha 4-8 adaptativa – superfície paramétrica

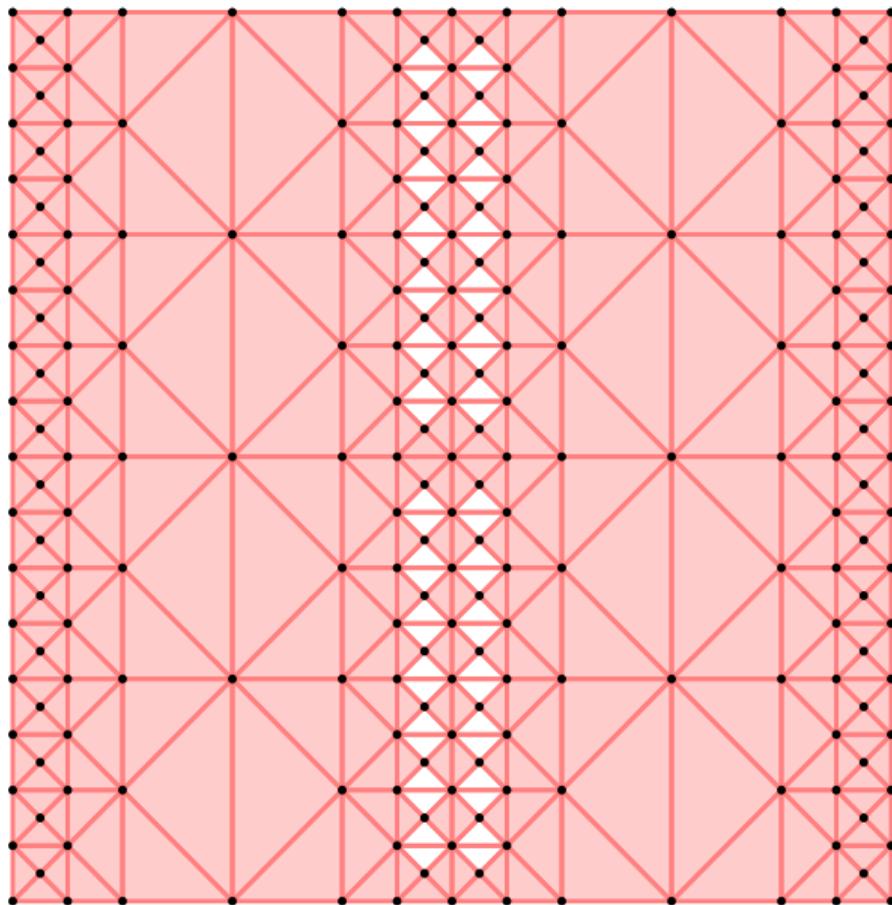


$$f(u, v) = (u, v, g(u)) \quad g(u) = -\frac{4}{9}x^6 + \frac{17}{9}x^4 - \frac{22}{9}x^2 + 1$$

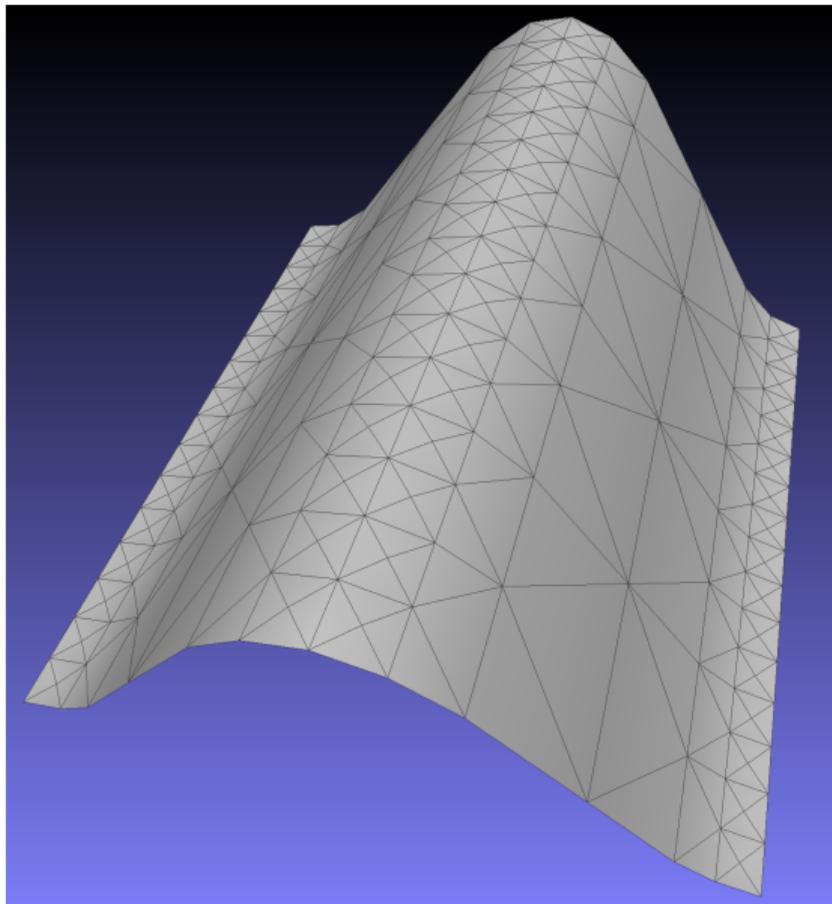
## Malha 4-8 adaptativa – superfície paramétrica



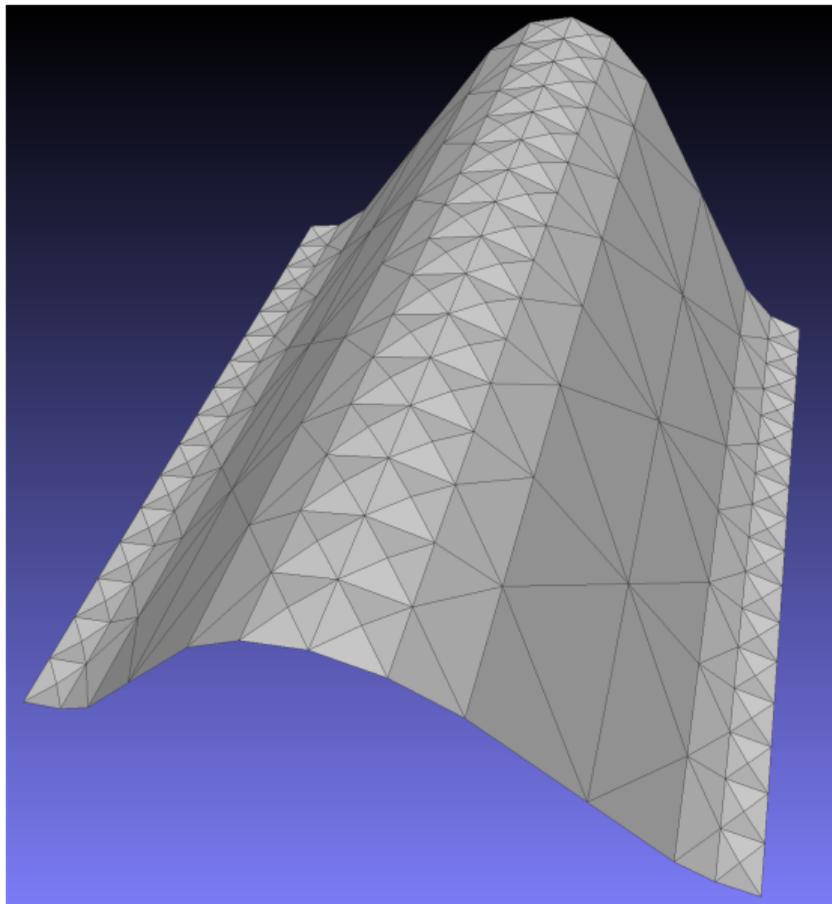
## Malha 4-8 adaptativa – superfície paramétrica



## Malha 4-8 adaptativa – superfície paramétrica

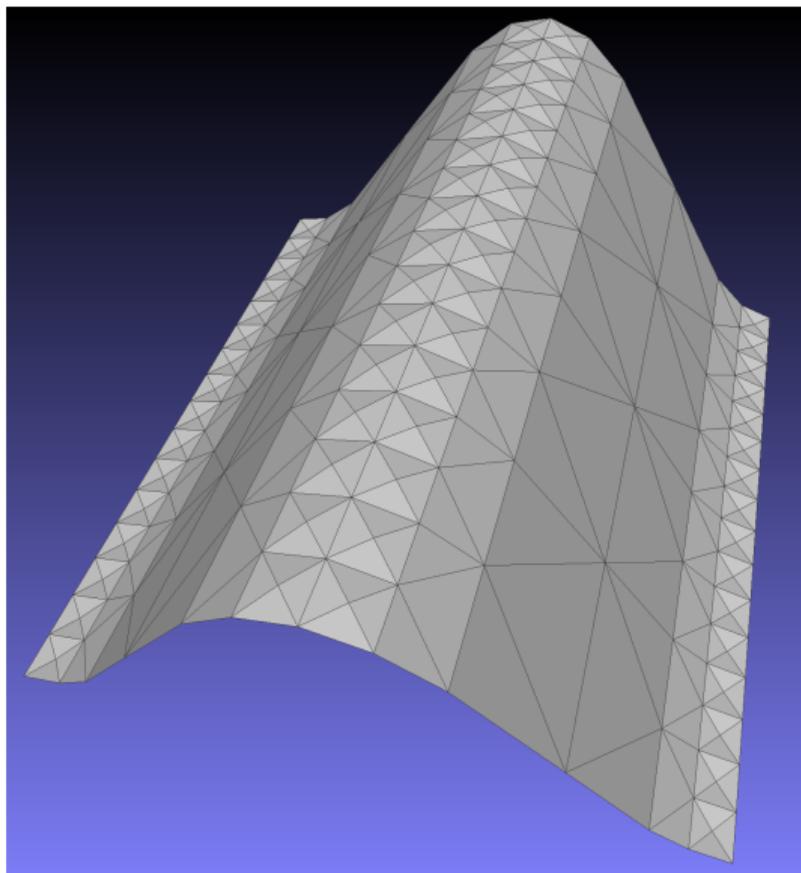


## Malha 4-8 adaptativa – superfície paramétrica



# Superfície paramétrica – representação

OBJ



geometria

v -0.25 -0.75 0.8544921875

v -0.5 -0.5 0.5

v -0.5 -1.0 0.5

v -0.25 -0.25 0.8544921875

v -0.5 0.0 0.5

v 0.25 -0.75 0.8544921875

v 0.5 -1.0 0.5

v 0.5 -0.5 0.5

v 0.25 -0.25 0.8544921875

v 0.5 0.0 0.5

v -0.75 -0.25 0.1435546875

v -0.75 -0.75 0.1435546875

v 0.75 -0.75 0.1435546875

v 0.75 -0.25 0.1435546875

v 0.25 0.75 0.8544921875

...

topologia

f 1 2 3

f 4 5 2

f 6 7 8

f 9 8 10

f 11 2 5

f 12 3 2

f 13 8 7

f 14 10 8

f 15 16 17

f 18 10 16

f 19 20 21

f 22 21 5

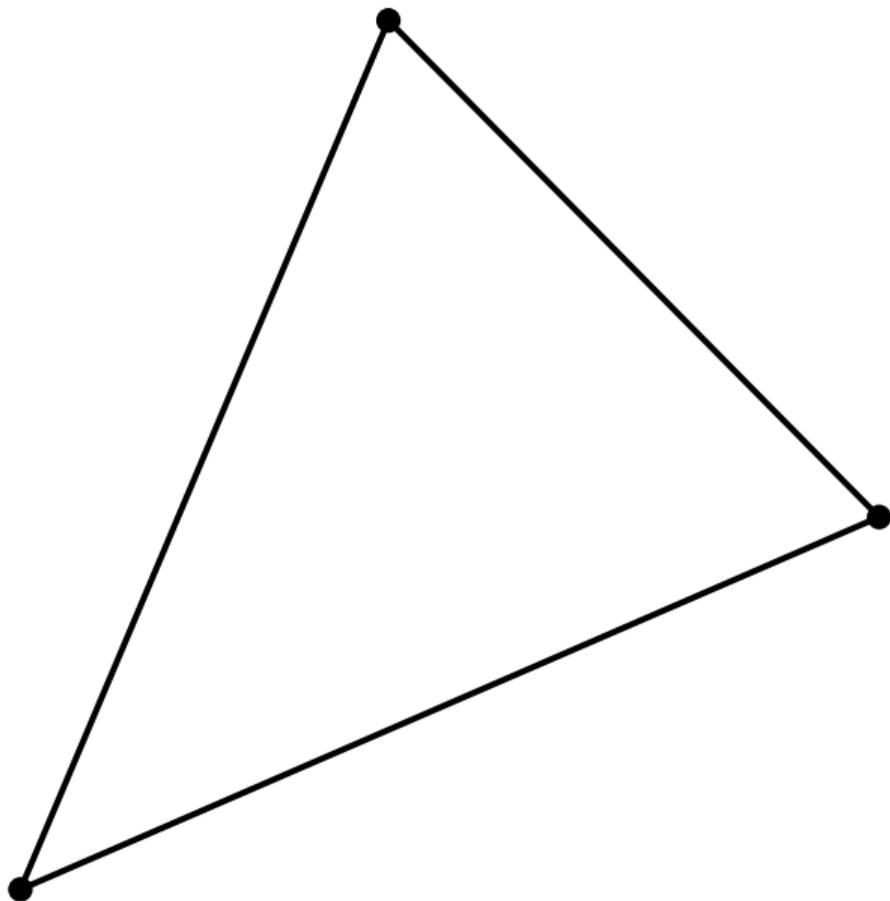
f 23 16 10

f 24 17 16

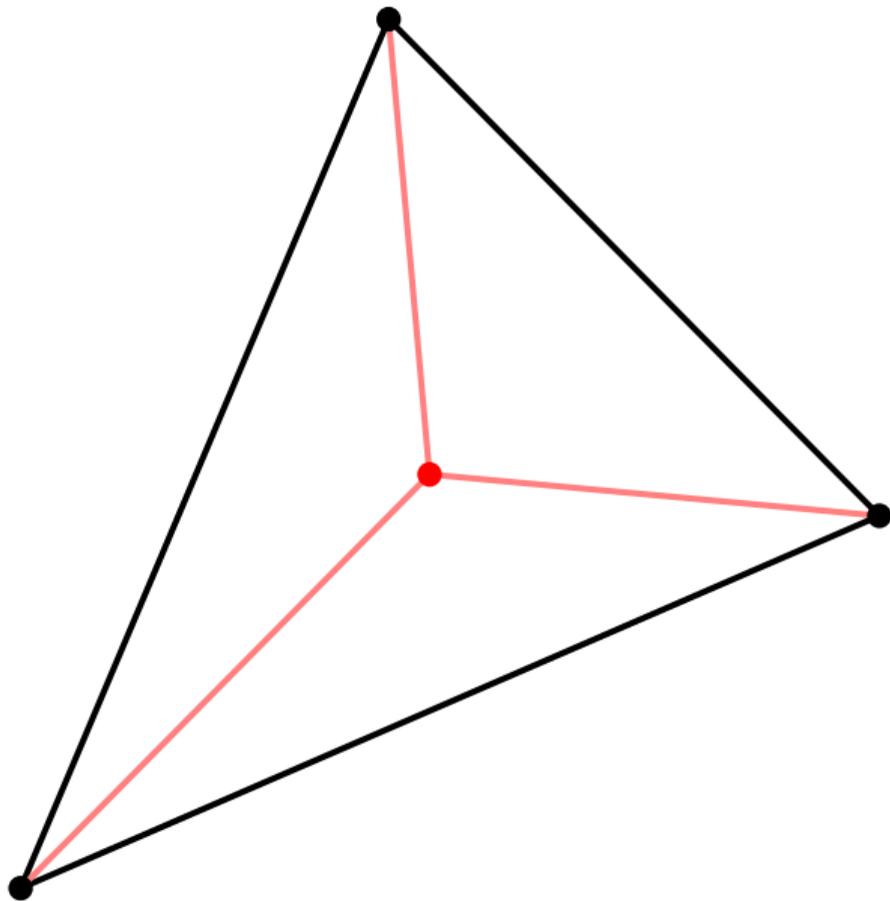
f 25 21 20

...

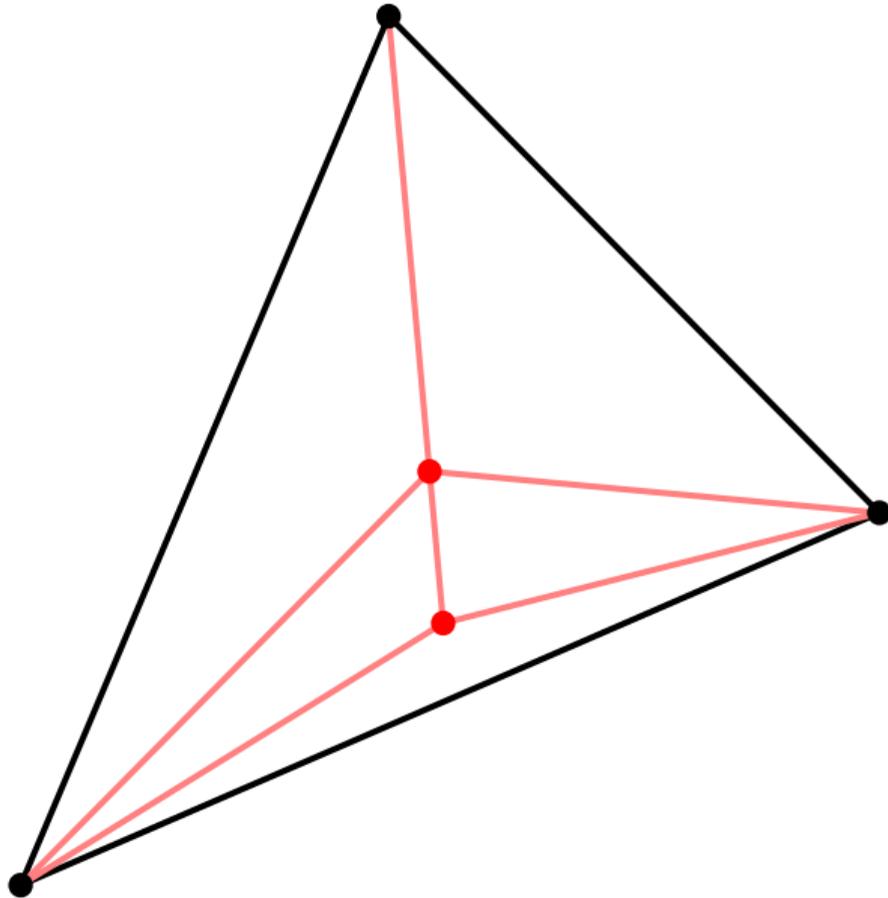
## Subdivisão de triângulos



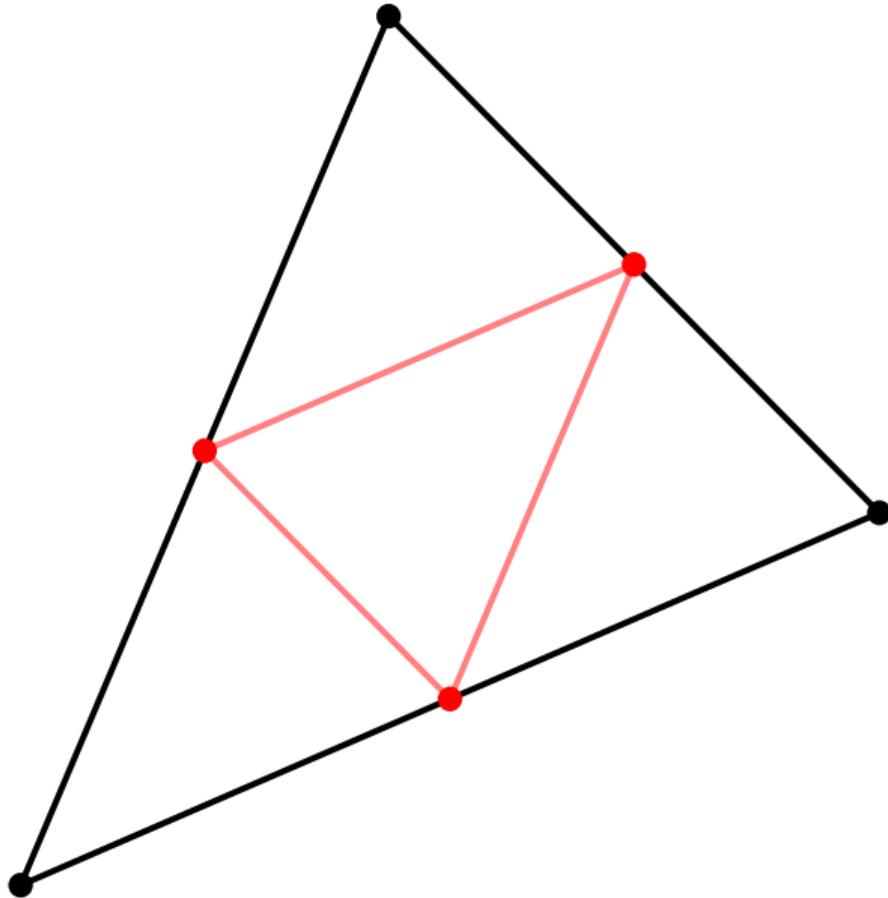
## Subdivisão de triângulos



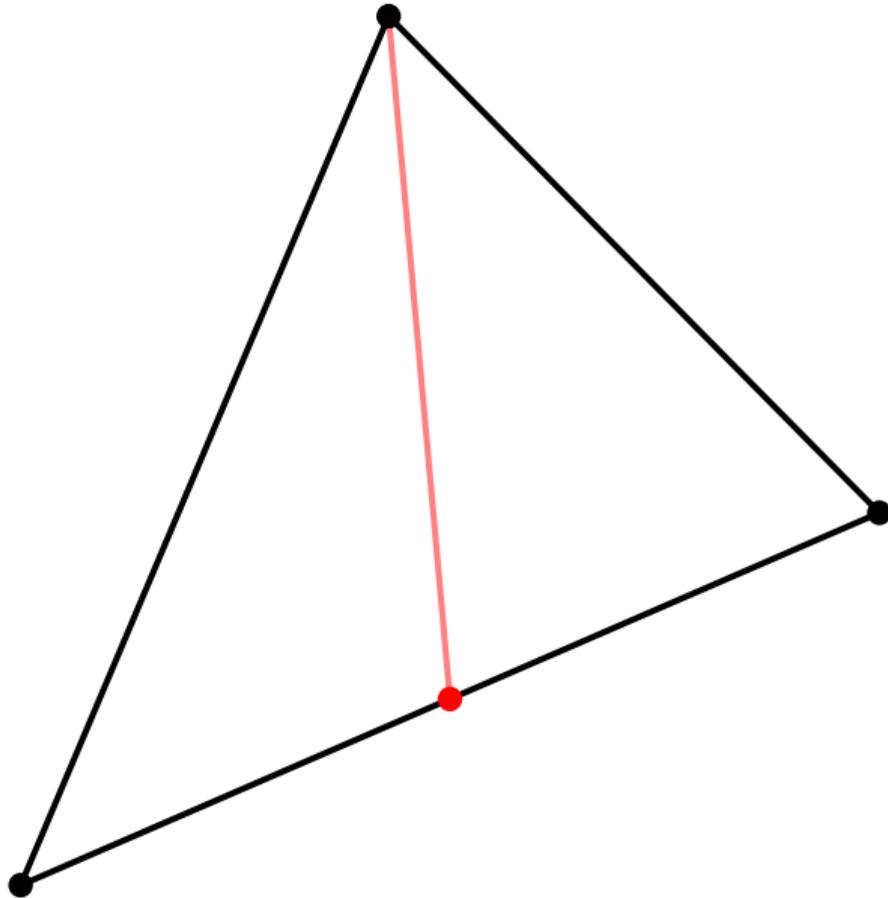
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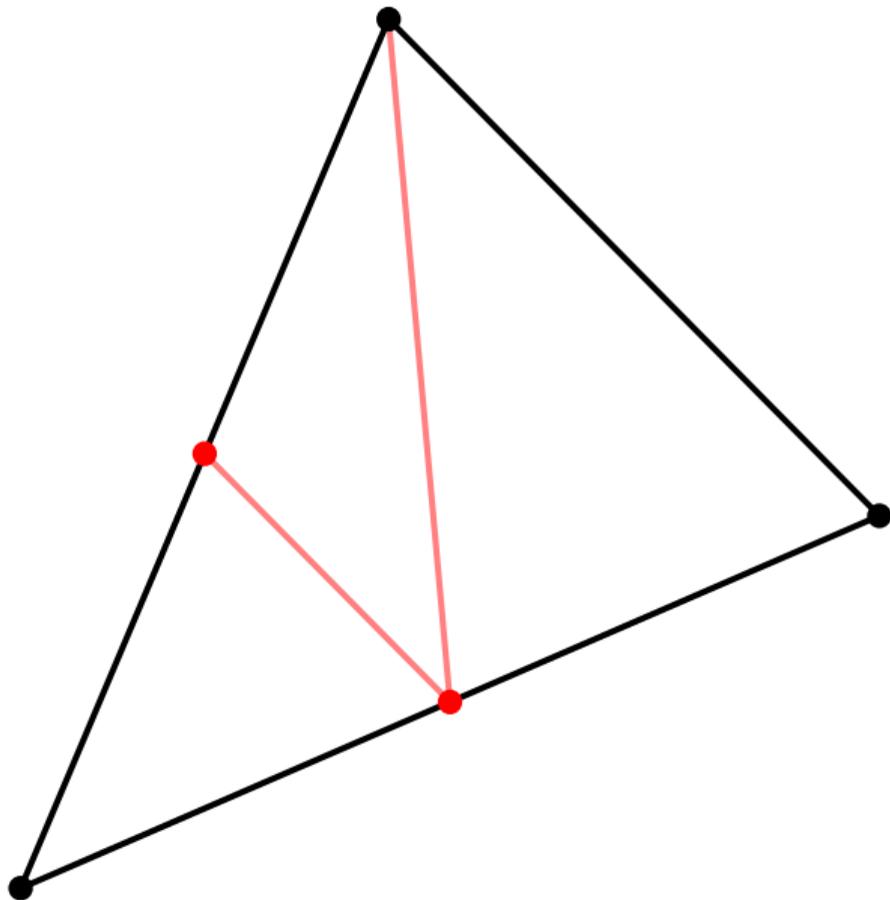
## Subdivisão de triângulos



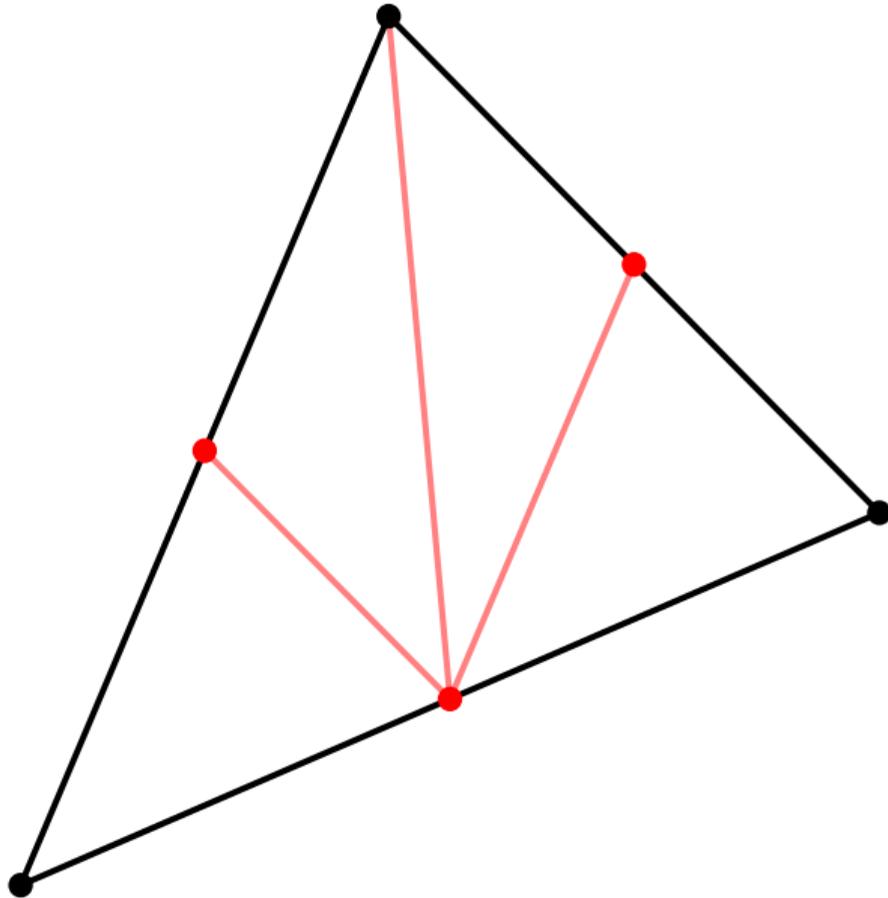
## Subdivisão de triângulos



## Subdivisão de triângulos



## Subdivisão de triângulos



curvas e superfícies de forma livre

## Requisitos para representação de formas

- pontos de controle
- intrínseca
- versatilidade
- controle de continuidade
- controle local
- redução de variação



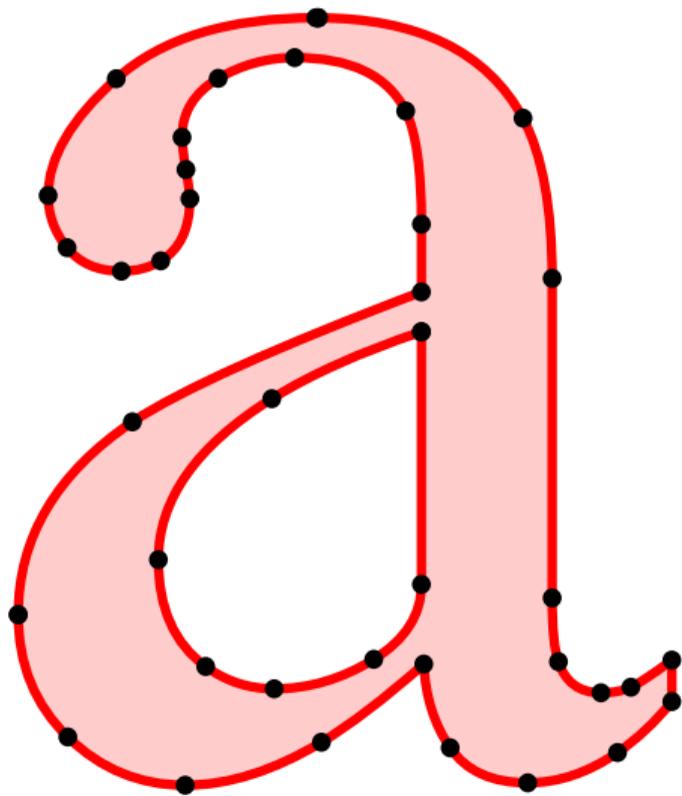
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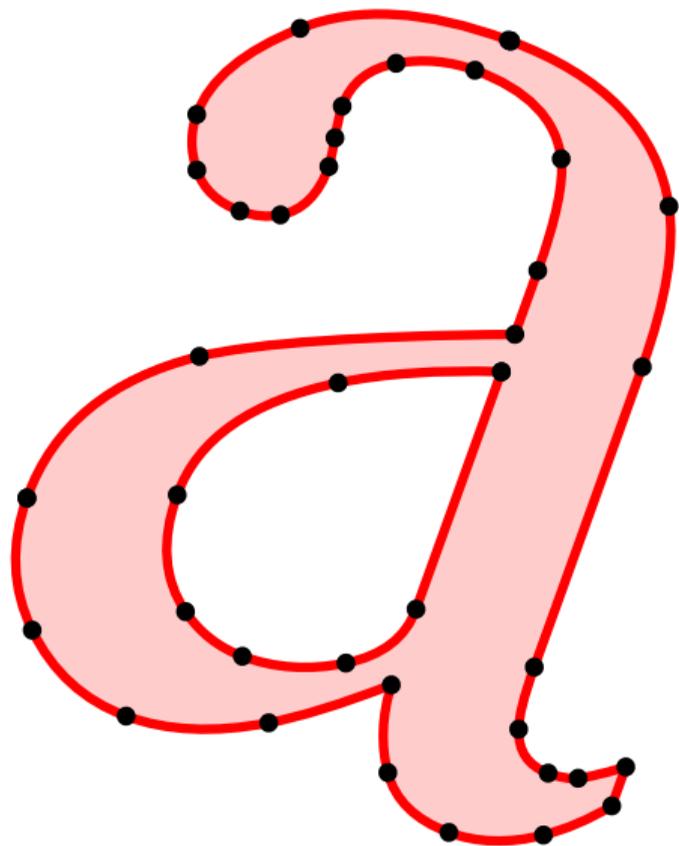
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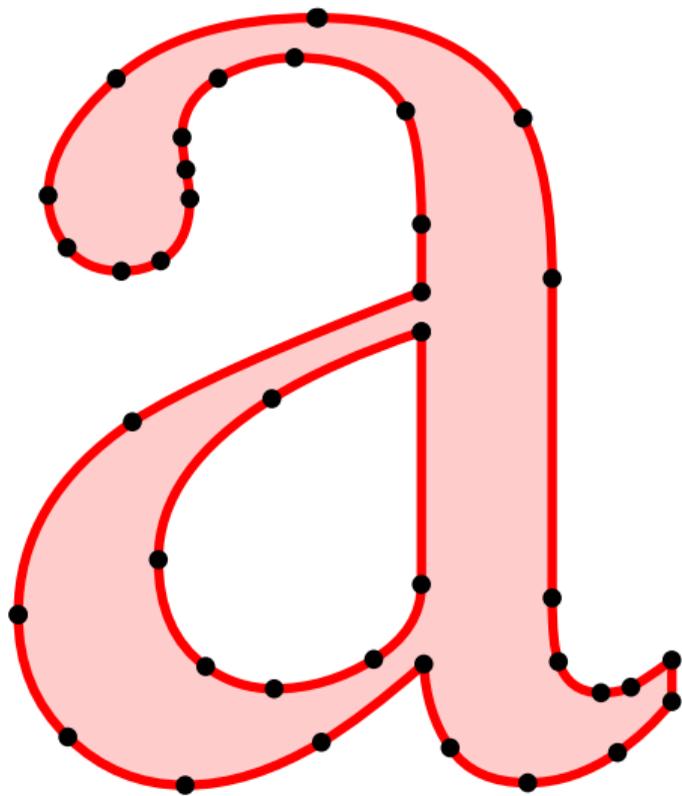
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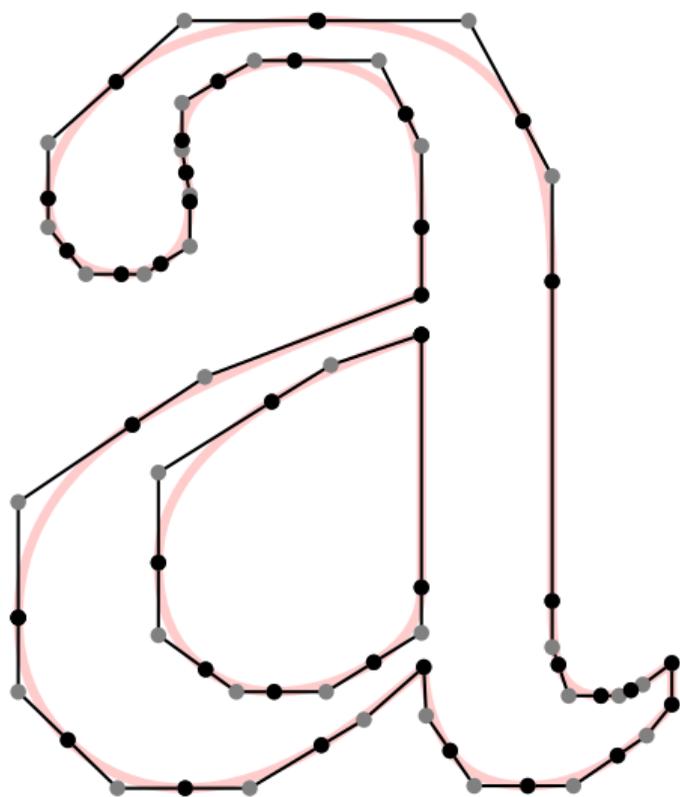
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## Requisitos para representação de formas

- pontos de controle
- intrínseca
- versatilidade
- controle de continuidade
- controle local
- redução de variação



## Requisitos para representação de formas

■ pontos de controle

M 146.75 138.5

Q 118.0 129.0 99.25 117.5

Q 63.25 95.25 63.25 67.0

■ intrínseca

Q 63.25 44.25 78.25 33.5

Q 88.0 26.5 100.0 26.5

■ versatilidade

Q 116.5 26.5 131.625 35.75

Q 146.75 45.0 146.75 59.25

■ controle de continuidade

L 146.75 138.5

M 18.75 49.75

Q 18.75 86.0 55.0 110.25

■ controle local

Q 78.0 125.25 146.75 151.0

L 146.75 172.25

■ redução de variação

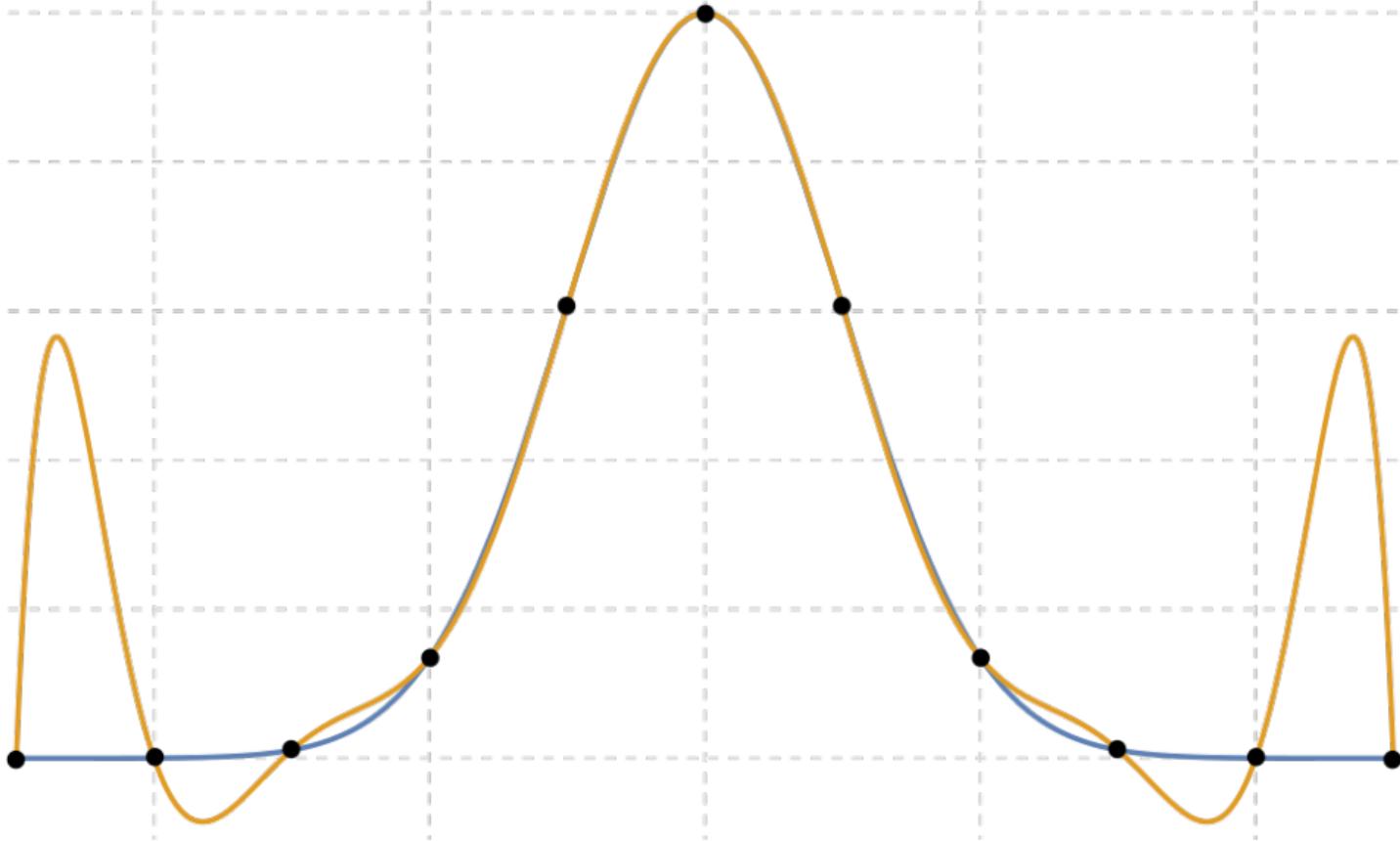
Q 146.75 197.75 141.75 207.75

Q 133.25 224.5 106.5 224.5

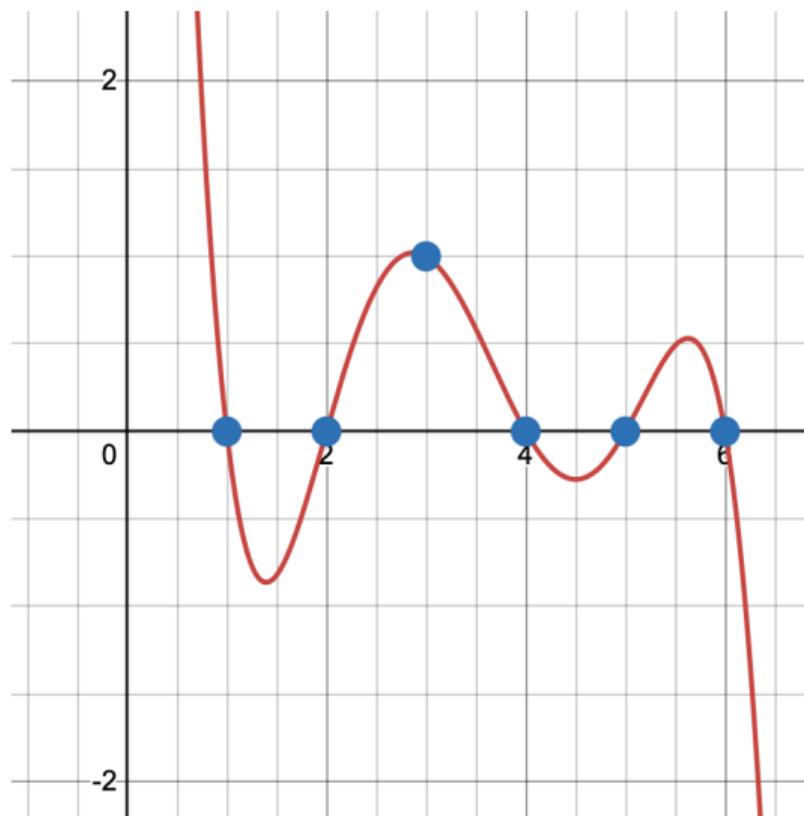
Q 93.75 224.5 82.25 218.0

...

# Aumento de variação



## Aumento de variação



## Objetos paramétricos

curvas paramétricas

$$\gamma: [a, b] \rightarrow \mathbf{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

$$\gamma: [a, b] \rightarrow \mathbf{R}^3$$

$$\gamma(t) = (x(t), y(t), z(t))$$

superficies paramétricas

$$\sigma: [a, b] \times [c, d] \rightarrow \mathbf{R}^3$$

$$\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$$

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$$\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$$

funções polinomiais e racionais

avaliação computacional exata e eficiente

## Funções polinomiais

avaliação ingênua

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_n t^n$$

$$v \leftarrow 0$$

$$v \leftarrow v + a_0$$

$$v \leftarrow v + a_1 t$$

$$v \leftarrow v + a_2 t^2$$

$$v \leftarrow v + a_3 t^3$$

...

$$v \leftarrow v + a_n t^n$$

$$v = f(t)$$

$$t^k = \text{pow}(t, k)$$

## Funções polinomiais

avaliação menos ingênua

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_n t^n$$

$v \leftarrow 0$	$s \leftarrow 1$
$v \leftarrow v + a_0 s$	$s \leftarrow s t$
$v \leftarrow v + a_1 s$	$s \leftarrow s t$
$v \leftarrow v + a_2 s$	$s \leftarrow s t$
$v \leftarrow v + a_3 s$	$s \leftarrow s t$
$\dots$	
$v \leftarrow v + a_n s$	$s \leftarrow s t$
$v = f(t)$	

$$t^{k+1} = t^k \cdot t, \quad t^0 = 1$$

## Funções polinomiais

avaliação eficiente

Horner

$$\begin{aligned}f(t) &= a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots + a_nt^n \\ &= a_0 + t(a_1 + t(a_2 + t(a_3 + \cdots + t(a_{n-1} + ta_n) \cdots)))\end{aligned}$$

$$v \leftarrow a_n$$

$$v \leftarrow a_{n-1} + tv$$

...

$$v \leftarrow a_3 + tv$$

$$v \leftarrow a_2 + tv$$

$$v \leftarrow a_1 + tv$$

$$v \leftarrow a_0 + tv$$

$$v = f(t)$$

fused multiply-add (FMA)

## Funções polinomiais

interpretação dos coeficientes

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_n t^n$$

expansão de Taylor em torno de  $t = 0$

$$a_k = \frac{f^{(k)}(0)}{k!}$$

informação pontual profunda determina forma global

não intuitivo

## Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots + a_nt^n$$

$$y(t) = b_0 + b_1t + b_2t^2 + b_3t^3 + \cdots + b_nt^n$$

$$\gamma(t) = P_0 + P_1t + P_2t^2 + P_3t^3 + \cdots + P_nt^n \quad P_k = (a_k, b_k)$$

## Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + \dots + a_nt^n$$

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$$\gamma(t) = P_0 + P_1t + P_2t^2 + P_3t^3 + \dots + P_nt^n \quad P_k = (a_k, b_k)$$

$$= P_0 + tP_1 + t^2P_2 + t^3P_3 + \dots + t^nP_n$$

combinação linear de **pontos de controle** com pesos  $1, t, t^2, t^3, \dots, t^n$

não intuitivo

## Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + \dots + a_nt^n$$

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combinação linear de **pontos de controle** com pesos  $1, t, t^2, t^3, \dots, t^n$

não intuitivo

como escolher pontos de controle e pesos intuitivos?

coeficientes geométricos

curvas de Hermite

## Continuidade geométrica

- $\alpha, \beta: [0, 1] \rightarrow \mathbf{R}^d$

- continuidade  $C^0$   
 $\alpha(1) = \beta(0)$

- continuidade  $C^1$   
 $\alpha'(1) = \beta'(0)$

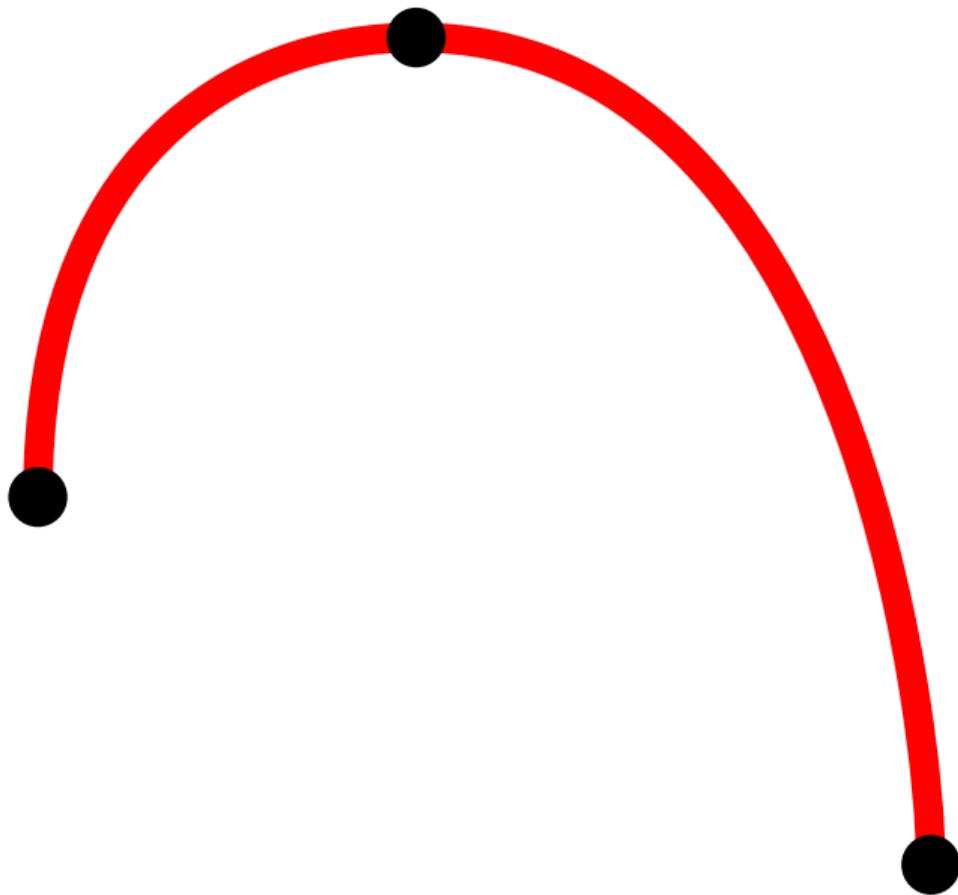
- continuidade  $G^1$   
 $\alpha'(1) \parallel \beta'(0)$

paramétrica

geométrica

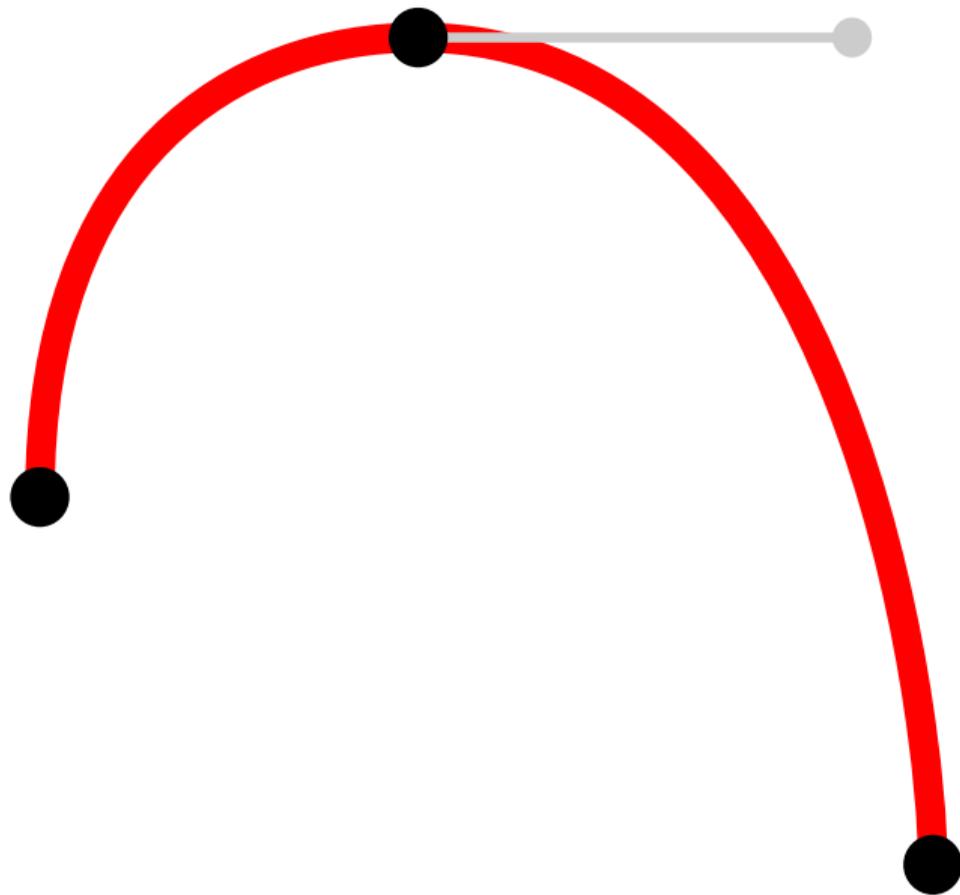
Continuidade  $C^0$

$$\alpha(1) = \beta(0)$$



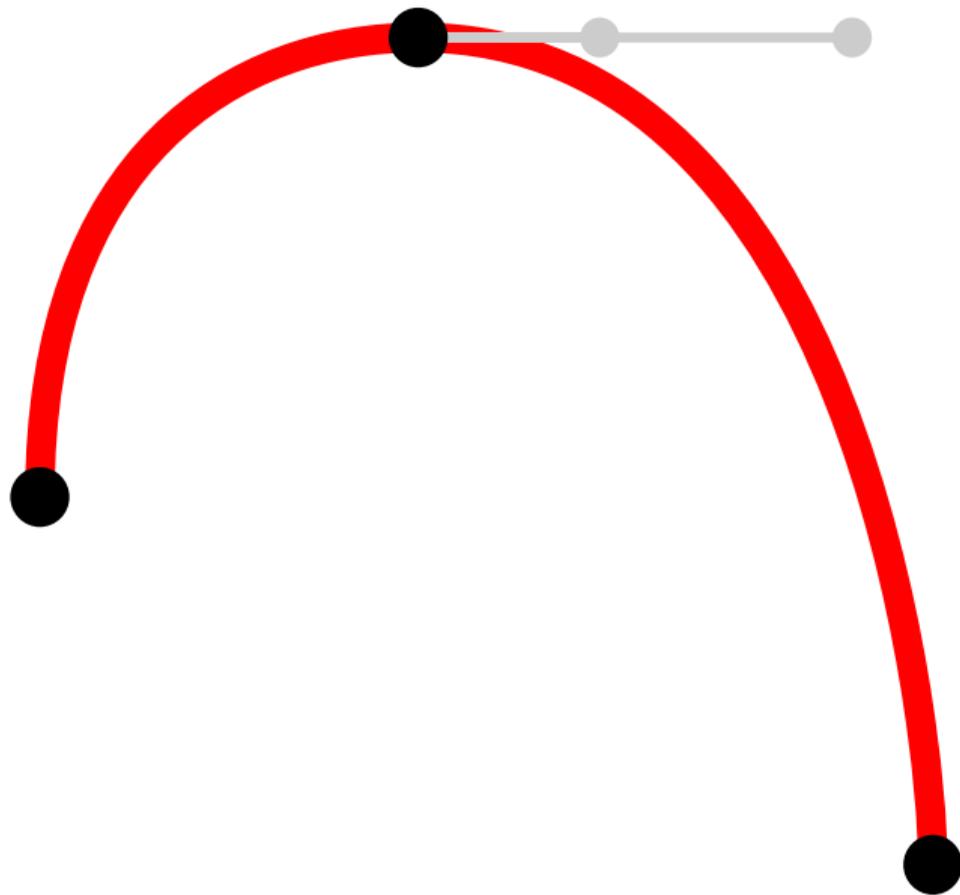
Continuidade paramétrica  $C^1$

$$\alpha'(1) = \beta'(0)$$



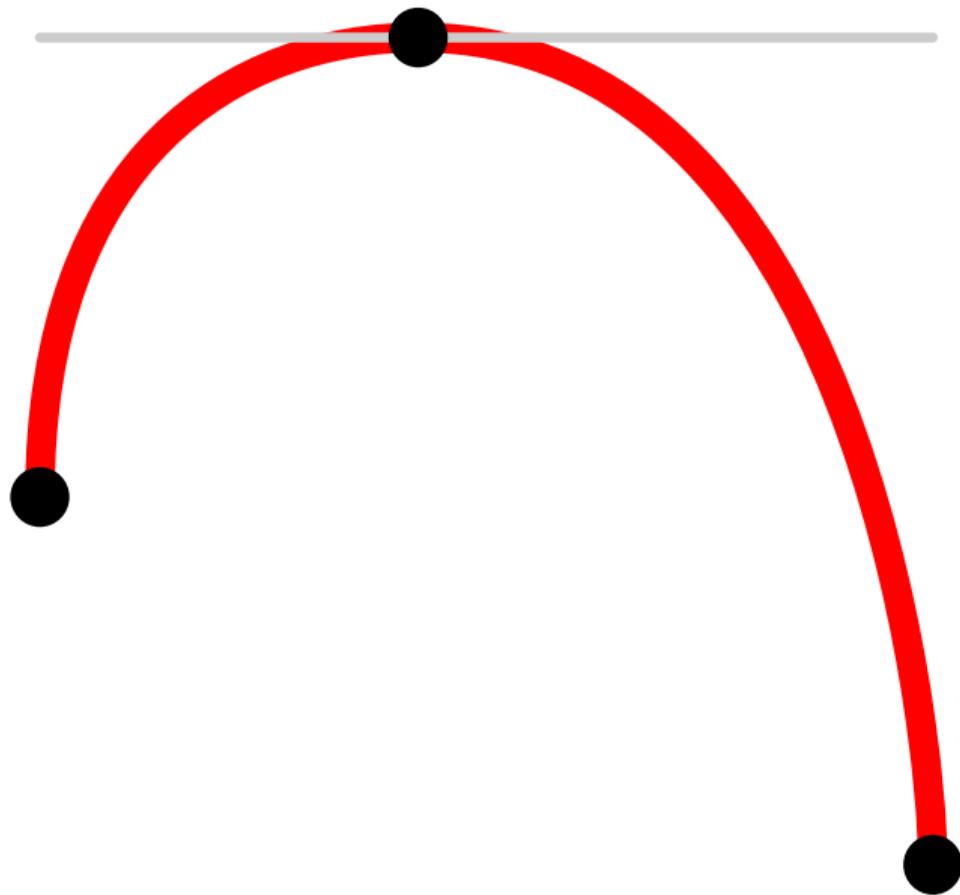
Continuidade geométrica  $G^1$

$\alpha'(1) \parallel \beta'(0)$

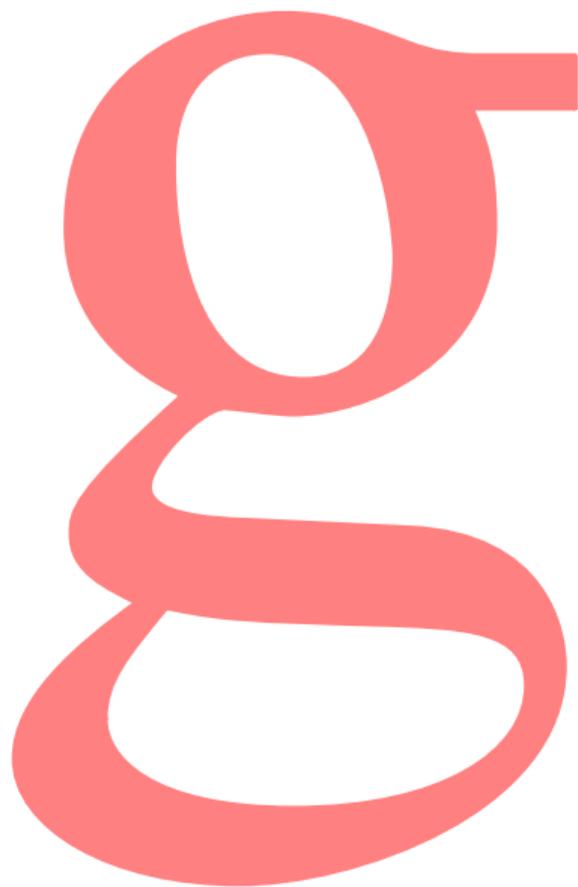


Continuidade geométrica  $G^1$

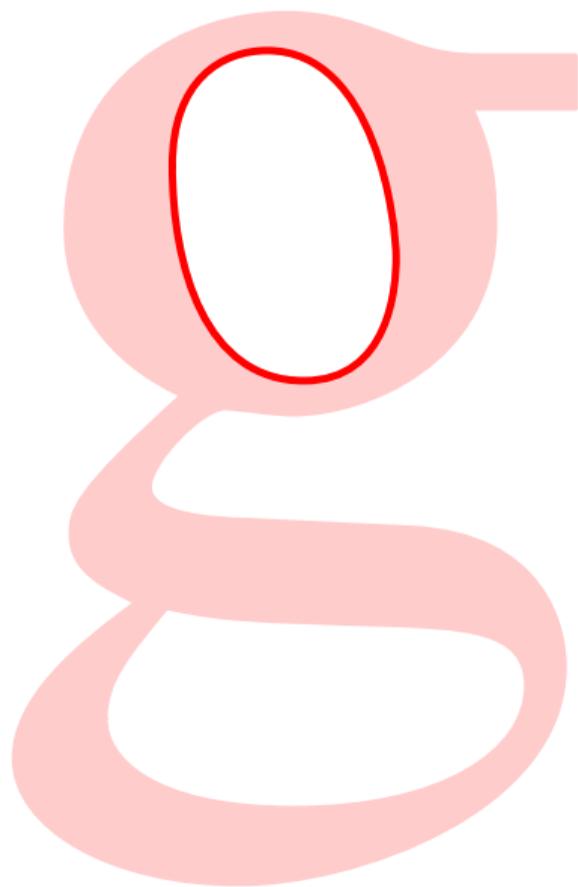
$\alpha'(1) \parallel \beta'(0)$



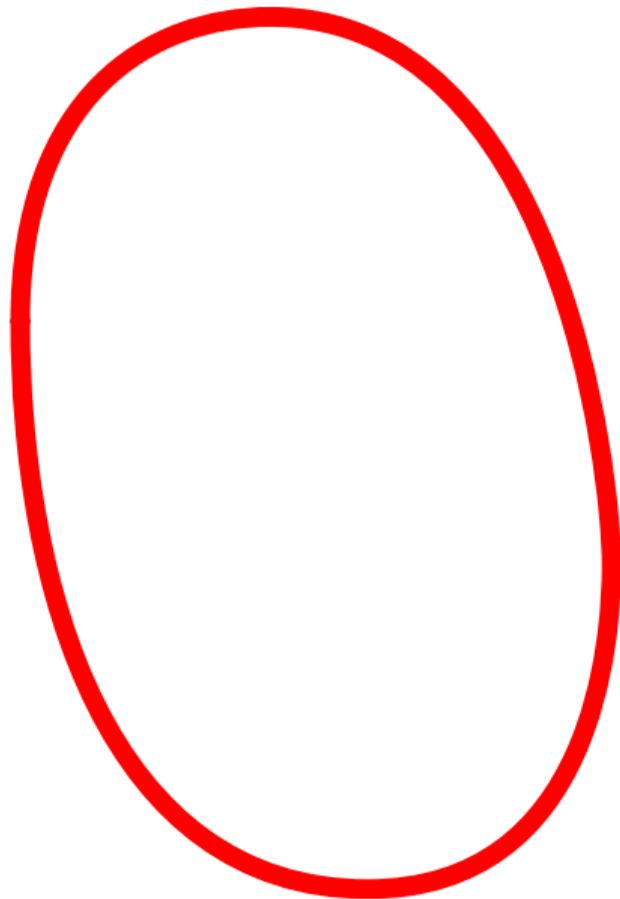
Continuidade geométrica



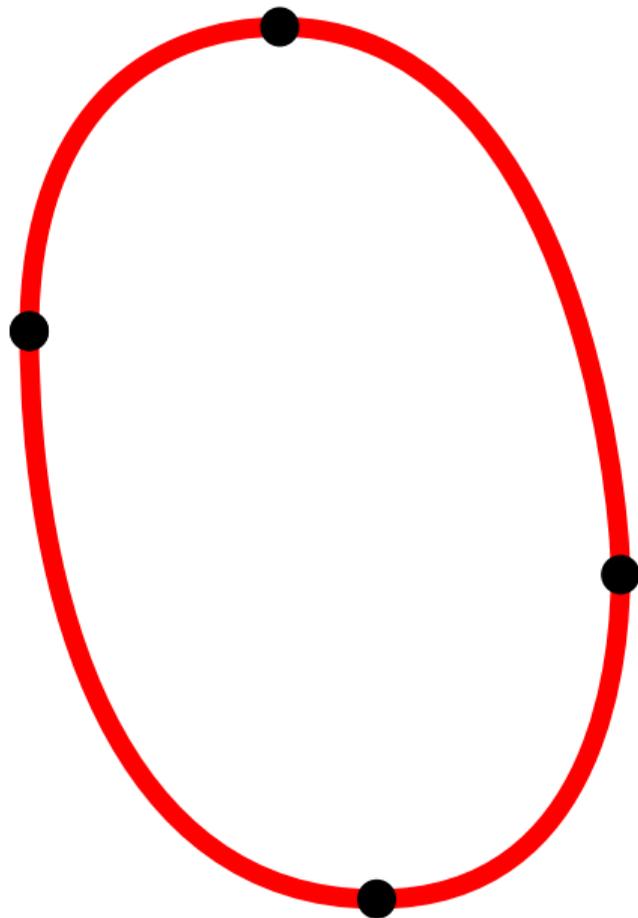
## Continuidade geométrica



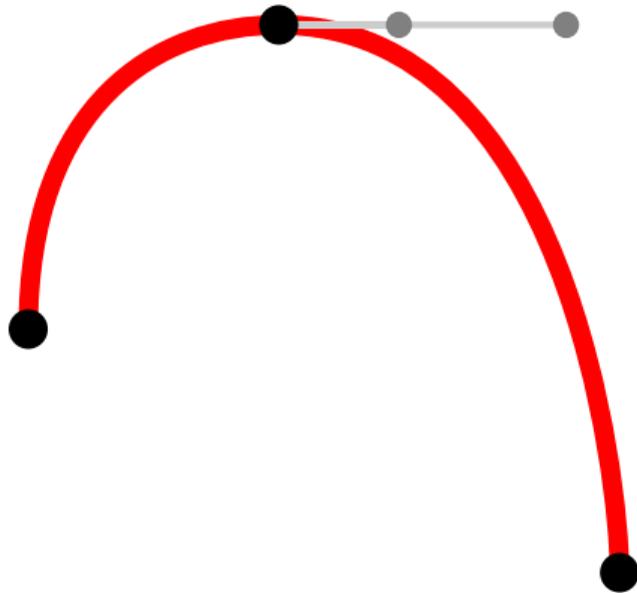
Continuidade geométrica



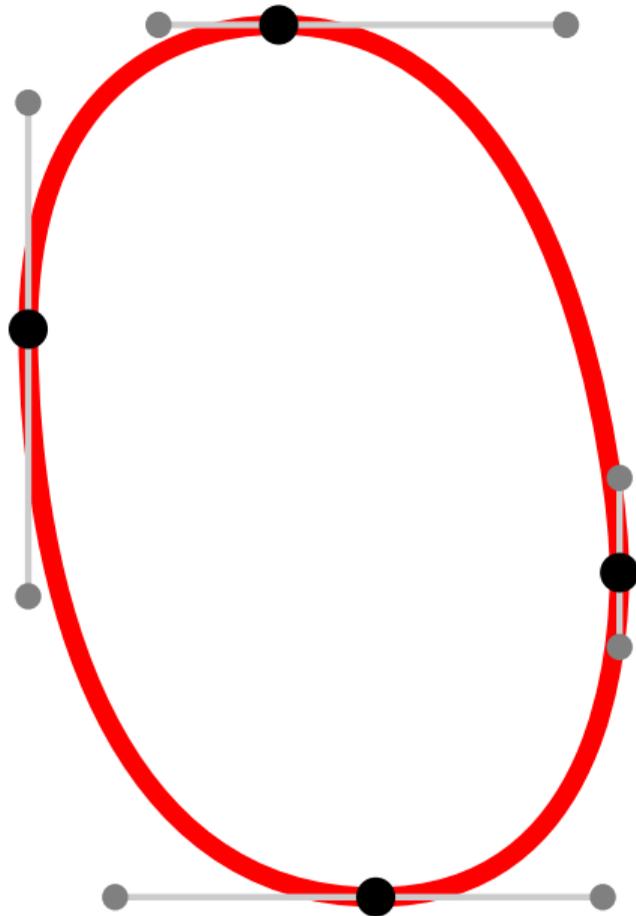
Continuidade geométrica



## Continuidade geométrica



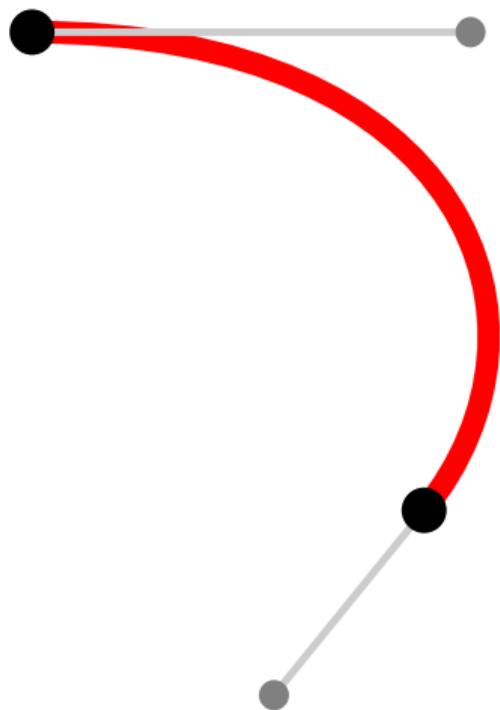
## Continuidade geométrica



## Cúbica de Hermite

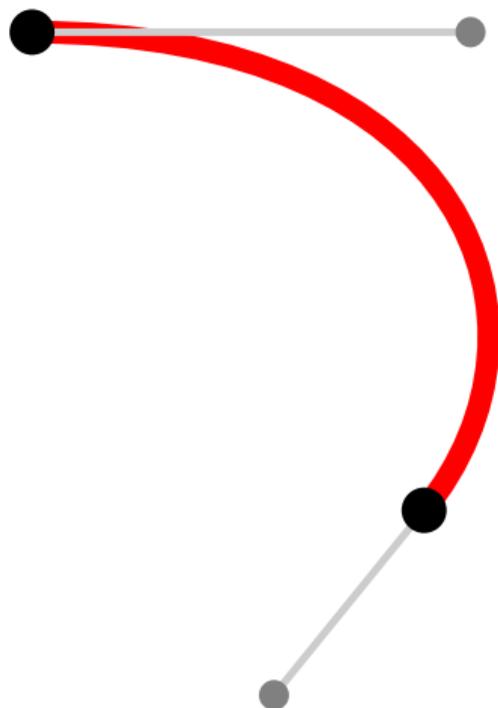
■ pontos extremos  $p_0, p_1$

■ vetores tangentes  $v_0, v_1$



## Cúbica de Hermite

- pontos extremos  $p_0, p_1$
- vetores tangentes  $v_0, v_1$
- $\gamma: [0, 1] \rightarrow \mathbf{R}^d$
- $\gamma(0) = p_0$
- $\gamma(1) = p_1$
- $\gamma'(0) = v_0$
- $\gamma'(1) = v_1$



## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = \mathbf{p}_0, \quad \gamma(1) = \mathbf{p}_1, \quad \gamma'(0) = \mathbf{v}_0, \quad \gamma'(1) = \mathbf{v}_1$$

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$$\gamma(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3 \quad \mathbf{a}_k \in \mathbf{R}^d$$

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$$\gamma \mapsto (\gamma(0), \gamma(1), \gamma'(0), \gamma'(1)) \quad \text{transformação linear}$$

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$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix}$$

## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = \mathbf{p}_0, \quad \gamma(1) = \mathbf{p}_1, \quad \gamma'(0) = \mathbf{v}_0, \quad \gamma'(1) = \mathbf{v}_1$$

$$\gamma(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3 \quad \mathbf{a}_k \in \mathbf{R}^d$$

$$\gamma \mapsto (\gamma(0), \gamma(1), \gamma'(0), \gamma'(1)) \quad \text{transformação linear}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix} \implies \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{pmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix}$$

## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = \mathbf{p}_0, \quad \gamma(1) = \mathbf{p}_1, \quad \gamma'(0) = \mathbf{v}_0, \quad \gamma'(1) = \mathbf{v}_1$$

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$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix} \implies \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{pmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix}$$

$$\gamma(t) = \mathbf{p}_0 + \mathbf{v}_0 t + (-3\mathbf{p}_0 + 3\mathbf{p}_1 - 2\mathbf{v}_0 - \mathbf{v}_1)t^2 + (2\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{v}_0 + \mathbf{v}_1)t^3$$

## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = \mathbf{p}_0, \quad \gamma(1) = \mathbf{p}_1, \quad \gamma'(0) = \mathbf{v}_0, \quad \gamma'(1) = \mathbf{v}_1$$

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monomial

## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = \mathbf{p}_0, \quad \gamma(1) = \mathbf{p}_1, \quad \gamma'(0) = \mathbf{v}_0, \quad \gamma'(1) = \mathbf{v}_1$$

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monomial

$$= (2t^3 - 3t^2 + 1)\mathbf{p}_0 + (-2t^3 + 3t^2)\mathbf{p}_1 + (t^3 - 2t^2 + t)\mathbf{v}_0 + (t^3 - t^2)\mathbf{v}_1$$

Hermite

## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = \mathbf{p}_0, \quad \gamma(1) = \mathbf{p}_1, \quad \gamma'(0) = \mathbf{v}_0, \quad \gamma'(1) = \mathbf{v}_1$$

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monomial

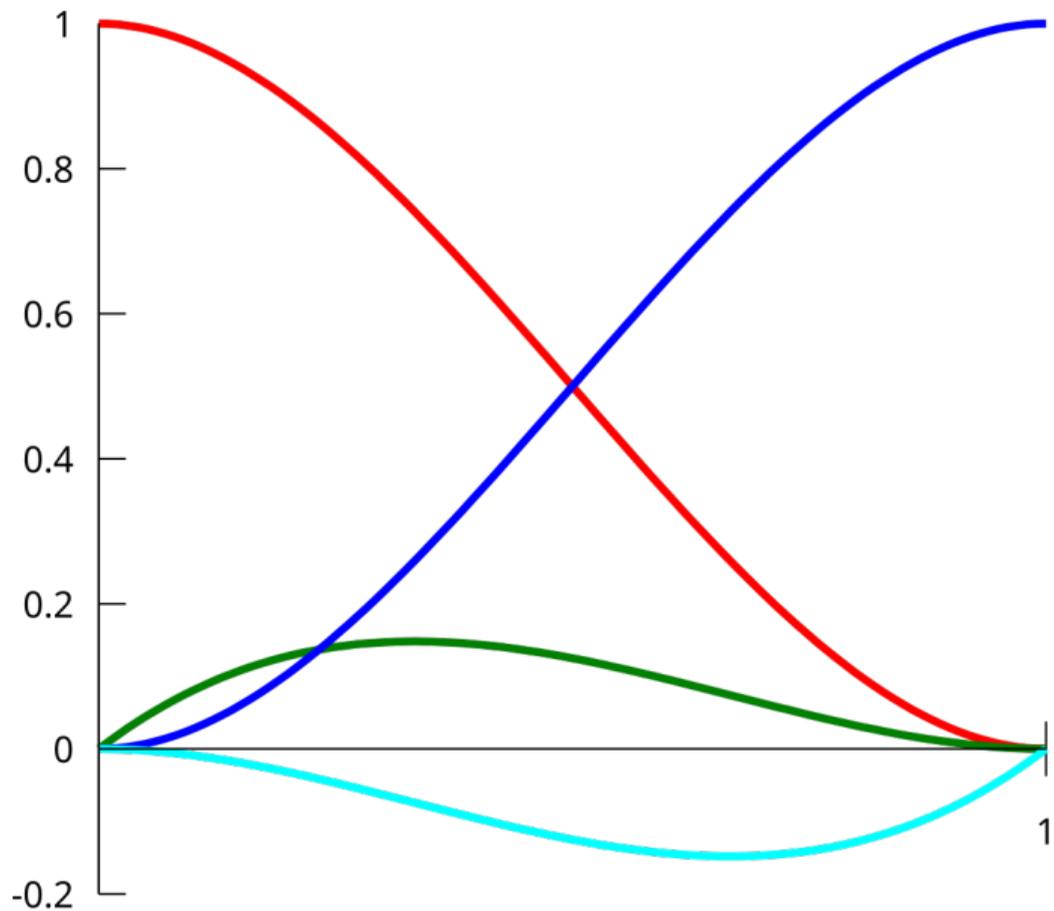
$$= (2t^3 - 3t^2 + 1)\mathbf{p}_0 + (-2t^3 + 3t^2)\mathbf{p}_1 + (t^3 - 2t^2 + t)\mathbf{v}_0 + (t^3 - t^2)\mathbf{v}_1$$

Hermite

$$= \mathbf{B}^3[\mathbf{p}_0, \mathbf{p}_0 + \frac{1}{3}\mathbf{v}_0, \mathbf{p}_1 - \frac{1}{3}\mathbf{v}_1, \mathbf{p}_1]$$

Bézier

## Base de Hermite



## Base de Hermite

$$\begin{aligned}\gamma(t) &= (2t^3 - 3t^2 + 1)p_0 + (-2t^3 + 3t^2)p_1 + (t^3 - 2t^2 + t)v_0 + (t^3 - t^2)v_1 \\ &= H_1(t) p_0 + H_2(t) p_1 + H_3(t) v_0 + H_4(t) v_1\end{aligned}$$

pontos de controle mistos

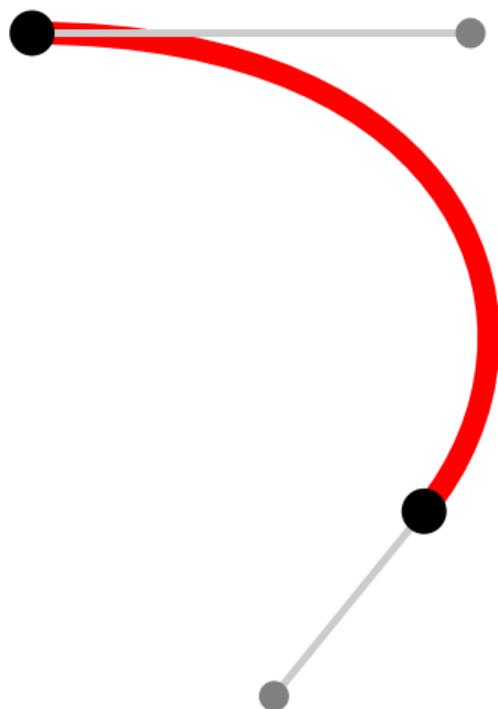
f	H <sub>1</sub>	H <sub>2</sub>	H <sub>3</sub>	H <sub>4</sub>
f(0)	1	0	0	0
f(1)	0	1	0	0
f'(0)	0	0	1	0
f'(1)	0	0	0	1

$$H_1(t) = H_2(1 - t), \quad H_3(t) = -H_4(1 - t)$$

base dual

## Cúbica de Hermite geométrica

- pontos extremos  $p_0, p_1$
- direções tangentes  $v_0, v_1$
- $\gamma: [0, 1] \rightarrow \mathbf{R}^d$
- $\gamma(0) = p_0$
- $\gamma(1) = p_1$
- $\gamma'(0) = \lambda_0 v_0$        $\lambda_0, \lambda_1 \in \mathbf{R}$  a determinar
- $\gamma'(1) = \lambda_1 v_1$



## Cúbica de Hermite geométrica

■ pontos extremos  $p_0, p_1$

■ direções tangentes  $v_0, v_1$

■  $\gamma: [0, 1] \rightarrow \mathbf{R}^d$

■  $\gamma(0) = p_0$

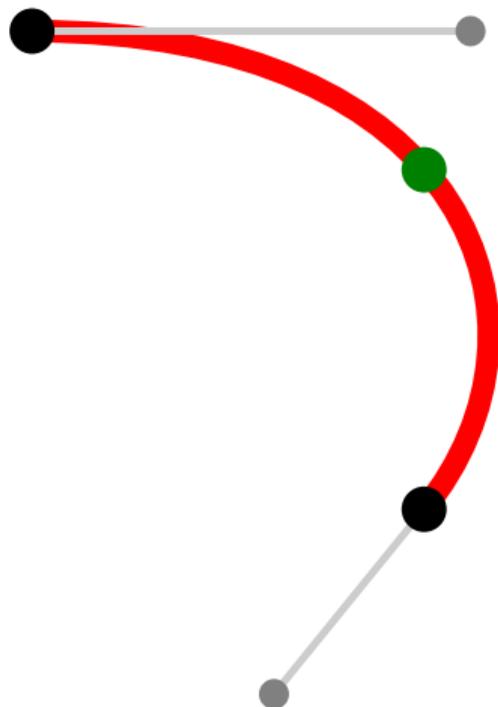
■  $\gamma(1) = p_1$

■  $\gamma'(0) = \lambda_0 v_0$        $\lambda_0, \lambda_1 \in \mathbf{R}$  a determinar

■  $\gamma'(1) = \lambda_1 v_1$

■  $\gamma(t^*) = p^*$

$d = 2$ :  $t^*$  dado /  $d = 3$ : equação cúbica em  $t^*$



curvas de Bézier

## Formulação geométrica

■ de Casteljau (Citroën, 1959)

interpolação linear repetida

■ Bézier (Renault, 1967)

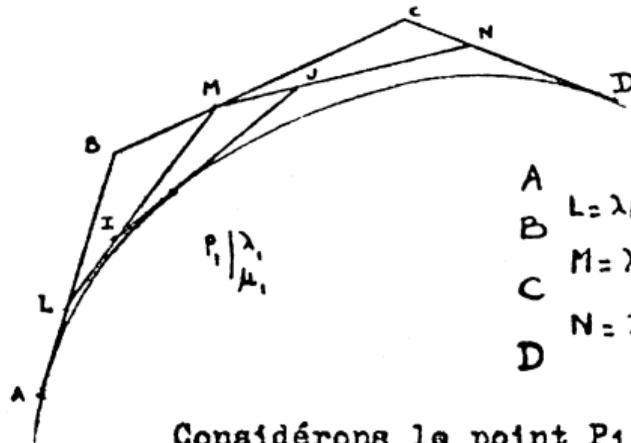
ponto + deslocamentos

■ Forrest (1972)

polinômios de Bernstein

■ Boehm (1975)

de Casteljau

1.5.- Sous-Pôles d'une courbe1.5.1.- Définition des sous-pôles.

Considérons une cubique des pôles A, B, C, D. Nous avons vu précédemment que la construction par la méthode des barycentres donnent les différents points L, M, N, I, J, P.

$$\begin{array}{l}
 A \\
 B \quad L = \lambda A + \mu B \\
 C \quad M = \lambda B + \mu C \\
 D \quad N = \lambda C + \mu D
 \end{array}
 \quad
 \begin{array}{l}
 I = \lambda^2 A + 2\lambda\mu B + \mu^2 C \\
 J = \lambda^2 B + 2\lambda\mu C + \mu^2 D \\
 P = \lambda^3 A + 3\lambda^2\mu B + 3\lambda\mu^2 C + \mu^3 D
 \end{array}$$

Considérons le point  $P_1$  de paramètres  $\lambda_1$  et  $\mu_1$  (avec  $\lambda_1 + \mu_1 = 1$ ) ( $\lambda_1$  varie de 0 à 1 et  $\mu_1$  1 à 0, lorsque P va en D).

Cherchons les pôles de la cubique  $P_1$  D. Cette courbe dérive de la cubique initiale AD par changement des paramètres.

$$P = \lambda^3 A + 3\lambda^2\mu B + 3\lambda\mu^2 C + \mu^3 D, \quad \lambda + \mu = 1, \quad \lambda = 1 - \mu, \quad \mu \in [0, 1]$$

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

# Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$

segmento se  $t \in [0, 1]$

# Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$

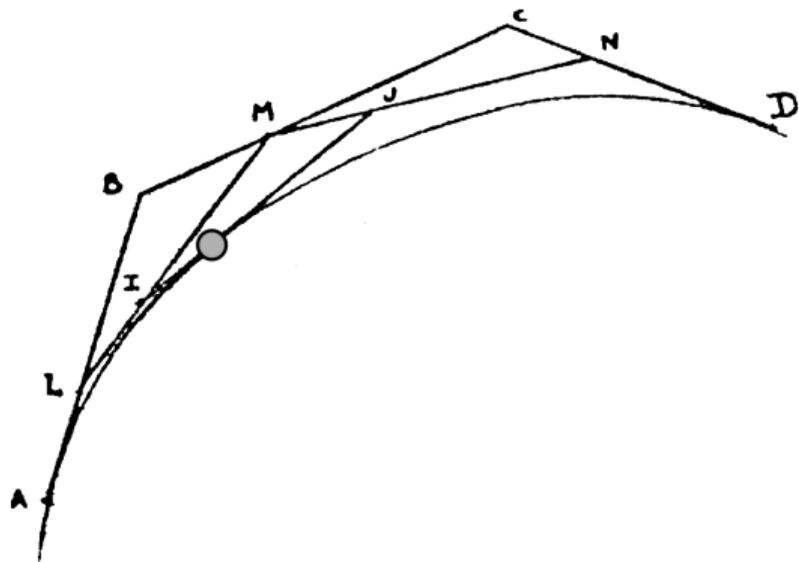
interpolação linear repetida

A

B

C

D



# Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$

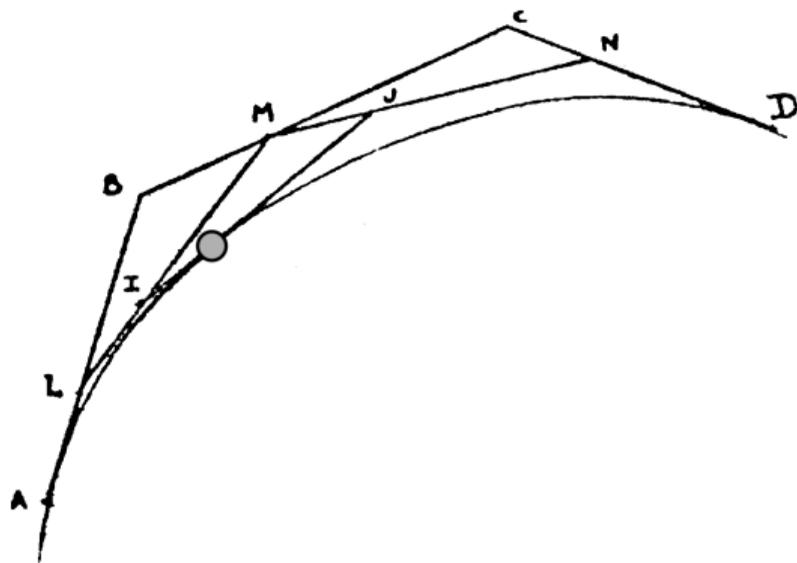
interpolação linear repetida

$$A \quad L = \text{lerp}(t, A, B)$$

$$B \quad M = \text{lerp}(t, B, C)$$

$$C \quad N = \text{lerp}(t, C, D)$$

D



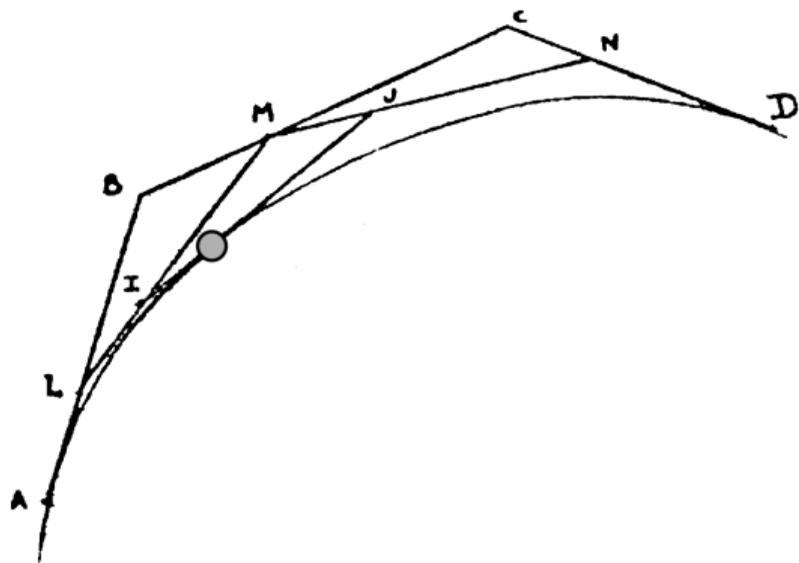
# Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$



interpolação linear repetida

$$A \quad L = \text{lerp}(t, A, B)$$

$$B \quad M = \text{lerp}(t, B, C)$$

$$C \quad N = \text{lerp}(t, C, D)$$

D

$$I = \text{lerp}(t, L, M)$$

$$J = \text{lerp}(t, M, N)$$

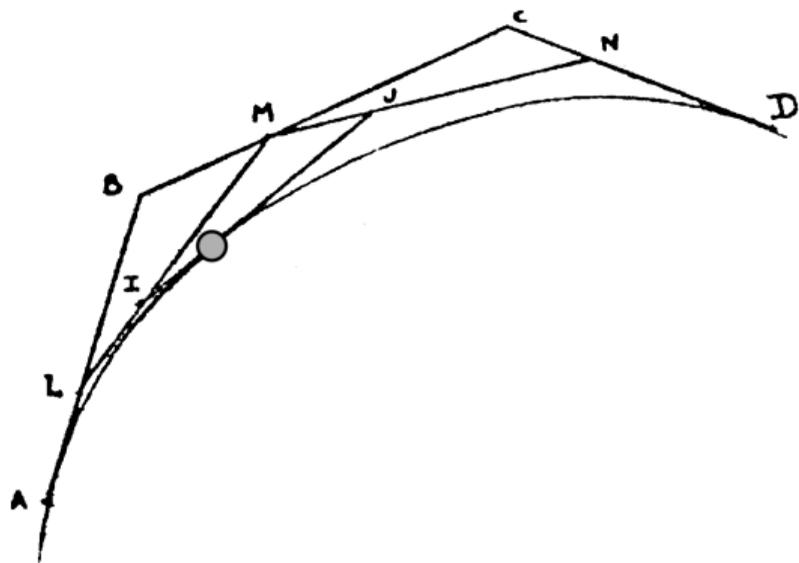
# Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$



interpolação linear repetida

$$A \quad L = \text{lerp}(t, A, B)$$

$$B \quad M = \text{lerp}(t, B, C)$$

$$C \quad N = \text{lerp}(t, C, D)$$

D

$$I = \text{lerp}(t, L, M)$$

$$J = \text{lerp}(t, M, N)$$

$$P = \text{lerp}(t, I, J)$$

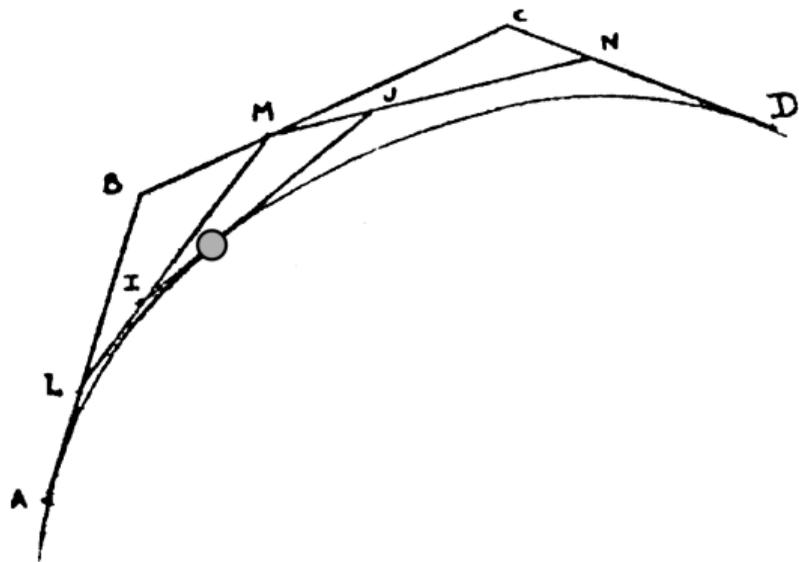
# Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$



interpolação linear repetida

A       $L = \text{lerp}(t, A, B)$

B       $M = \text{lerp}(t, B, C)$

C       $N = \text{lerp}(t, C, D)$

D

I = lerp(t, L, M)

J = lerp(t, M, N)

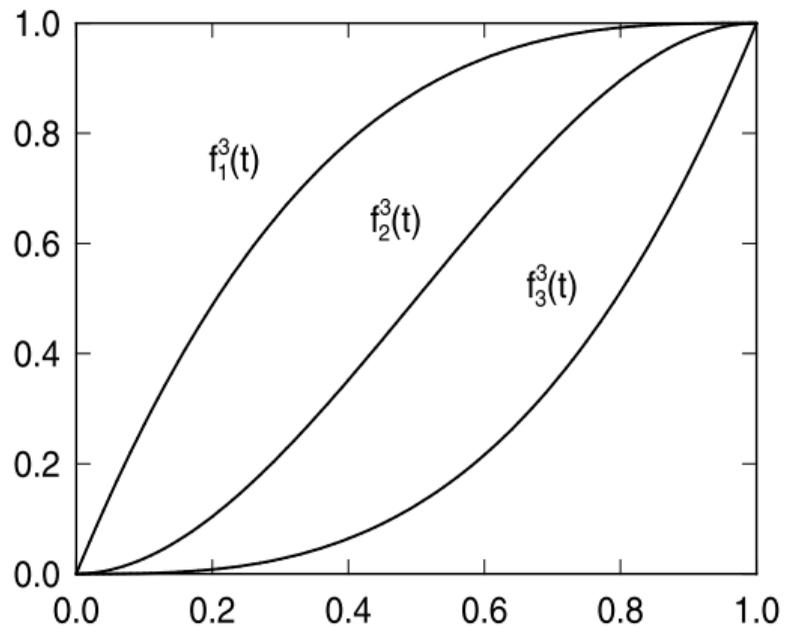
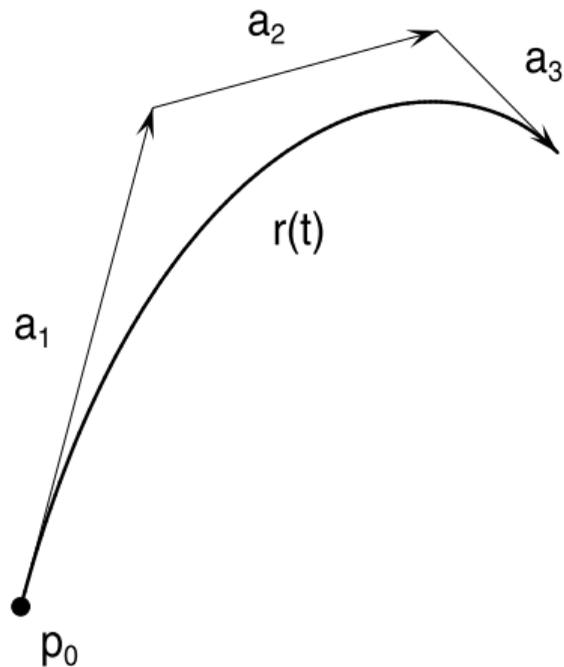
P = lerp(t, I, J)

$$P = (1 - t)^3 A + 3(1 - t)^2 t B + 3(1 - t) t^2 C + t^3 D$$

## Formulação geométrica

Bézier (Renault, 1967)

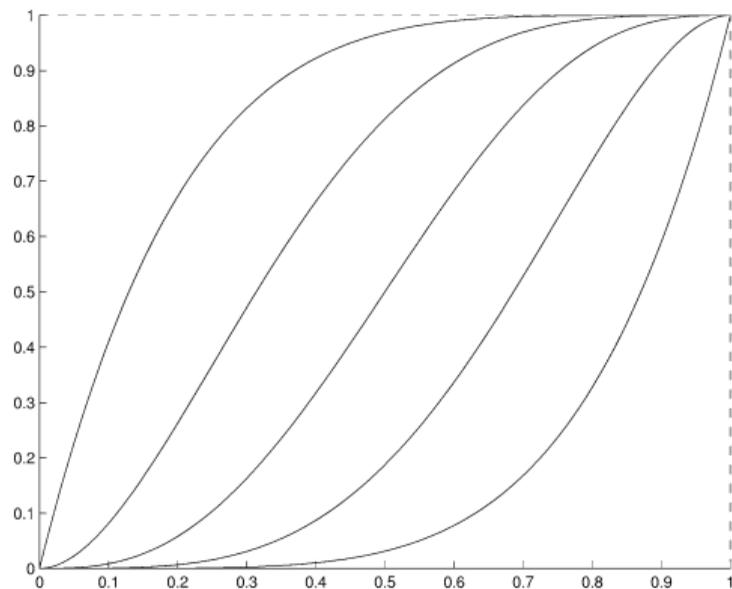
ponto + deslocamentos:  $B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$



ponto + deslocamentos: 
$$B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$$

- $f_k^n$  polinômio de grau  $n$
- $f_k^n(0) = 0$
- $f_k^n(1) = 1$
- $(f_k^n)^{(r)}(0) = 0, \quad r = 1, \dots, k-1$
- $(f_k^n)^{(r)}(1) = 0, \quad r = 1, \dots, n-k$

interpolação de Hermite–Birkhoff



## Formulação geométrica

Forrest (1972)

$$B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$$

$$p_k = p_0 + \sum_{j=1}^k a_j$$

$$a_k = p_k - p_{k-1}$$

$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

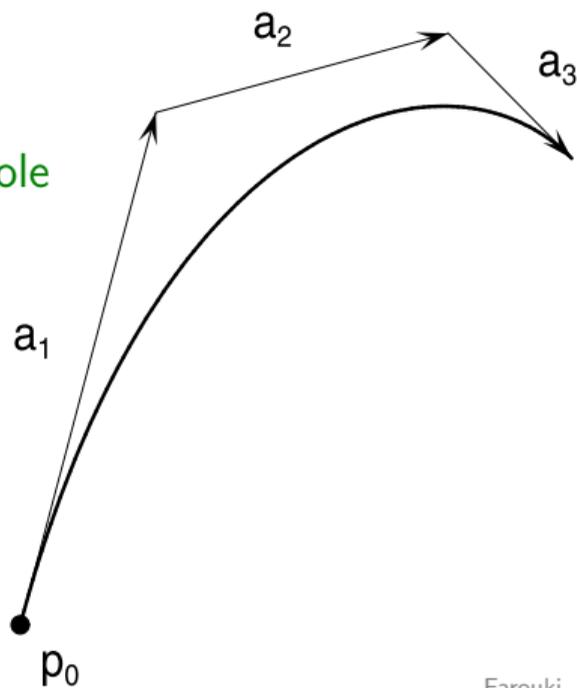
$$B_k^n(t) = f_k^n(t) - f_{k+1}^n(t)$$

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

polígono de controle

curva de Bézier

polinômios de Bernstein



Farouki

## Curvas de Bézier

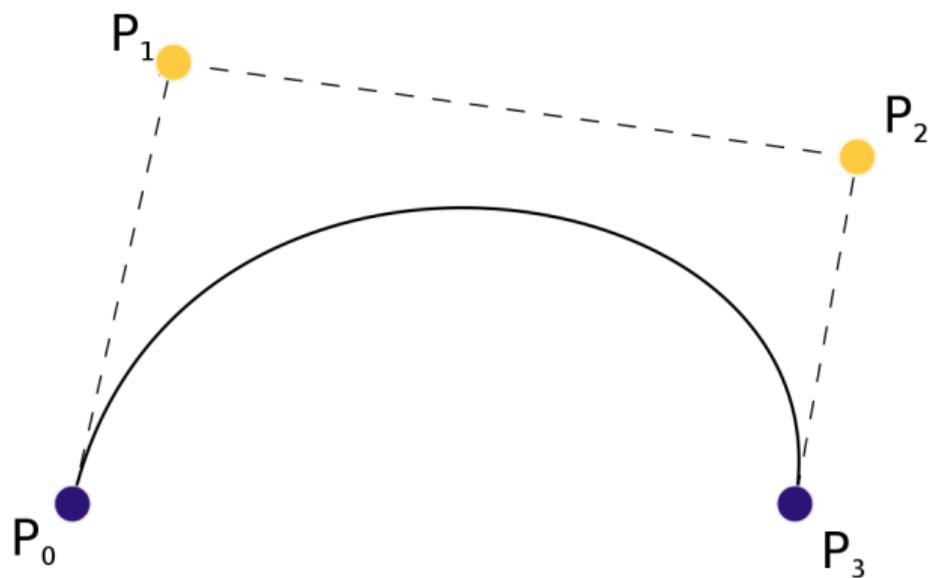
$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

$$B^0(t) = p_0$$

$$B^1(t) = (1-t)p_0 + tp_1$$

$$B^2(t) = (1-t)^2 p_0 + 2(1-t)t p_1 + t^2 p_2$$

$$B^3(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

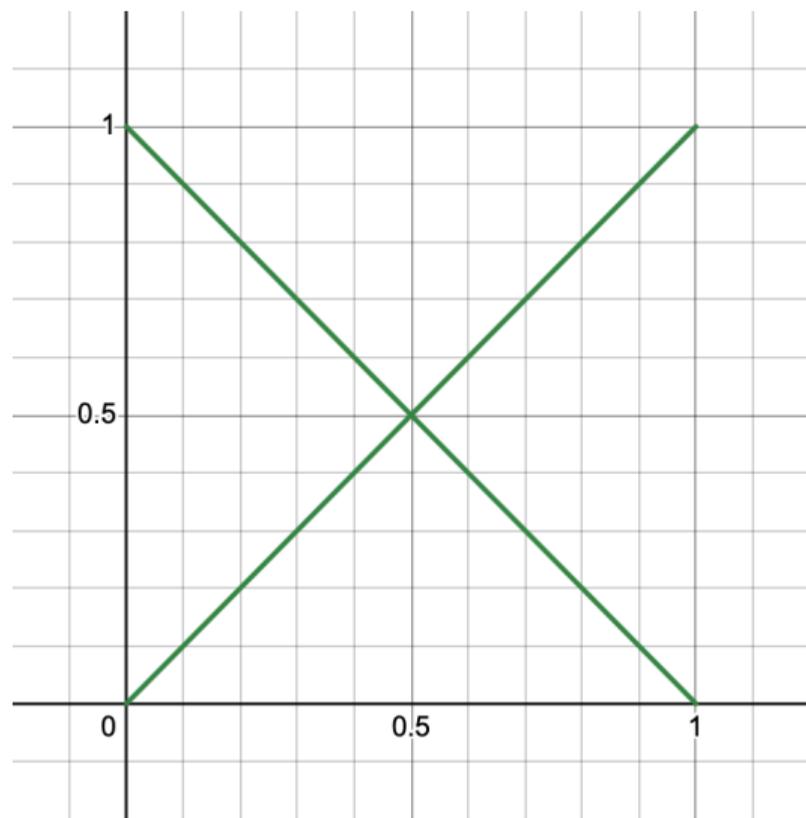


demo

## Polinômios de Bernstein

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

n				
0	1			
1	1-t	t		
2	(1-t) <sup>2</sup>	2(1-t)t	t <sup>2</sup>	
3	(1-t) <sup>3</sup>	3(1-t) <sup>2</sup> t	3(1-t)t <sup>2</sup>	t <sup>3</sup>
k	0	1	2	3

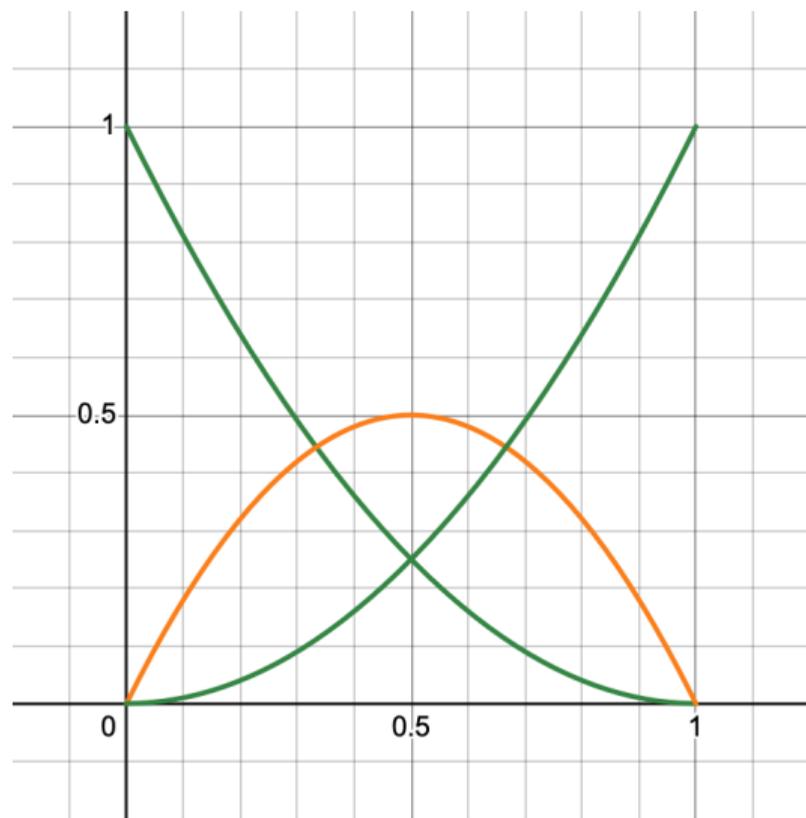


n = 1

## Polinômios de Bernstein

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

n				
0	1			
1	1-t	t		
2	(1-t) <sup>2</sup>	2(1-t)t	t <sup>2</sup>	
3	(1-t) <sup>3</sup>	3(1-t) <sup>2</sup> t	3(1-t)t <sup>2</sup>	t <sup>3</sup>
k	0	1	2	3

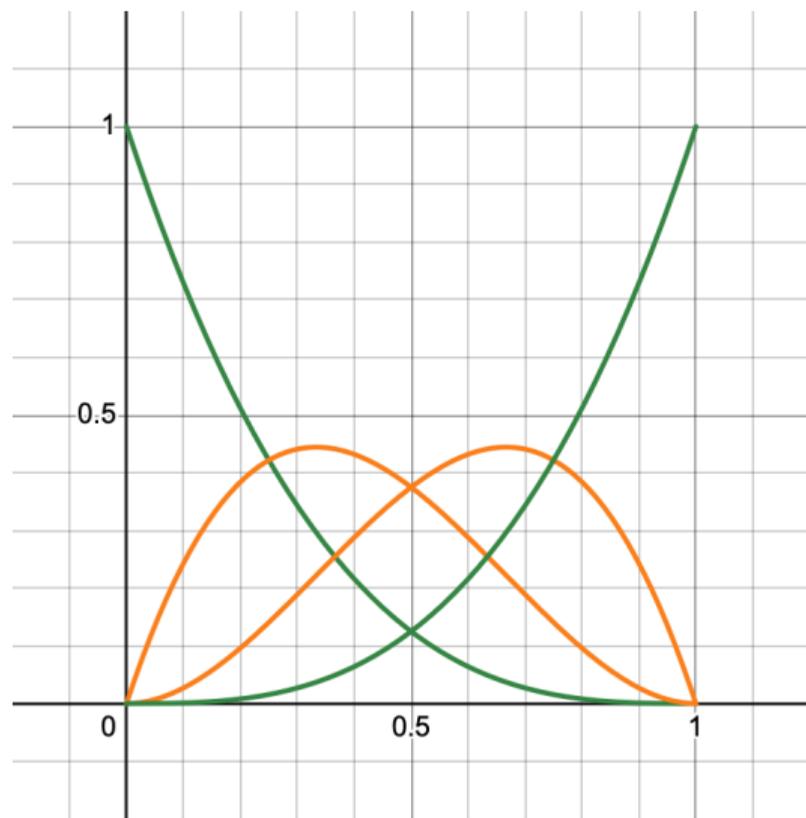


$n = 2$

# Polinômios de Bernstein

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

n				
0	1			
1	1-t	t		
2	(1-t) <sup>2</sup>	2(1-t)t	t <sup>2</sup>	
3	(1-t) <sup>3</sup>	3(1-t) <sup>2</sup> t	3(1-t)t <sup>2</sup>	t <sup>3</sup>
k	0	1	2	3

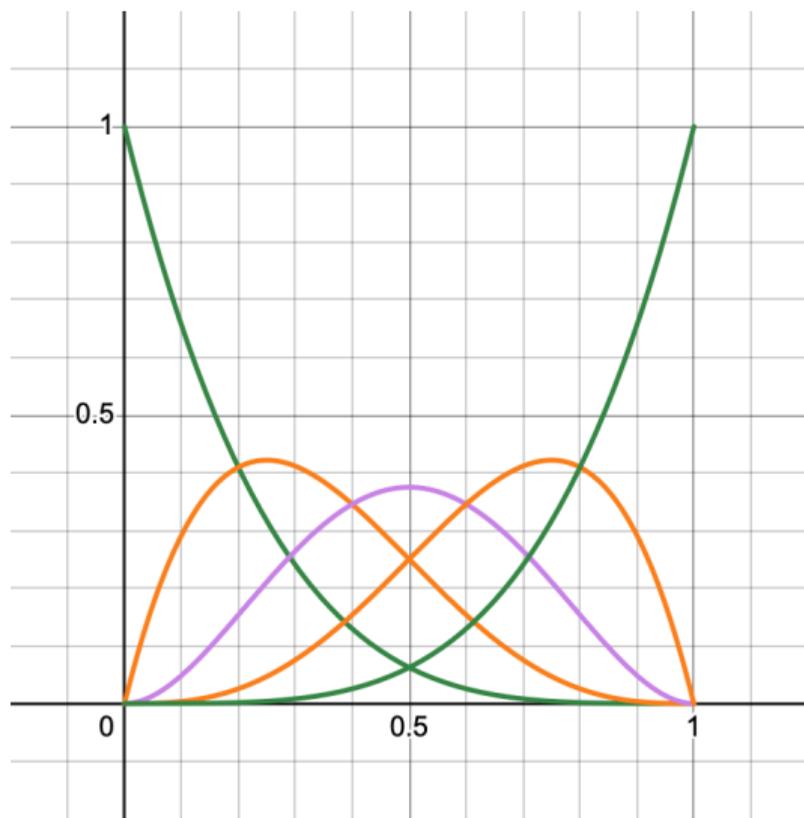


n = 3

# Polinômios de Bernstein

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

n				
0	1			
1	1-t	t		
2	(1-t) <sup>2</sup>	2(1-t)t	t <sup>2</sup>	
3	(1-t) <sup>3</sup>	3(1-t) <sup>2</sup> t	3(1-t)t <sup>2</sup>	t <sup>3</sup>
k	0	1	2	3



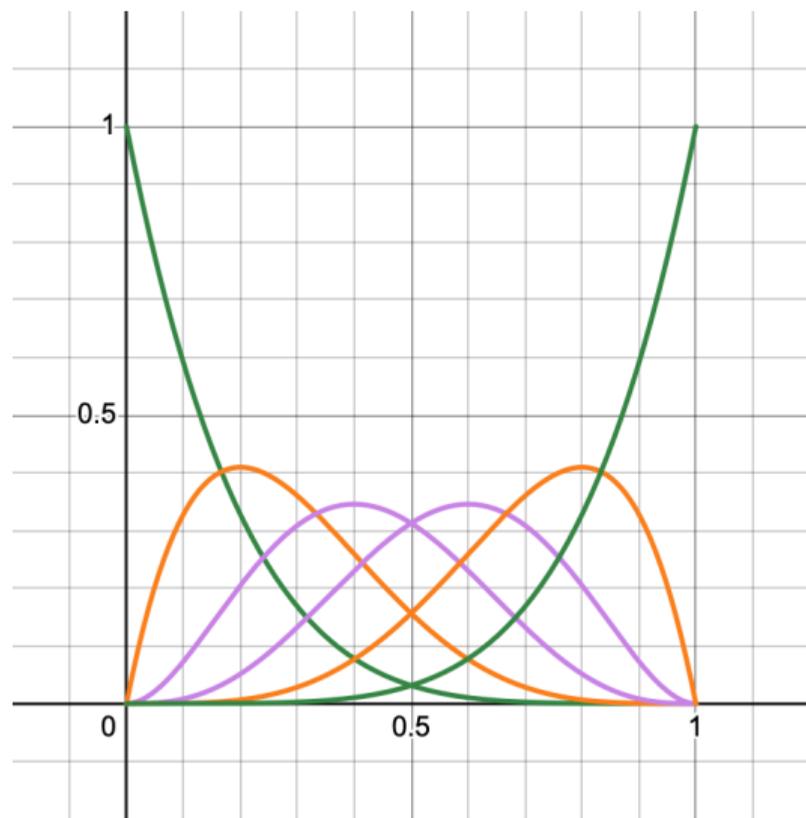
n = 4

# Polinômios de Bernstein



$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

n				
0	1			
1	1-t	t		
2	(1-t) <sup>2</sup>	2(1-t)t	t <sup>2</sup>	
3	(1-t) <sup>3</sup>	3(1-t) <sup>2</sup> t	3(1-t)t <sup>2</sup>	t <sup>3</sup>
k	0	1	2	3



n = 5

## Teorema de aproximação

Weierstrass (1885)

Dada  $f \in C[0, 1]$ , para todo  $\varepsilon > 0$  existe um polinômio  $p$  tal que

$$|f(t) - p(t)| < \varepsilon \quad \text{para todo } t \in [0, 1]$$

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$$\lim_{n \rightarrow \infty} p_n = f \quad \text{uniforme}$$

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$$\lim_{n \rightarrow \infty} p_n = f \quad \text{uniforme}$$

prova construtiva:

Bernstein (1912)

$$B_n(f)(t) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_k^n(t) \implies \lim_{n \rightarrow \infty} B_n(f) = f \quad \text{uniforme}$$

$$f \in C^r \implies \lim_{n \rightarrow \infty} B_n(f)^{(r)} = f^{(r)} \quad \text{uniforme}$$

## Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités.

Je me propose d'indiquer une démonstration fort simple du théorème suivant de Weierstrass:

Si  $F(x)$  est une fonction continue quelconque dans l'intervalle  $01$ , il est toujours possible, quel que petit que soit  $\varepsilon$ , de déterminer un polynôme  $E_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$  de degré  $n$  assez élevé, tel qu'on ait

$$|F(x) - E_n(x)| < \varepsilon$$

en tout point de l'intervalle considéré.

A cet effet, je considère un événement  $A$ , dont la probabilité est égale à  $x$ . Supposons qu'on effectue  $n$  expériences et que l'on convienne de payer à un joueur la somme  $F\left(\frac{m}{n}\right)$ , si l'événement  $A$  se produit  $m$  fois. Dans ces conditions, l'espérance mathématique  $E_n$  du joueur aura pour valeur

$$E_n = \sum_{m=0}^{m=n} F\left(\frac{m}{n}\right) \cdot C_n^m x^m (1-x)^{n-m}. \quad (1)$$

Or, il résulte de la continuité de la fonction  $F(x)$  qu'il est possible de fixer un nombre  $\delta$ , tel que l'inégalité

$$|x - x_0| \leq \delta$$

entraîne

$$\left|F(x) - F(x_0)\right| < \frac{\varepsilon}{2};$$

de sorte que, si  $\bar{F}(x)$  désigne le maximum et  $\underline{F}(x)$  le minimum de  $F(x)$  dans l'intervalle  $(x - \delta, x + \delta)$ , on a

$$\bar{F}(x) - F(x) < \frac{\varepsilon}{2}, \quad F(x) - \underline{F}(x) < \frac{\varepsilon}{2}. \quad (2)$$

Soit de plus  $\eta$  la probabilité de l'inégalité  $\left|x - \frac{m}{n}\right| > \delta$ , et  $L$  le maximum de  $|F(x)|$  dans l'intervalle  $01$ .

On aura alors

$$\underline{F}(x) \cdot (1 - \eta) - L \cdot \eta < E_n < \bar{F}(x) \cdot (1 - \eta) + L \cdot \eta. \quad (3)$$

Mais, en vertu du théorème de Bernoulli, on pourra prendre  $n$  assez grand pour avoir

$$\eta < \frac{\varepsilon}{4L}. \quad (4)$$

L'inégalité (3) se mettra donc successivement sous la forme

$$F(x) + (\underline{F}(x) - F(x)) - \eta(L + \underline{F}(x)) < E_n < F(x) + (\bar{F}(x) - F(x)) + \eta(L - \bar{F}(x))$$

et ensuite

$$F(x) - \frac{\varepsilon}{2} - \frac{2L}{4L} \varepsilon < E_n < F(x) + \frac{\varepsilon}{2} + \frac{2L}{4L} \varepsilon;$$

donc

$$|F(x) - E_n| < \varepsilon \quad (5)$$

Or  $E_n$  est manifestement un polynôme de degré  $n$ .

Le théorème est donc démontré.

J'ajouterai seulement deux remarques.

Les polynômes approchés  $E_n(x)$  sont surtout commodes, il me semble, lorsqu'on connaît exactement ou approximativement les valeurs de  $F(x)$

pour  $x = \frac{m}{n}$  ( $m = 0, 1, \dots, n$ ).

La formule (1) et l'inégalité (5) montrent que, quelle que soit la fonction continue  $F(x)$ , on a

$$F(x) = \lim_{n \rightarrow \infty} \sum_{m=0}^{m=n} F\left(\frac{m}{n}\right) \cdot C_n^m x^m (1-x)^{n-m}.$$

## Polinômios de Bernstein – base do espaço de polinômios

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} t^{k+j}$$

base canônica = base monomial:  $1, t, t^2, t^3, \dots, t^n$

## Polinômios de Bernstein – base do espaço de polinômios

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} t^{k+j}$$

$$\begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix} = \begin{bmatrix} (1-t)^3 \\ 3(1-t)^2t \\ 3(1-t)t^2 \\ t^3 \end{bmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

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$$t^k = \frac{1}{\binom{n}{k}} \sum_{j=k}^n \binom{j}{k} B_j^n(t) \quad \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

## Polinômios de Bernstein – propriedades

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

- $B_k^n(t) = B_{n-k}^n(1-t)$  simetria
- $B_k^n(0) = 0, \quad B_k^n(1) = 0, \quad B_0^n(0) = 1, \quad B_n^n(1) = 1$  pontos extremos
- $B_k^n(t)$  tem único máximo em  $[0, 1]$ , em  $t = k/n$  unimodal
- $B_k^n(t) \geq 0$  para  $t \in [0, 1]$  não negativo
- $\sum_{k=0}^n B_k^n(t) = ((1-t) + t)^n = 1$  partição da unidade
- $\sum_{k=0}^n \frac{k}{n} B_k^n(t) = t$  precisão linear

## Curvas de Bézier – propriedades

$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

- $B(t) = B(1 - t)$  simetria  
 $B[p_0, \dots, p_n](t) = B[p_n, \dots, p_0](1 - t)$
- $B(0) = p_0, \quad B(1) = p_n$  pontos extremos
- a influência de  $p_k$  é maior perto de  $t = k/n$  unimodal
- $B(t)$  é **combinação convexa** de  $p_0, \dots, p_n$  não negativo  
curva contida no fecho convexo + partição da unidade
- $B(t)$  é gráfico de função quando  $x_k = k/n$  precisão linear  
segmento de reta se pontos de controle uniformemente espaçados

## Curvas de Bézier – invariância afim

- Combinação afim

$$\alpha_1 p_1 + \cdots + \alpha_n p_n \text{ com } \alpha_1 + \cdots + \alpha_n = 1$$

- T transformação afim

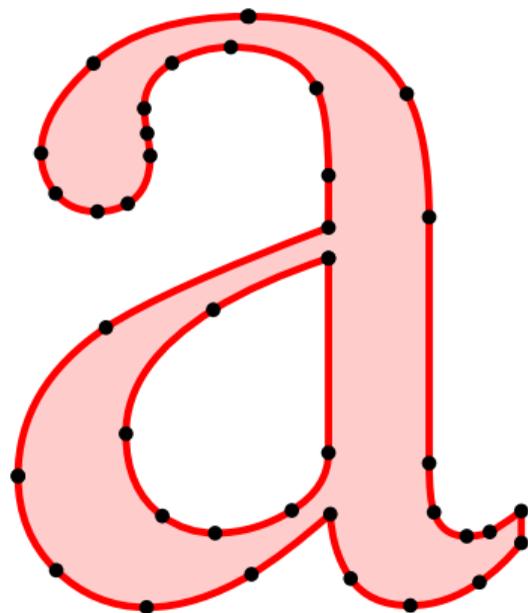
$$T(\alpha_1 p_1 + \cdots + \alpha_n p_n) = \alpha_1 T(p_1) + \cdots + \alpha_n T(p_n)$$

- invariância afim

$$T(B[p_0, \dots, p_n]) = B[T(p_0), \dots, T(p_n)]$$

$$T(p) = L(p) + v$$

$$\sum_{k=0}^n B_k^n(t)v = v$$



## Curvas de Bézier – invariância afim

- Combinação afim

$$\alpha_1 p_1 + \dots + \alpha_n p_n \text{ com } \alpha_1 + \dots + \alpha_n = 1$$

- T transformação afim

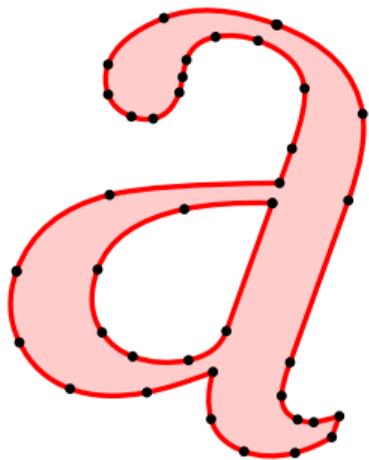
$$T(\alpha_1 p_1 + \dots + \alpha_n p_n) = \alpha_1 T(p_1) + \dots + \alpha_n T(p_n)$$

- invariância afim

$$T(B[p_0, \dots, p_n]) = B[T(p_0), \dots, T(p_n)]$$

$$T(p) = L(p) + v$$

$$\sum_{k=0}^n B_k^n(t)v = v$$



## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Pascal

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

$$B_k^{n+1}(t) = \binom{n+1}{k} (1-t)^{n+1-k} t^k$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

$$\begin{aligned} B_k^{n+1}(t) &= \binom{n+1}{k} (1-t)^{n+1-k} t^k \\ &= \binom{n}{k} (1-t)^{n+1-k} t^k + \binom{n}{k-1} (1-t)^{n+1-k} t^k \end{aligned}$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

$$\begin{aligned} B_k^{n+1}(t) &= \binom{n+1}{k} (1-t)^{n+1-k} t^k \\ &= \binom{n}{k} (1-t)^{n+1-k} t^k + \binom{n}{k-1} (1-t)^{n+1-k} t^k \\ &= (1-t) \binom{n}{k} (1-t)^{n-k} t^k + t \binom{n}{k-1} (1-t)^{n-(k-1)} t^{k-1} \end{aligned}$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

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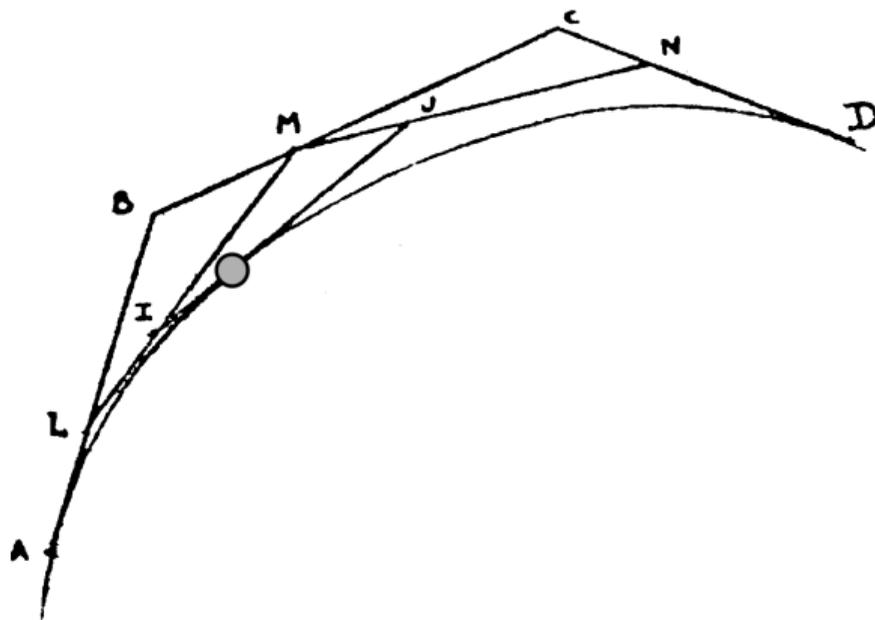
$$B_k^{n+1}(t) = (1-t) B_k^n(t) + t B_{k-1}^n(t) \quad \text{interpolação linear repetida}$$

## Curvas de Bézier – algoritmo de de Casteljaeu

$$B_k^{n+1}(t) = (1-t) B_k^n(t) + t B_{k-1}^n(t)$$

recursão

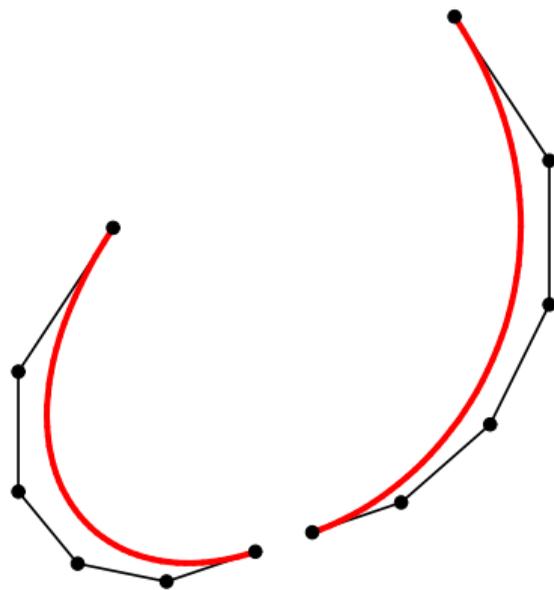
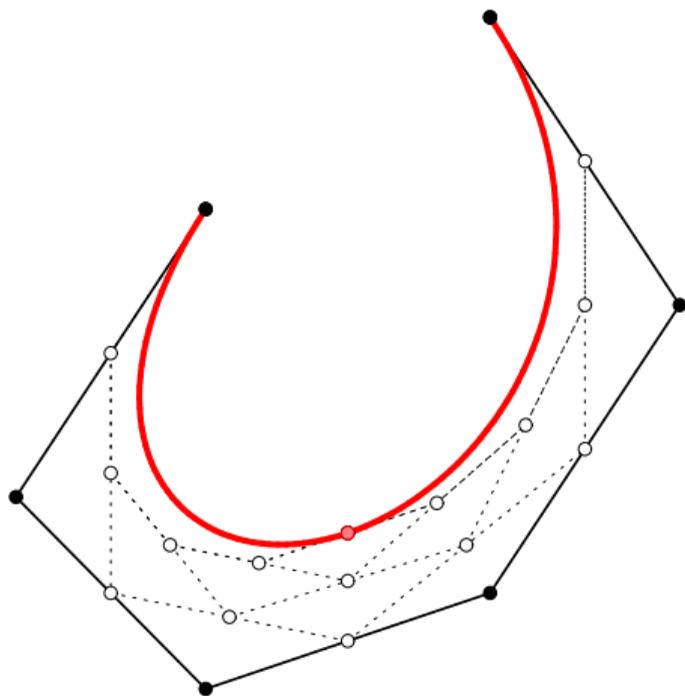
$$B^{n+1}[p_0, \dots, p_{n+1}](t) = (1-t) B^n[p_0, \dots, p_n](t) + t B^n[p_1, \dots, p_{n+1}](t)$$



## Curvas de Bézier – subdivisão adaptativa

$$B_k^{n+1}(t) = (1-t) B_k^n(t) + t B_{k-1}^n(t)$$

recursão



## Curvas de Bézier – presença

grau	PDF	SVG	JavaScript
1	●	●	●
2	●	●	●
3	●	●	●
4+	●	●	●

## Curvas de Bézier – elevação de grau

$$B^2[q_0, q_1, q_2] = B^3[p_0, p_1, p_2, p_3] ?$$

- Bézier quadrática

$$\gamma(t) = (1-t)^2 q_0 + 2(1-t)t q_1 + t^2 q_2$$

... multiplicando por  $((1-t) + t) = 1$

- Bézier cúbica

$$\gamma(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

$$p_0 = q_0, \quad p_1 = \frac{1}{3}q_0 + \frac{2}{3}q_1, \quad p_2 = \frac{1}{3}q_2 + \frac{2}{3}q_1, \quad p_3 = q_2$$

- Bézier de grau  $n$

$$p_k = (1-t) q_k + t q_{k-1} \quad t = \frac{k}{n+1}$$

## Curvas de Bézier – elevação de grau

$$B^2[q_0, q_1, q_2] = B^3[p_0, p_1, p_2, p_3] ?$$

- Bézier quadrática

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$$\gamma(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

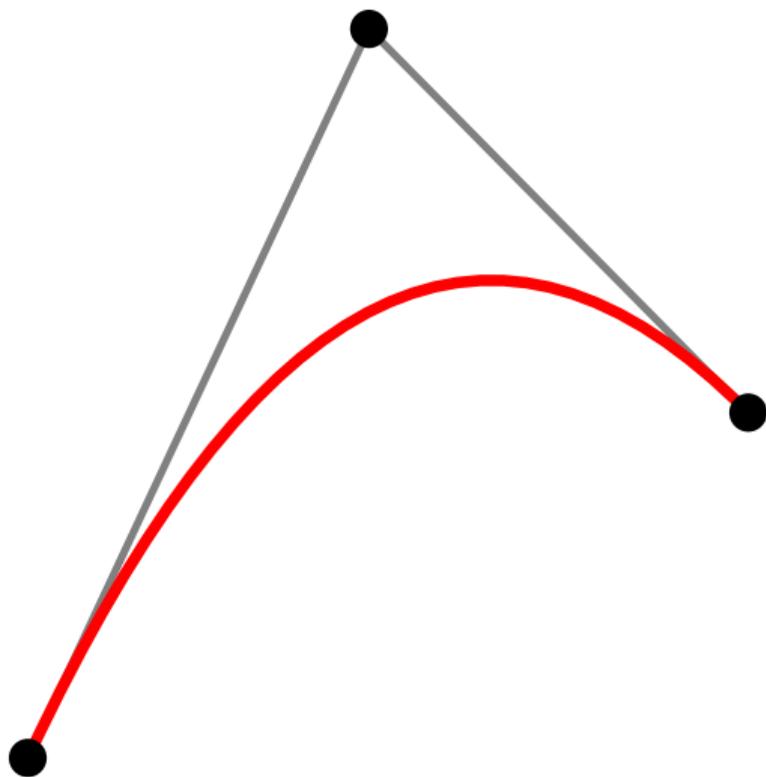
$$p_0 = q_0, \quad p_1 = \frac{1}{3}q_0 + \frac{2}{3}q_1, \quad p_2 = \frac{1}{3}q_2 + \frac{2}{3}q_1, \quad p_3 = q_2$$

- Bézier de grau  $n$

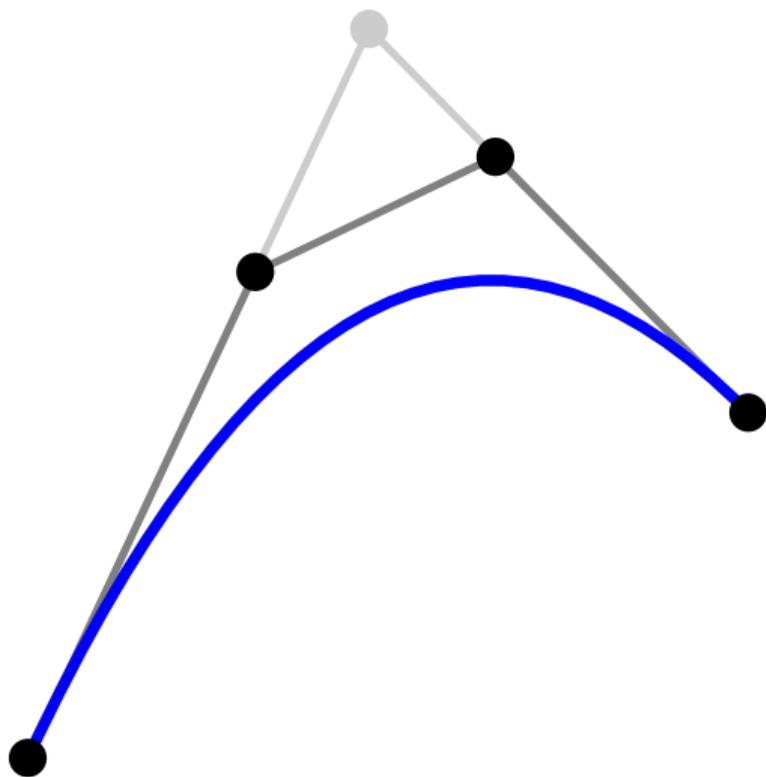
$$p_k = (1-t) q_k + t q_{k-1} \quad t = \frac{k}{n+1}$$

$$P^{n+r} \rightarrow B^n$$

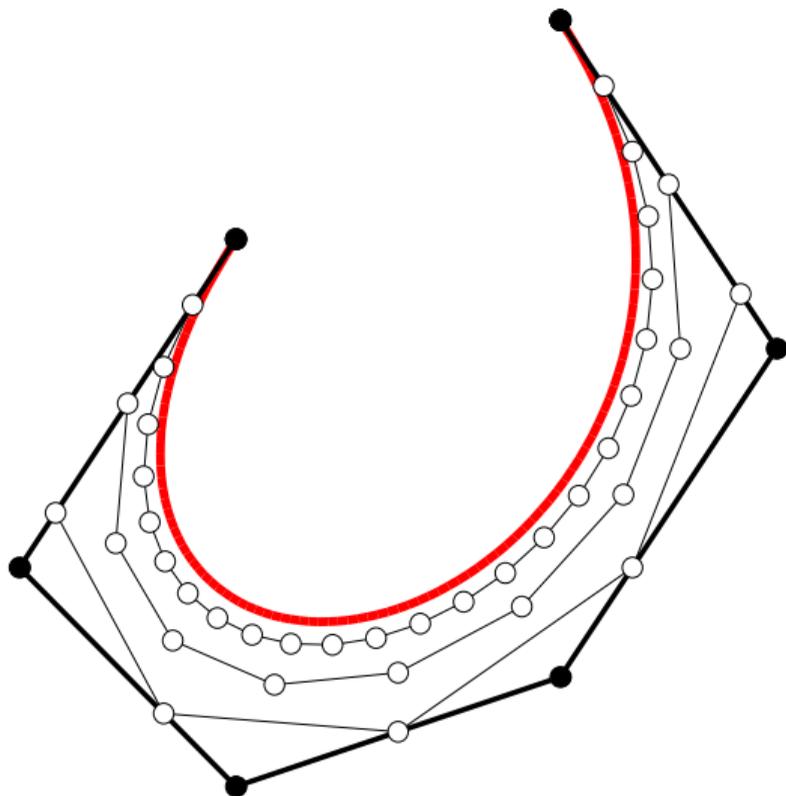
## Curvas de Bézier – elevação de grau



## Curvas de Bézier – elevação de grau



## Curvas de Bézier – elevação de grau



$n = 5 \rightarrow 6, 10, 25$

retalhos de Bézier

## Retalho bilinear

- pontos de controle = vértices de quadrilátero

$p_{00}, p_{10}, p_{01}, p_{11}$

- interpolação bilinear

$$f(u, v) = (1 - u)(1 - v) p_{00} + u(1 - v) p_{10} + (1 - u)v p_{01} + uv p_{11}$$

$$u, v \in [0, 1]$$

## Retalho bilinear

- pontos de controle = vértices de quadrilátero

$p_{00}, p_{10}, p_{01}, p_{11}$

- interpolação bilinear

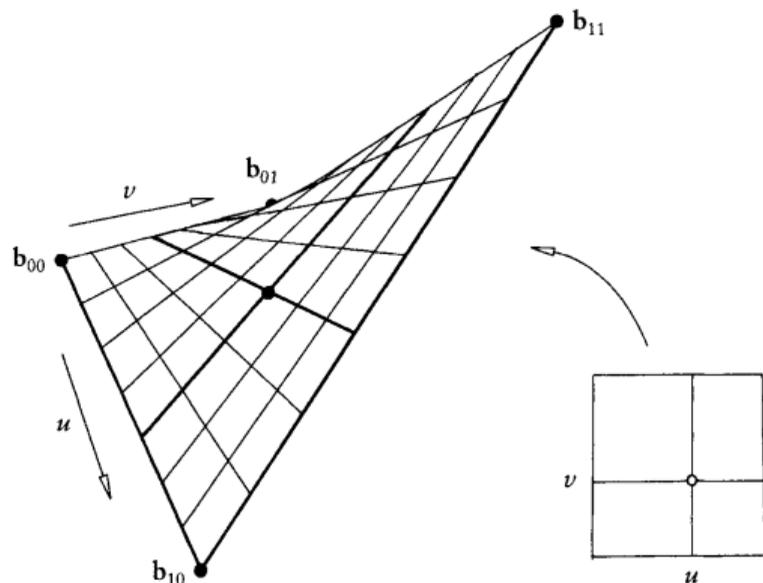
$$f(u, v) = (1 - u)(1 - v) p_{00} + u(1 - v) p_{10} + (1 - u)v p_{01} + uv p_{11}$$

$u, v \in [0, 1]$

- parabolóide hiperbólico

- superfície regrada curva

$\gamma(t) = f(t, t)$  é Bézier quadrática



## Retalho bilinear

- pontos de controle = vértices de quadrilátero

$p_{00}, p_{10}, p_{01}, p_{11}$

- interpolação bilinear

$$f(u, v) = (1 - u)(1 - v) p_{00} + u(1 - v) p_{10} + (1 - u)v p_{01} + uv p_{11}$$

$$u, v \in [0, 1]$$

- formulação matricial

$$f(u, v) = \begin{bmatrix} B_0^1(u) & B_1^1(u) \end{bmatrix} \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \begin{bmatrix} B_0^1(v) \\ B_1^1(v) \end{bmatrix}$$

$$f(u, v) = \sum_{i=0}^1 \sum_{j=0}^1 B_i^1(u) B_j^1(v) p_{ij} \quad \text{produto tensorial}$$

## Retalho de Bézier retangular

- pontos de controle = vértices de grade  $(m + 1) \times (n + 1)$  de pontos  $p_{ij}$ ,  $i = 0, \dots, m$ ,  $j = 0, \dots, n$

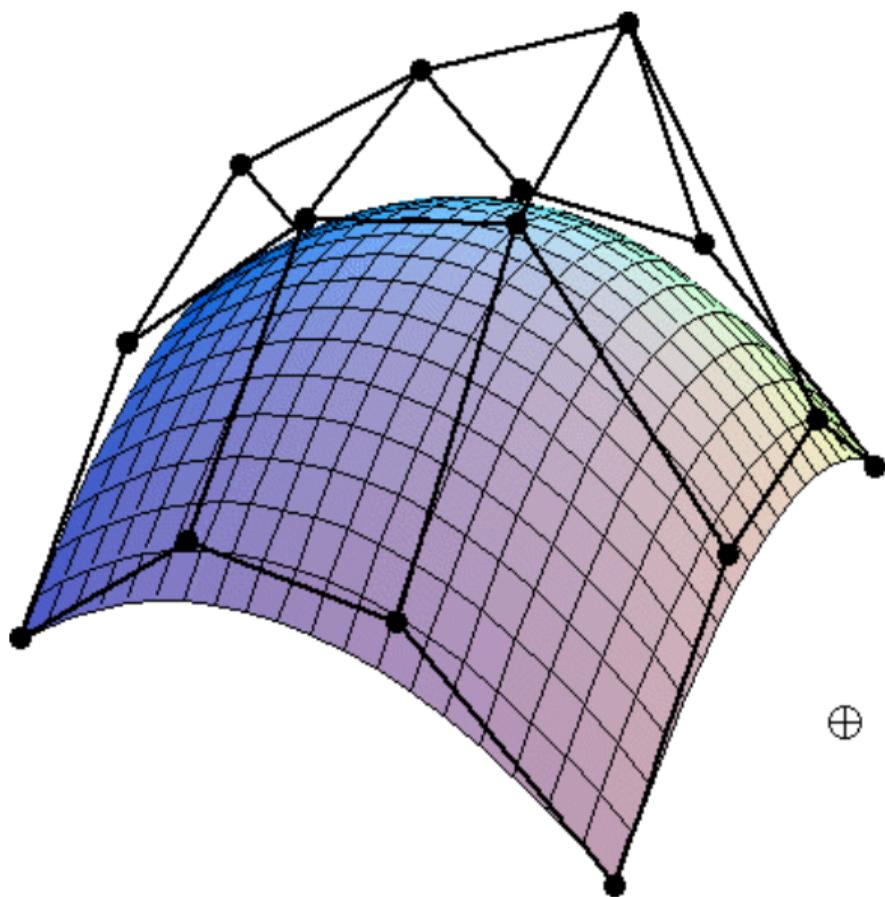
- produto tensorial de grau  $m + n$

$$f(u, v) = \sum_{i=0}^m \sum_{j=0}^n B_i^m(u) B_j^n(v) p_{ij} \quad u, v \in [0, 1]$$

- formulação matricial

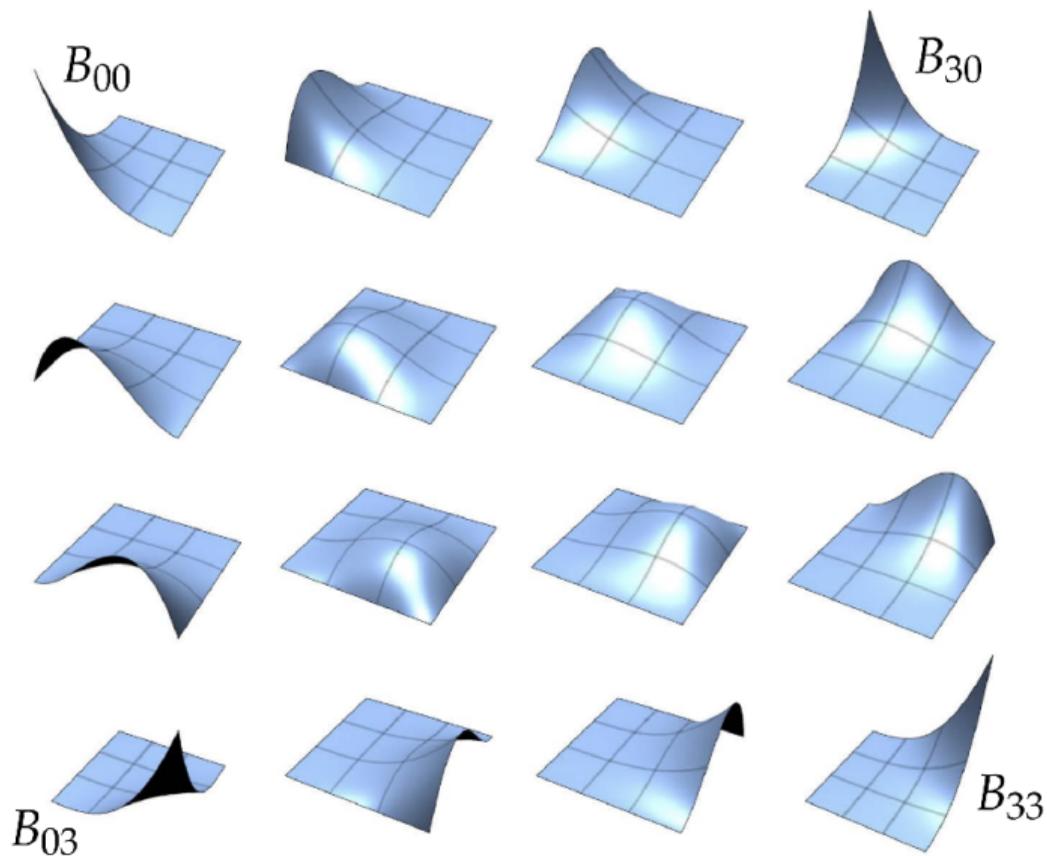
$$f(u, v) = \begin{bmatrix} B_0^m(u) & \cdots & B_m^m(u) \end{bmatrix} \begin{bmatrix} p_{00} & \cdots & p_{0n} \\ p_{m0} & \cdots & p_{mn} \end{bmatrix} \begin{bmatrix} B_0^n(v) \\ \cdots \\ B_n^n(v) \end{bmatrix}$$

## Retalho de Bézier bicúbico

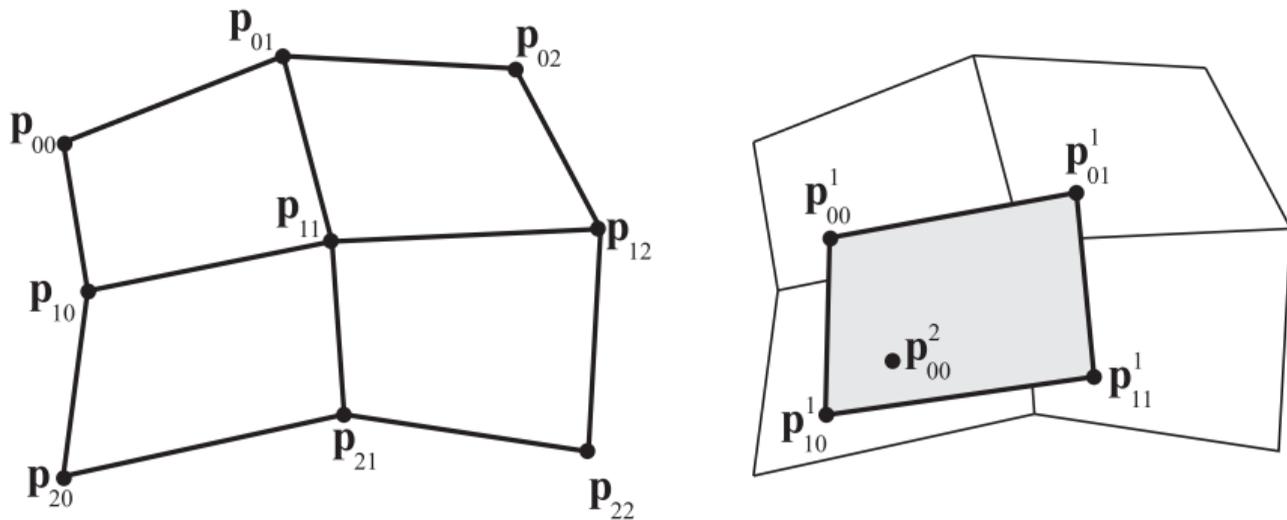


# Base de Bézier bicúbica

$$B_{ij}^{mn}(u, v) = B_i^m(u)B_j^n(v)$$

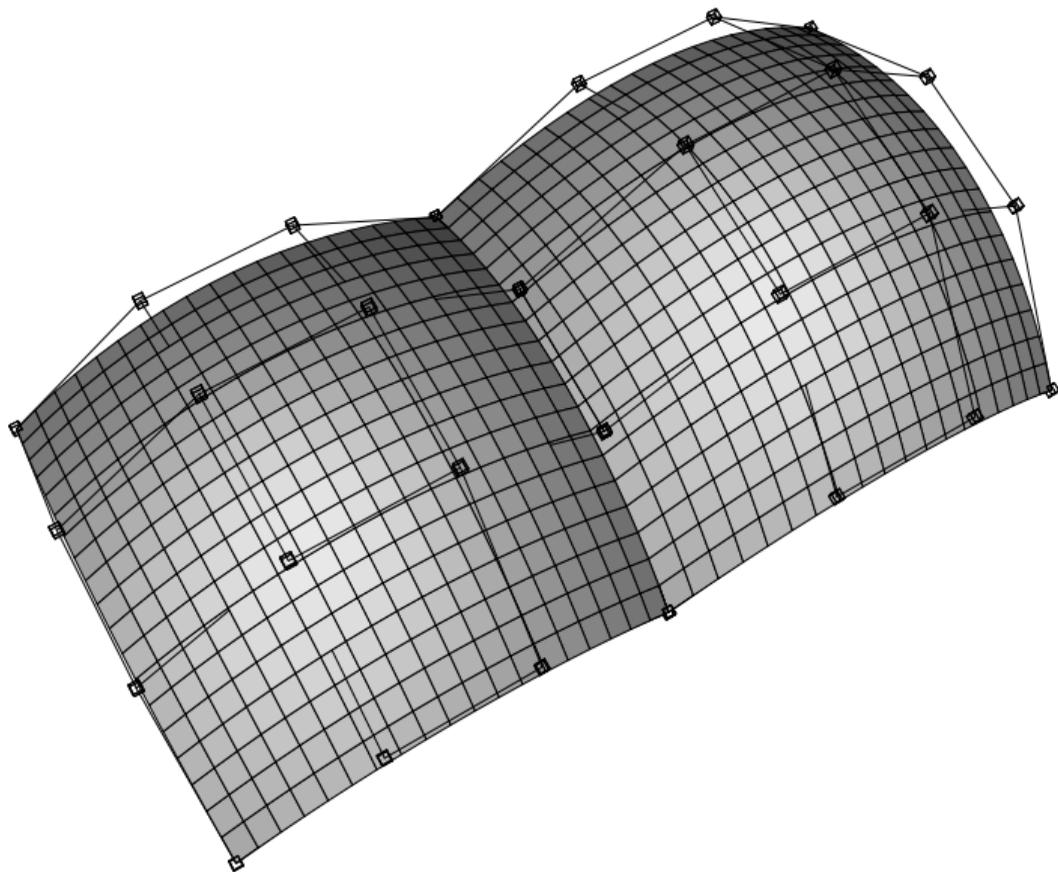


## Retalho de Bézier – avaliação via de Casteljau



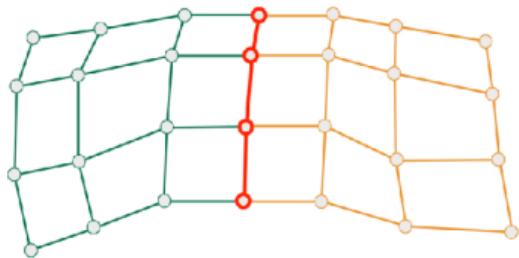
**Figure 13.15.** Left: a biquadratic Bézier surface, defined by nine control points,  $\mathbf{p}_{ij}$ . Right: To generate a point on the Bézier surface, four points  $\mathbf{p}_{ij}^1$  are first created using bilinear interpolation from the nearest control points. Finally, the point surface  $\mathbf{p}(u, v) = \mathbf{p}_{00}^2$  is bilinearly interpolated from these created points.

## Composição de retalhos de Bézier

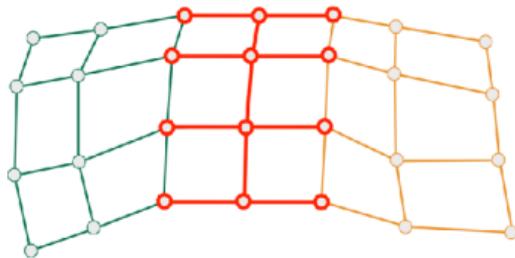


# Continuity

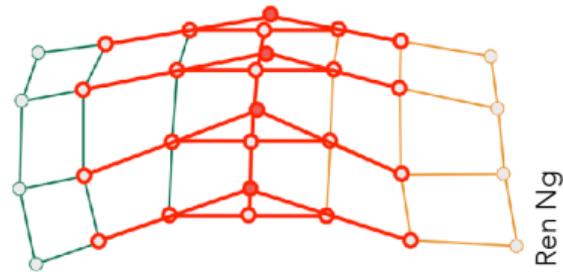
Continuity is now determined along each **boundary edge** between two patches.



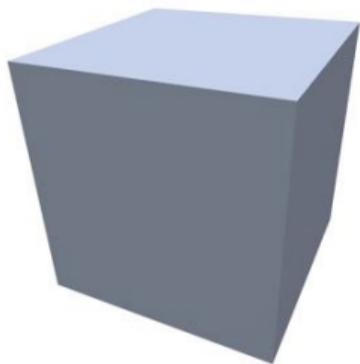
$C^0$  continuity:  
Boundary points agree



$C^1$  continuity:  
Adjacent edges equal



$C^2$  continuity:  
A-frames



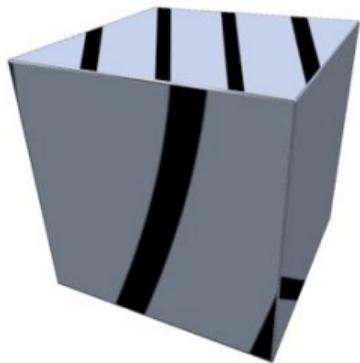
$C^0$  continuity



$C^1$  continuity



$C^2$  continuity



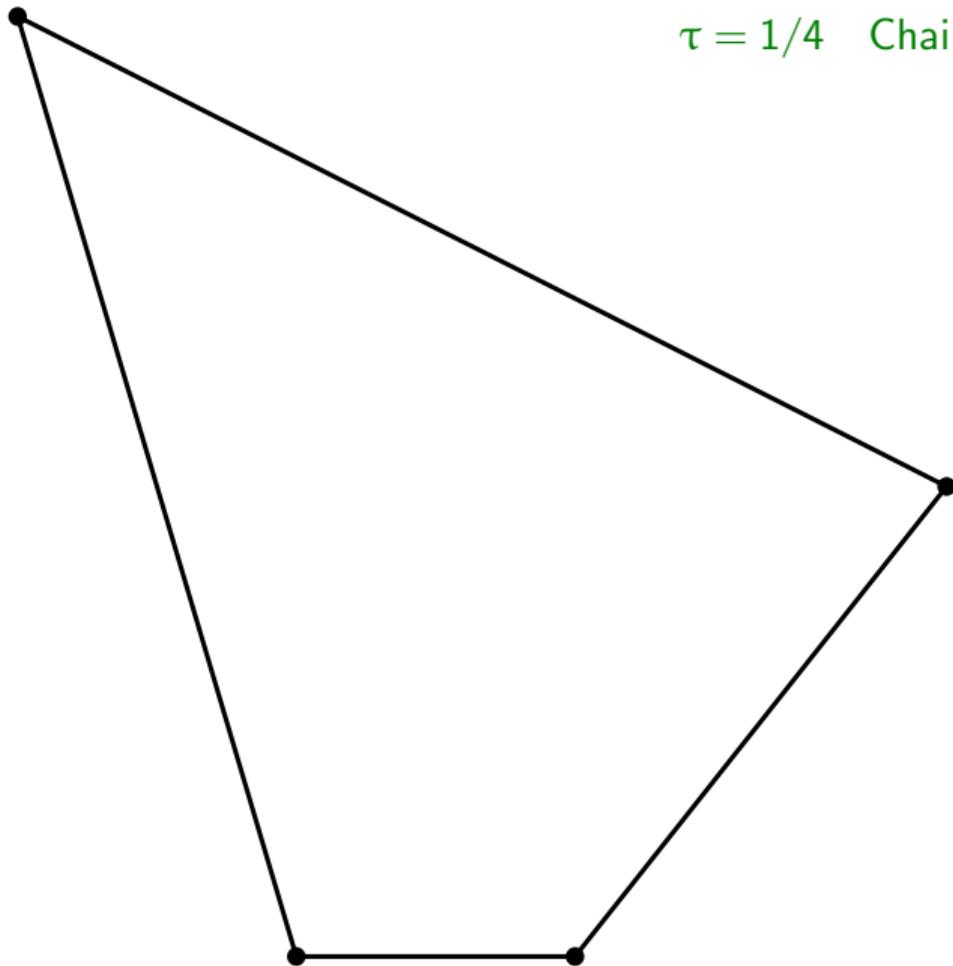
subdivisão

## Curvas de subdivisão

- de Rham (1947, 1956) cantos a  $1/3$
- de Casteljau (1963) Bézier quadrática
- Chaikin (1974) cantos a  $1/4$
- Riesenfeld (1975) spline de Bézier quadráticas
- de Boor (1987) Comer cutting always works

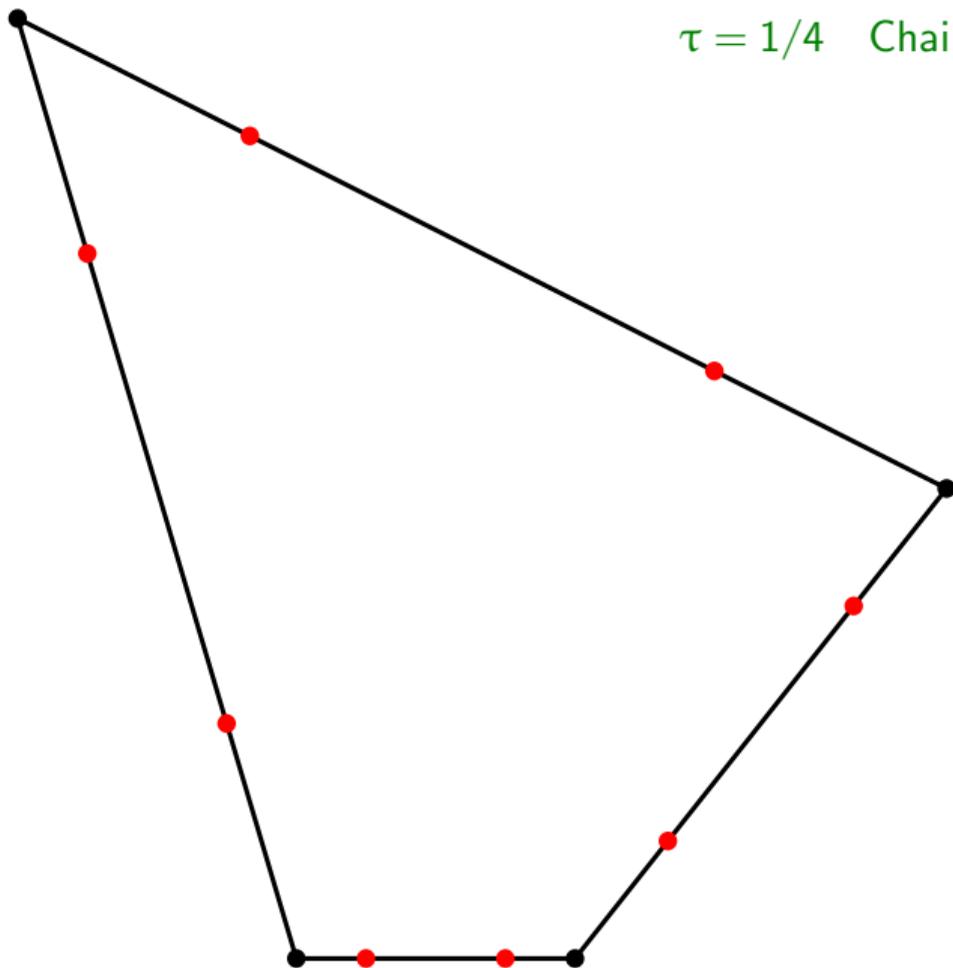
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



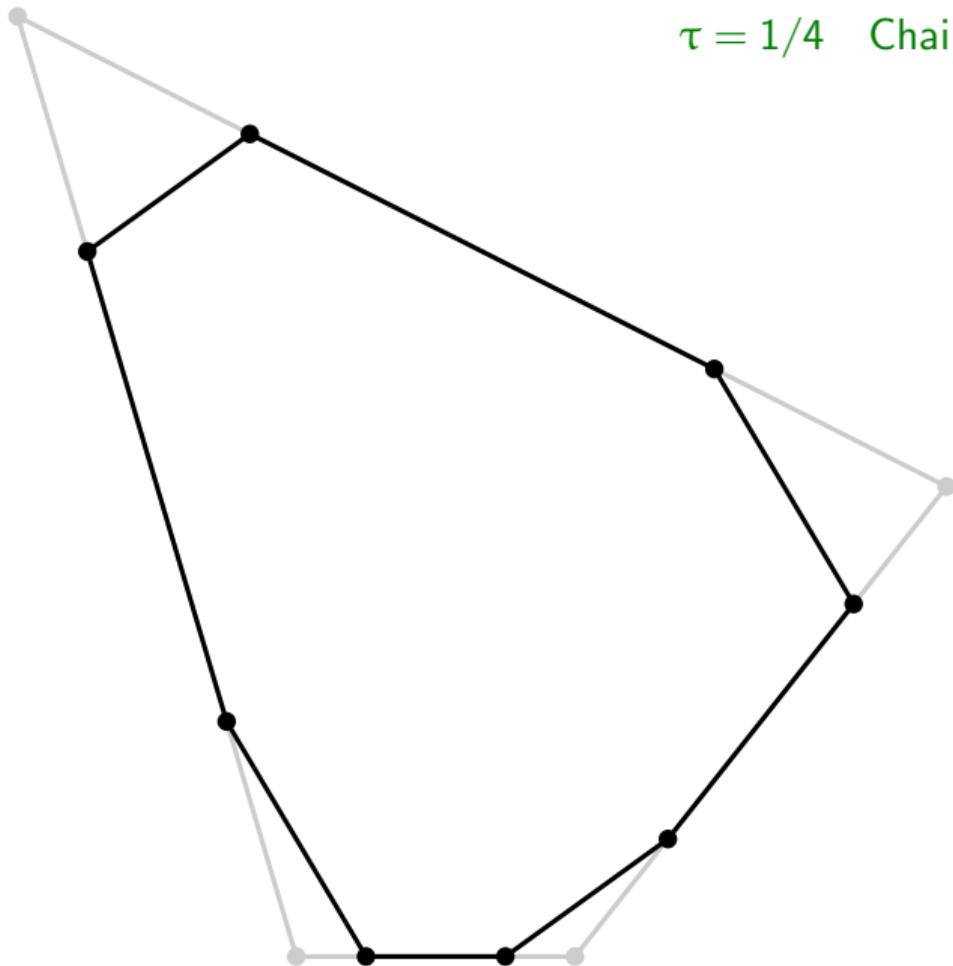
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



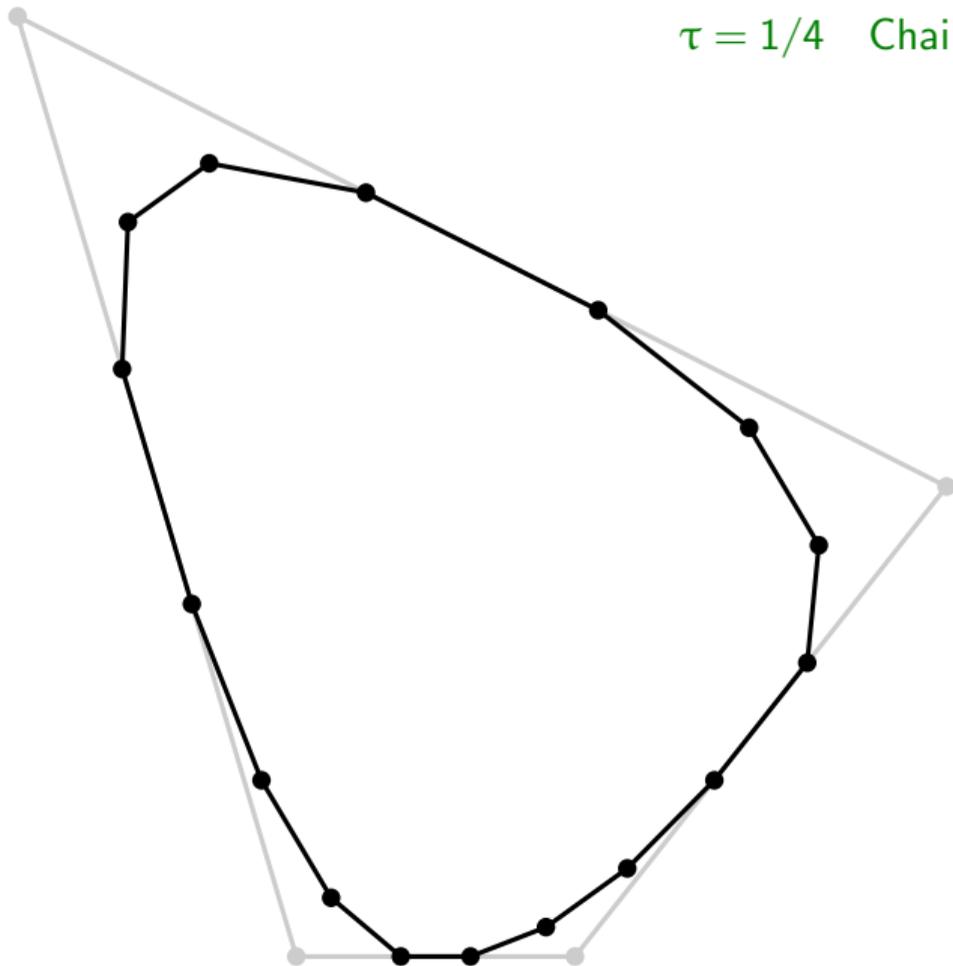
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



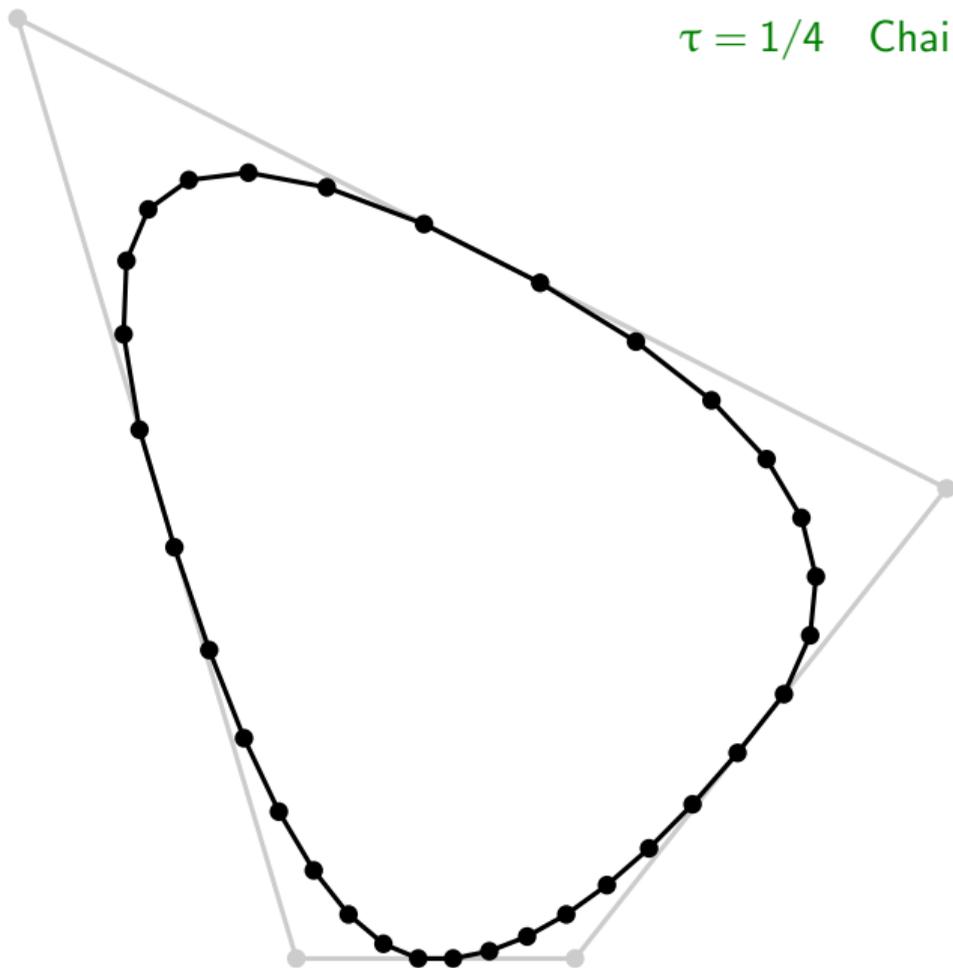
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



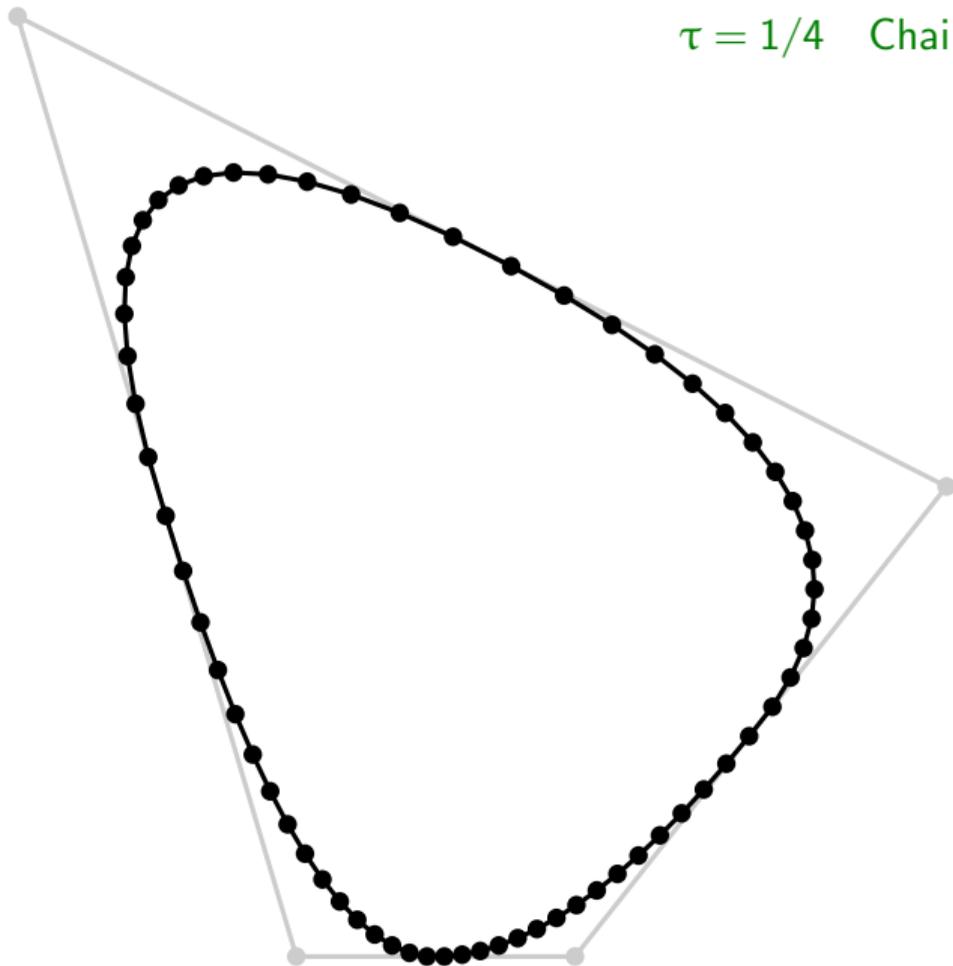
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



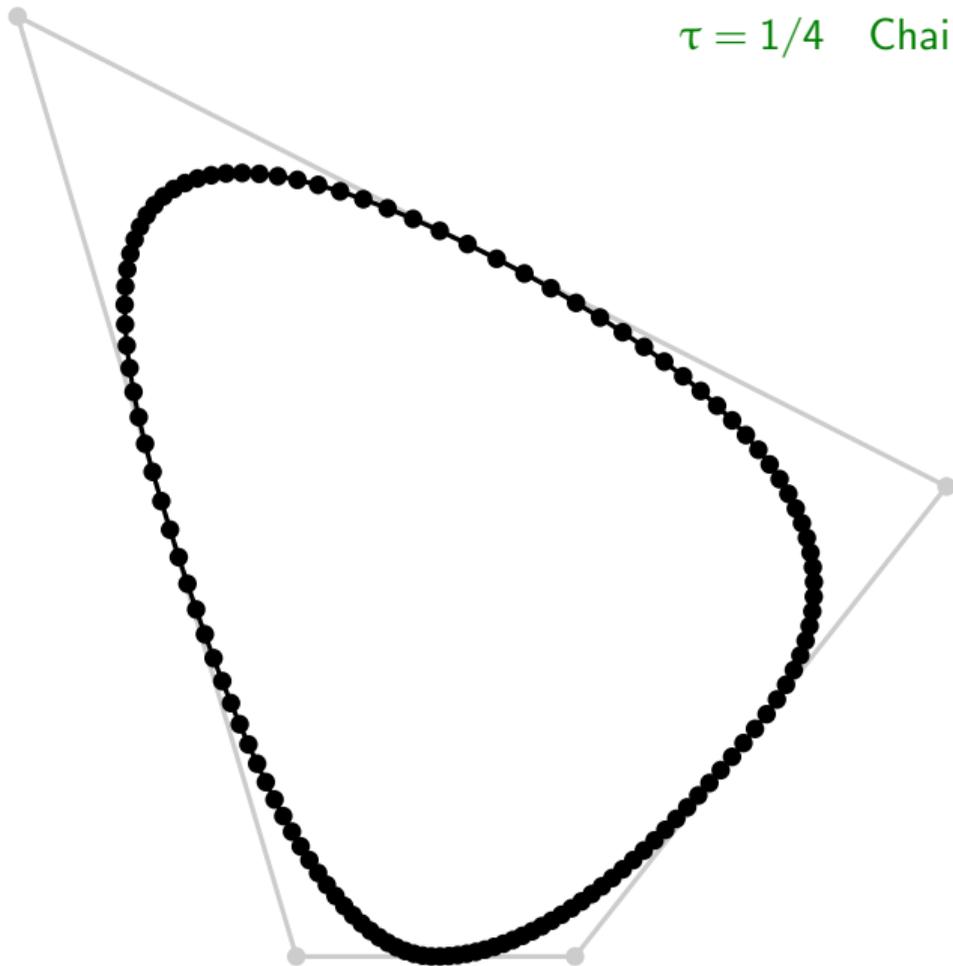
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



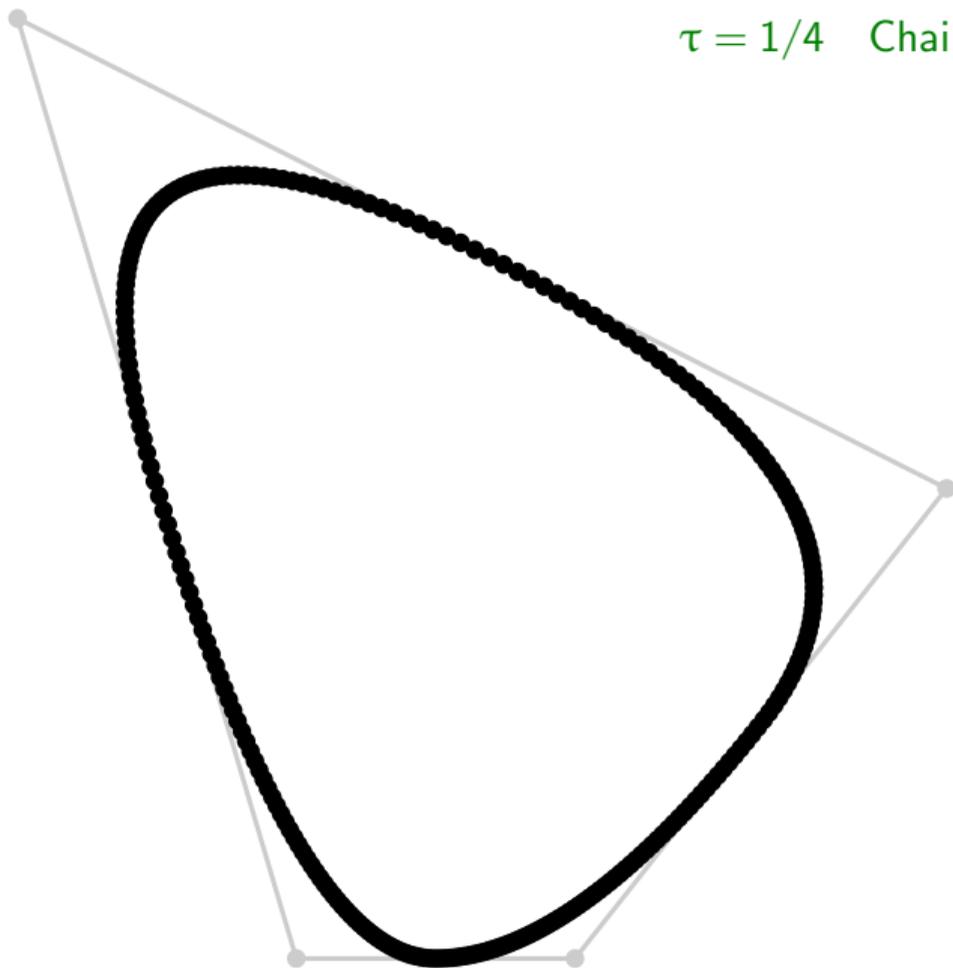
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



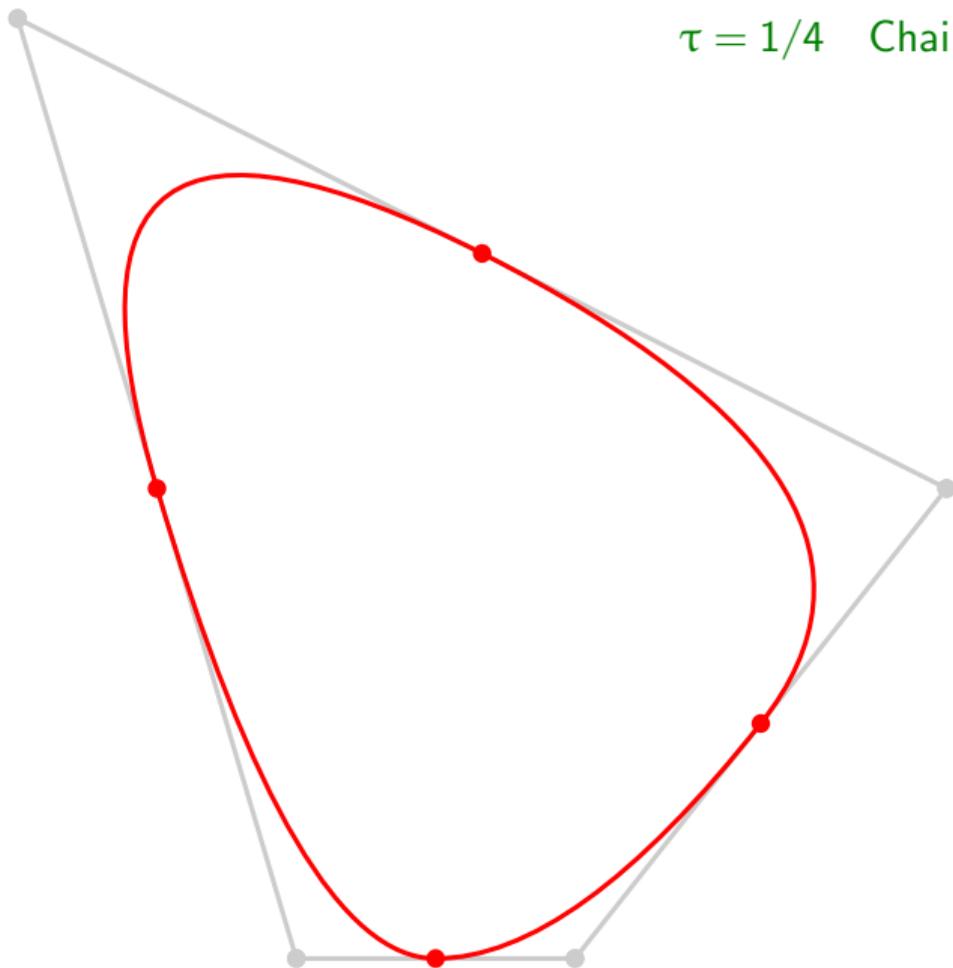
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



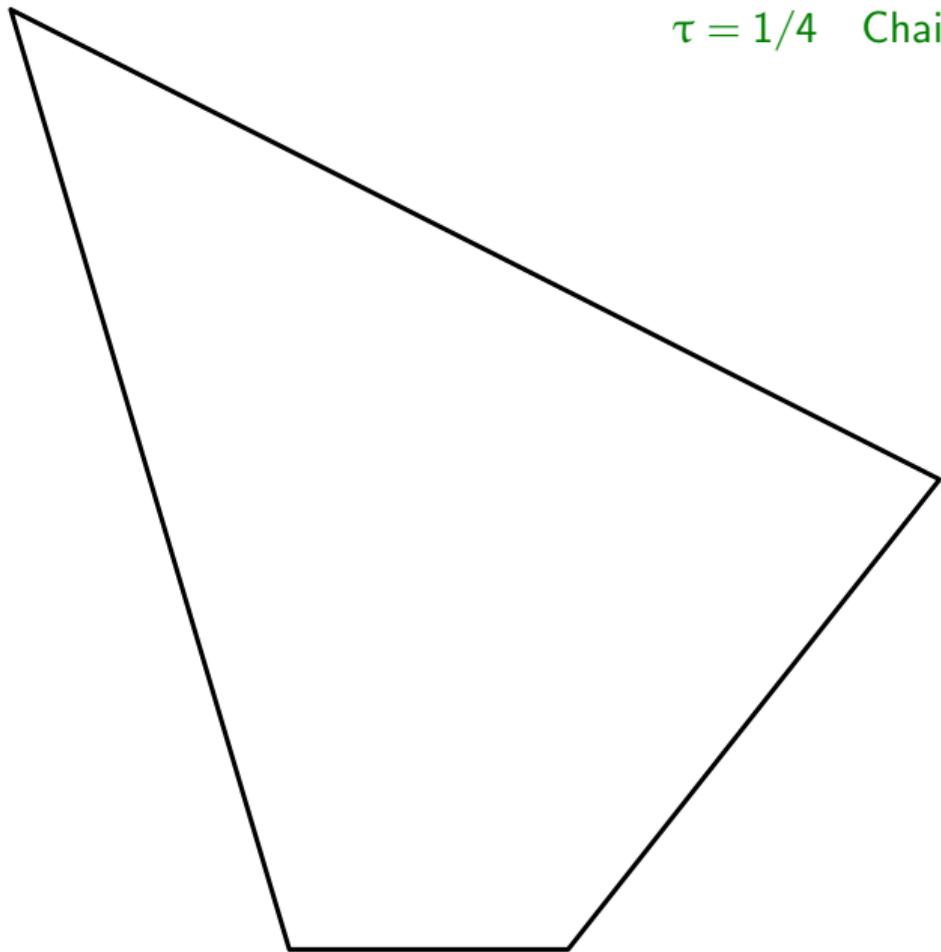
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



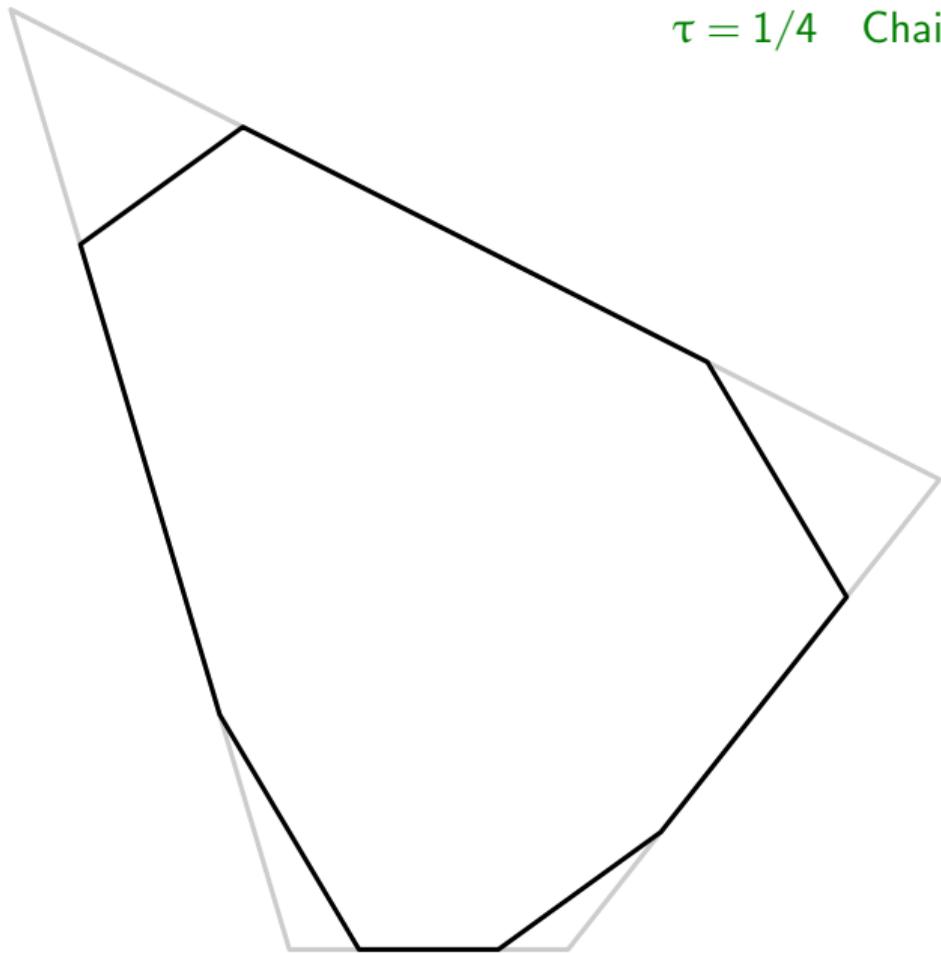
Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



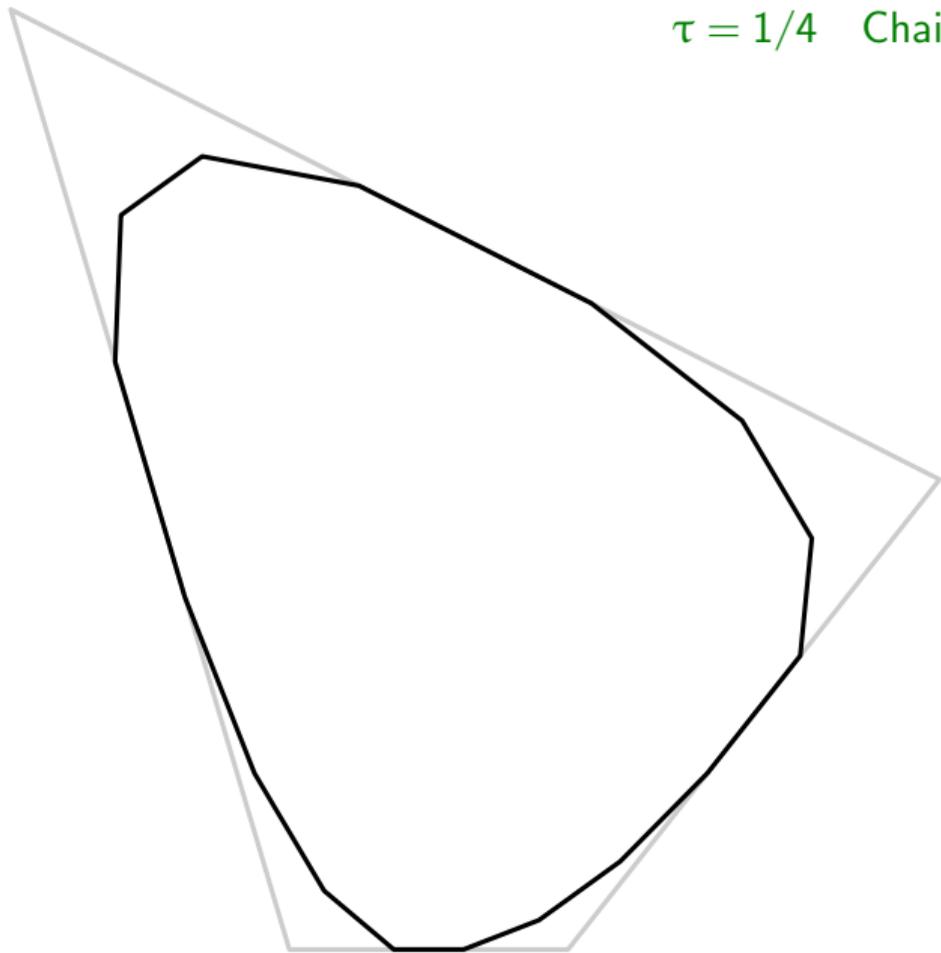
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



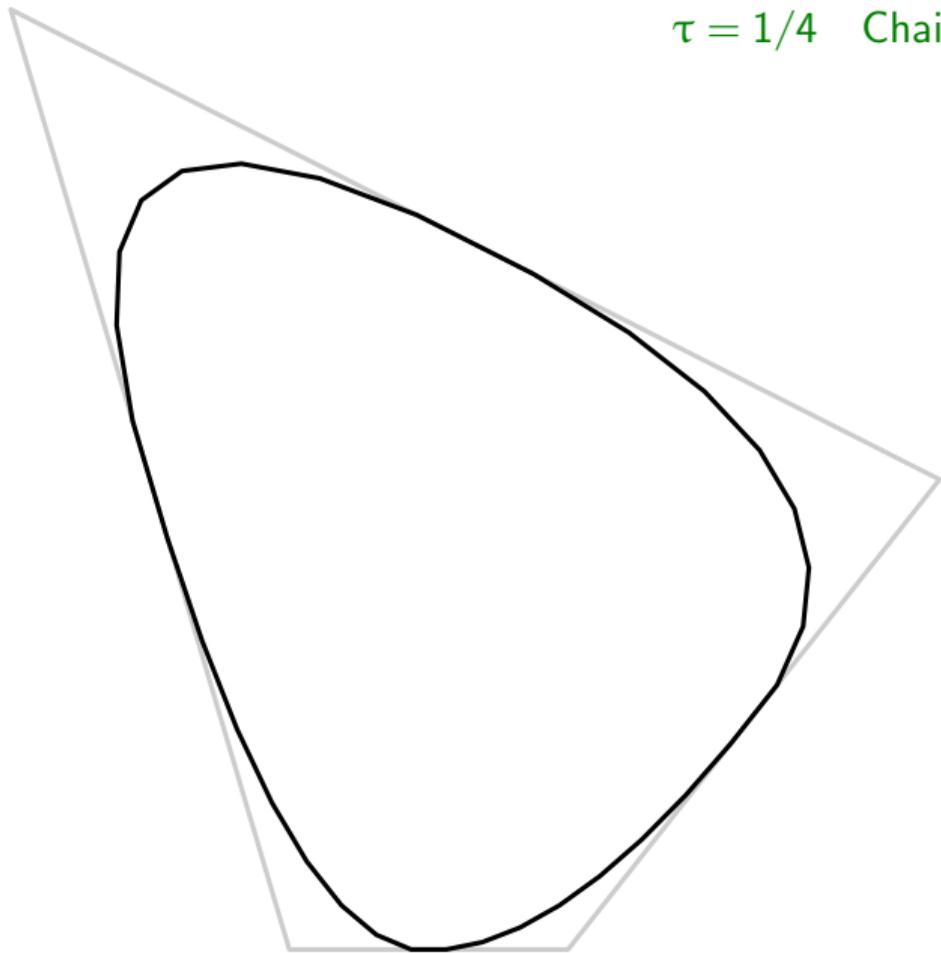
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



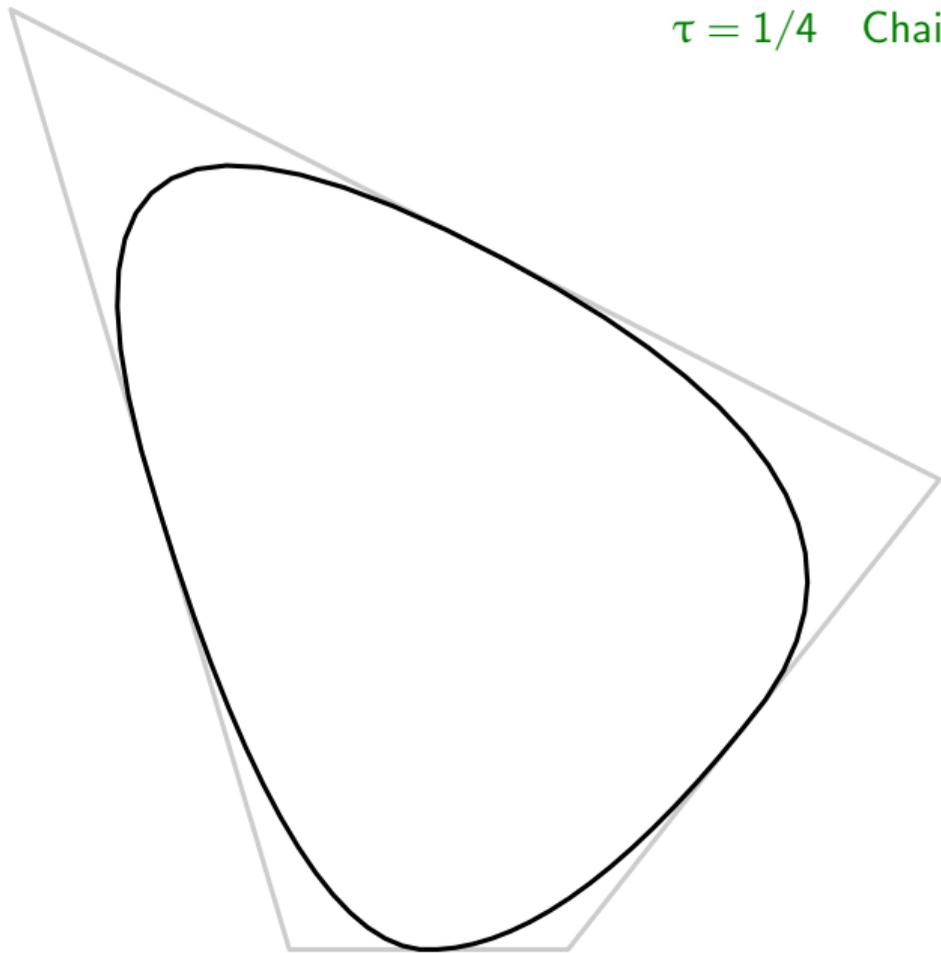
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



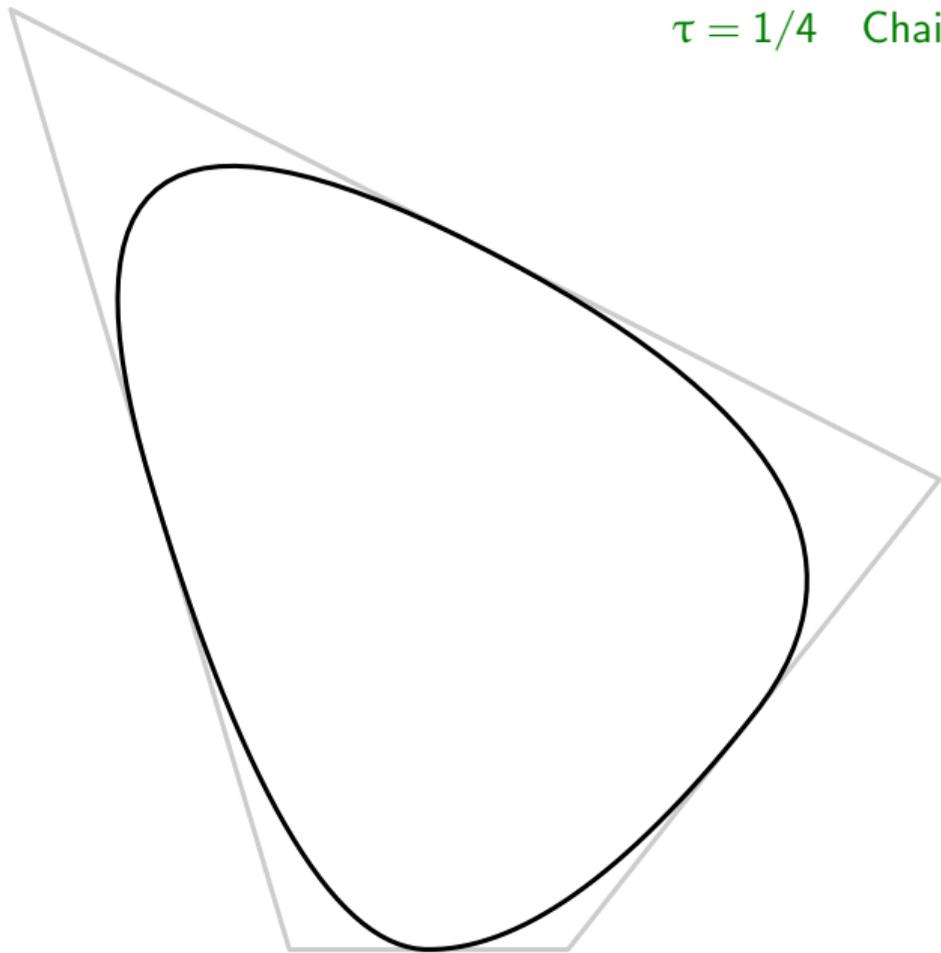
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



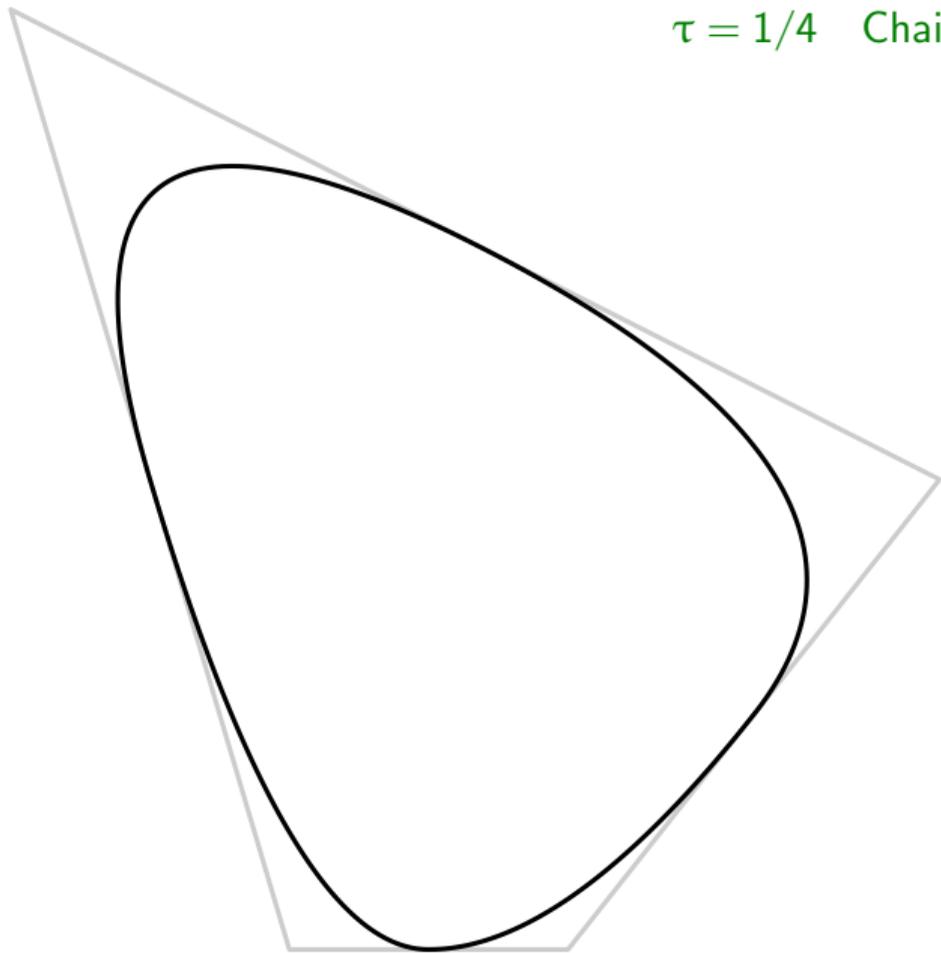
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



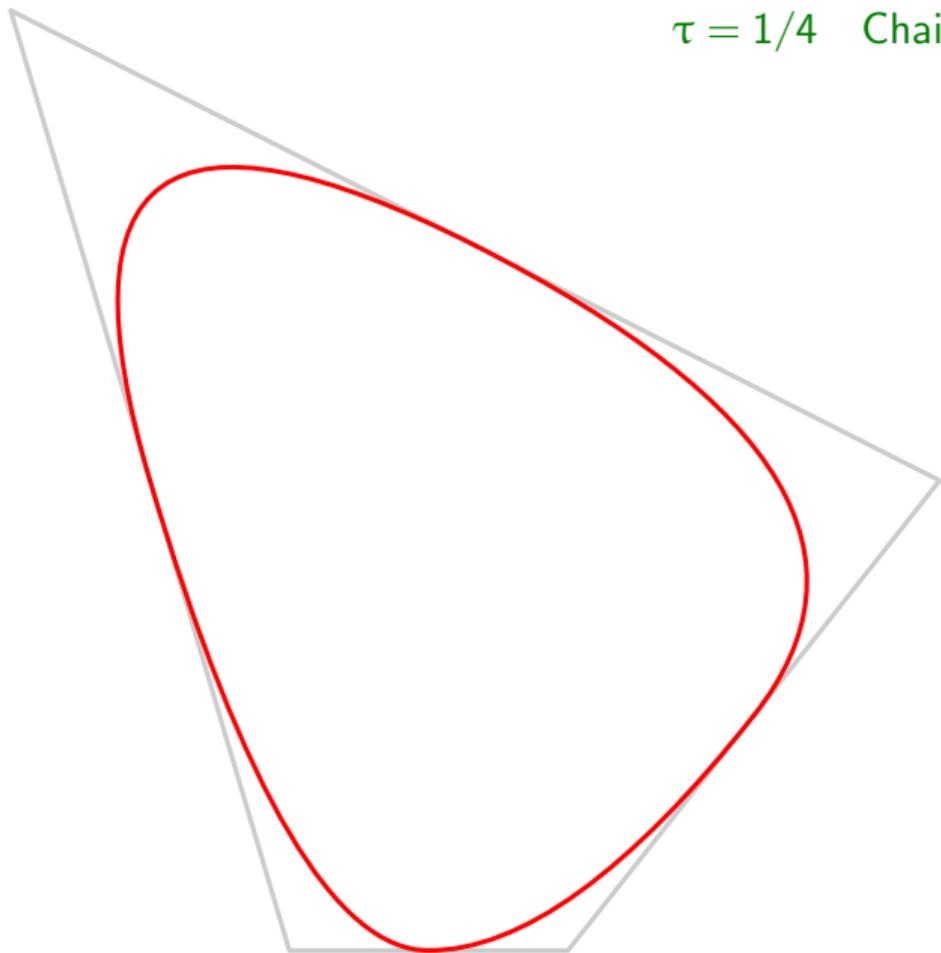
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



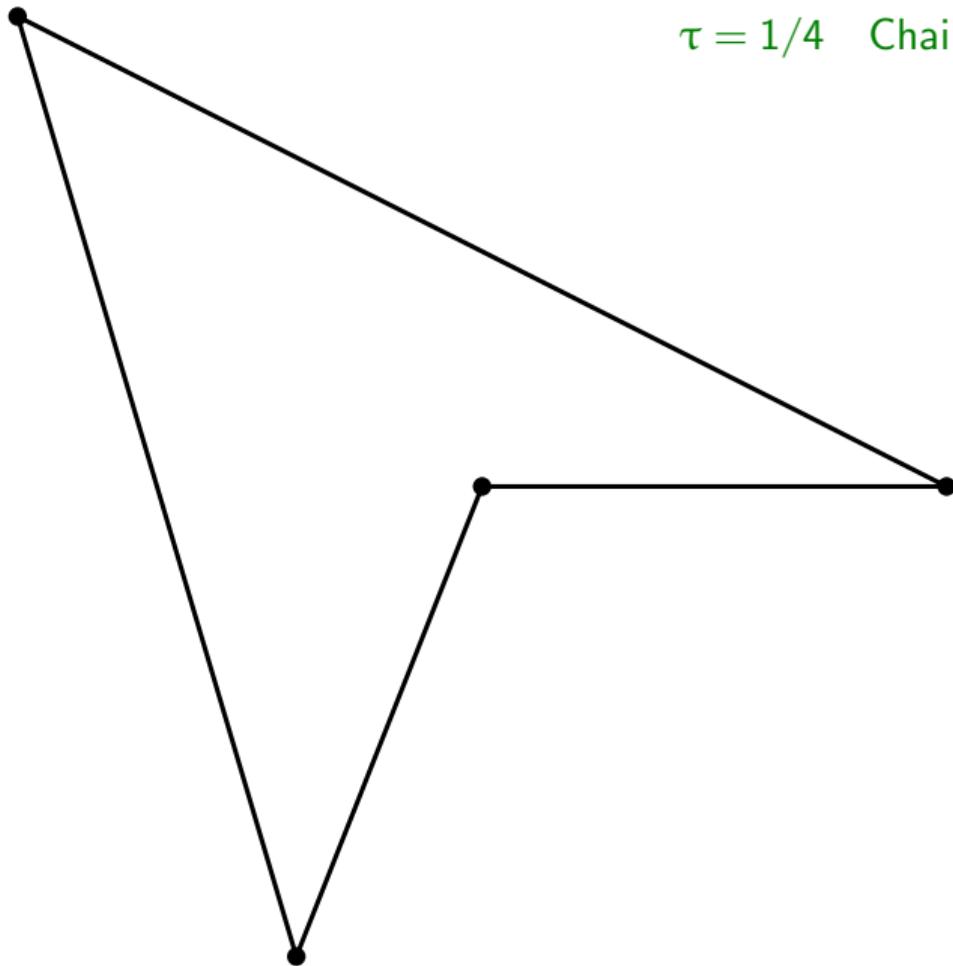
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



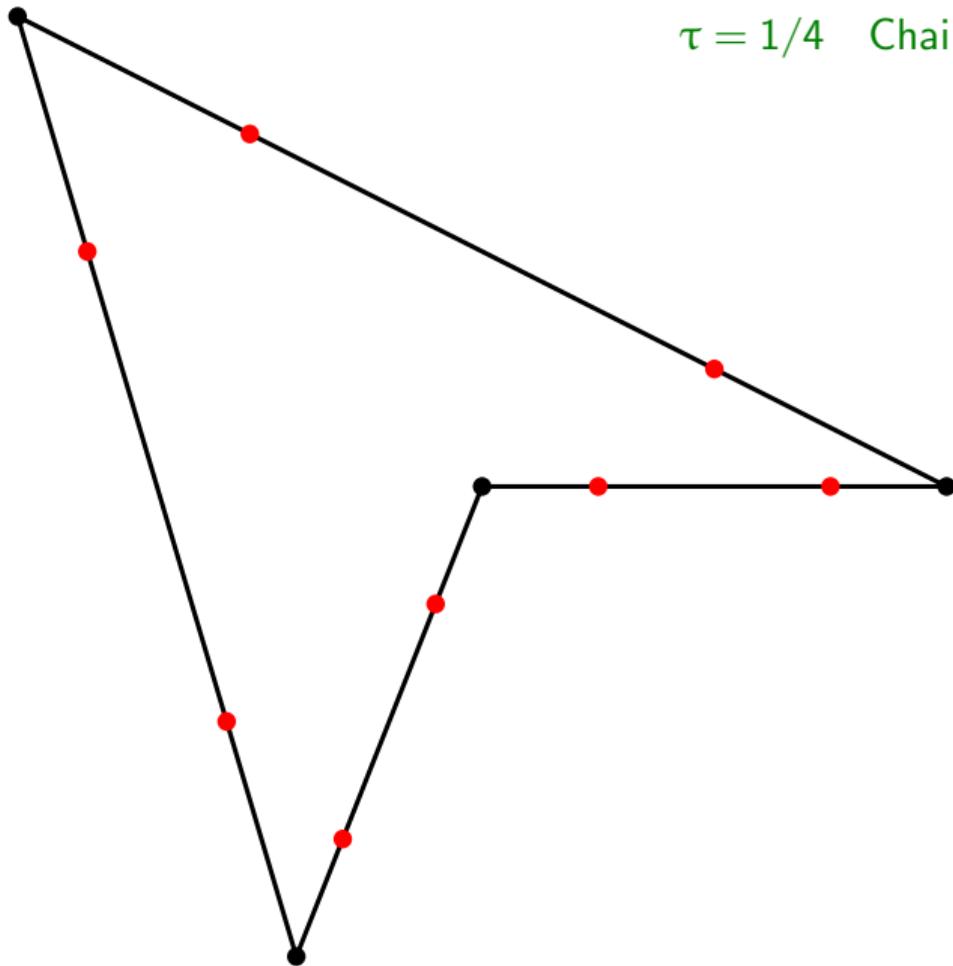
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



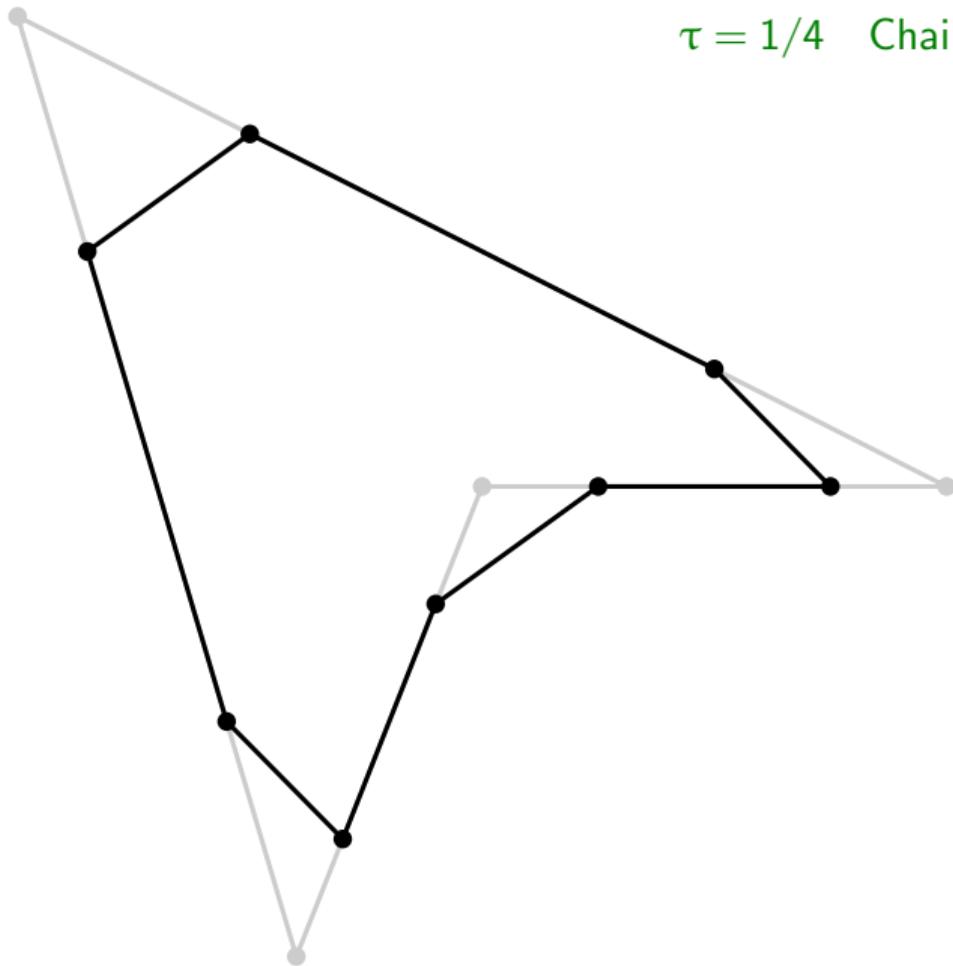
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



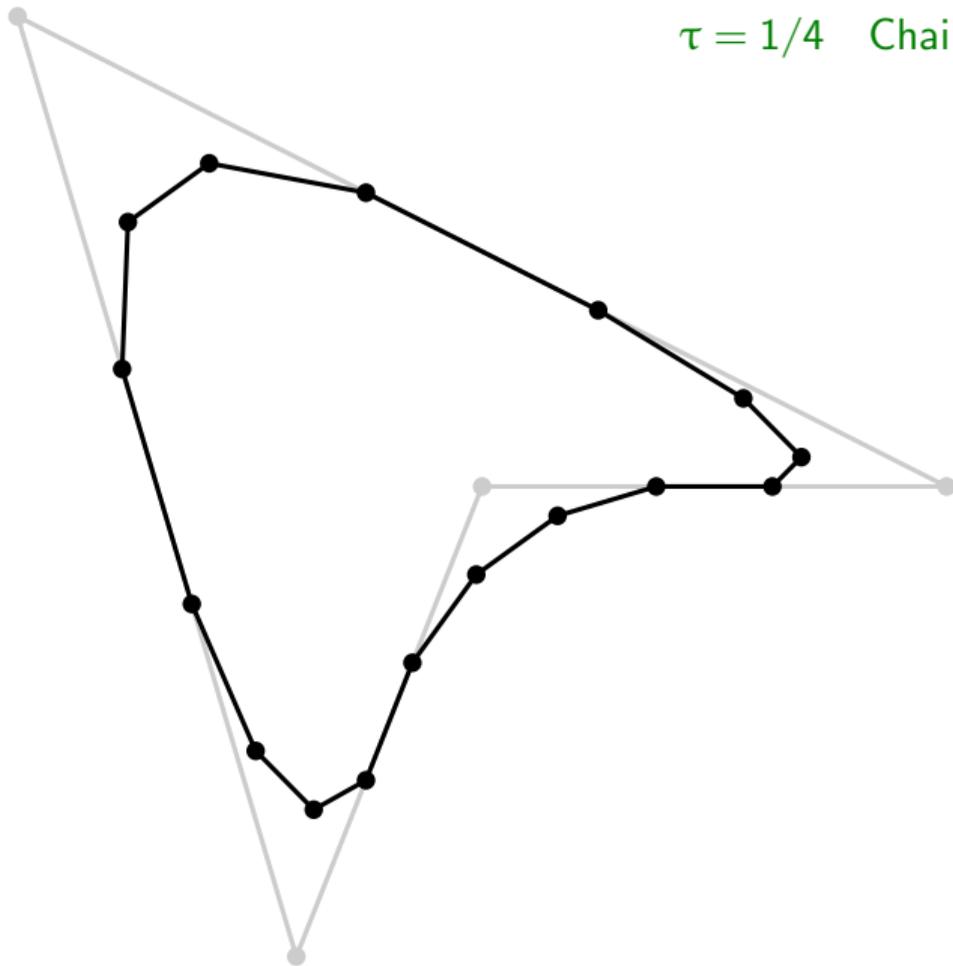
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



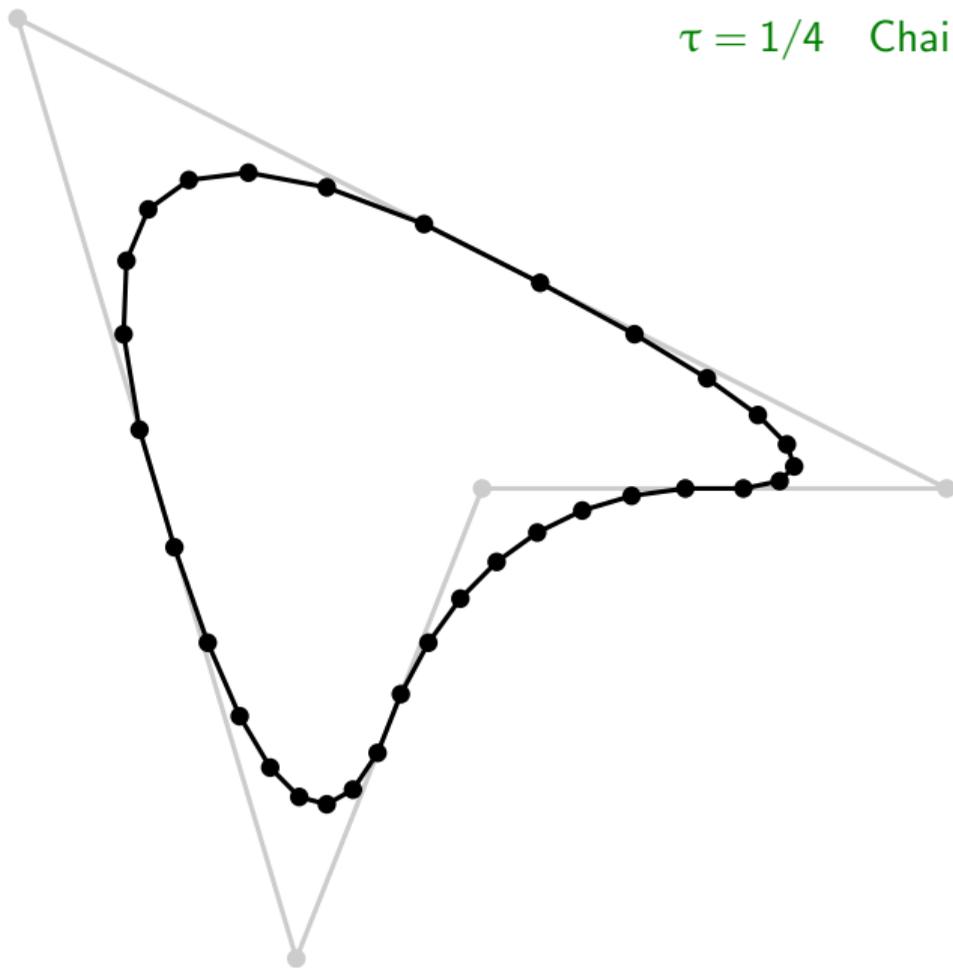
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



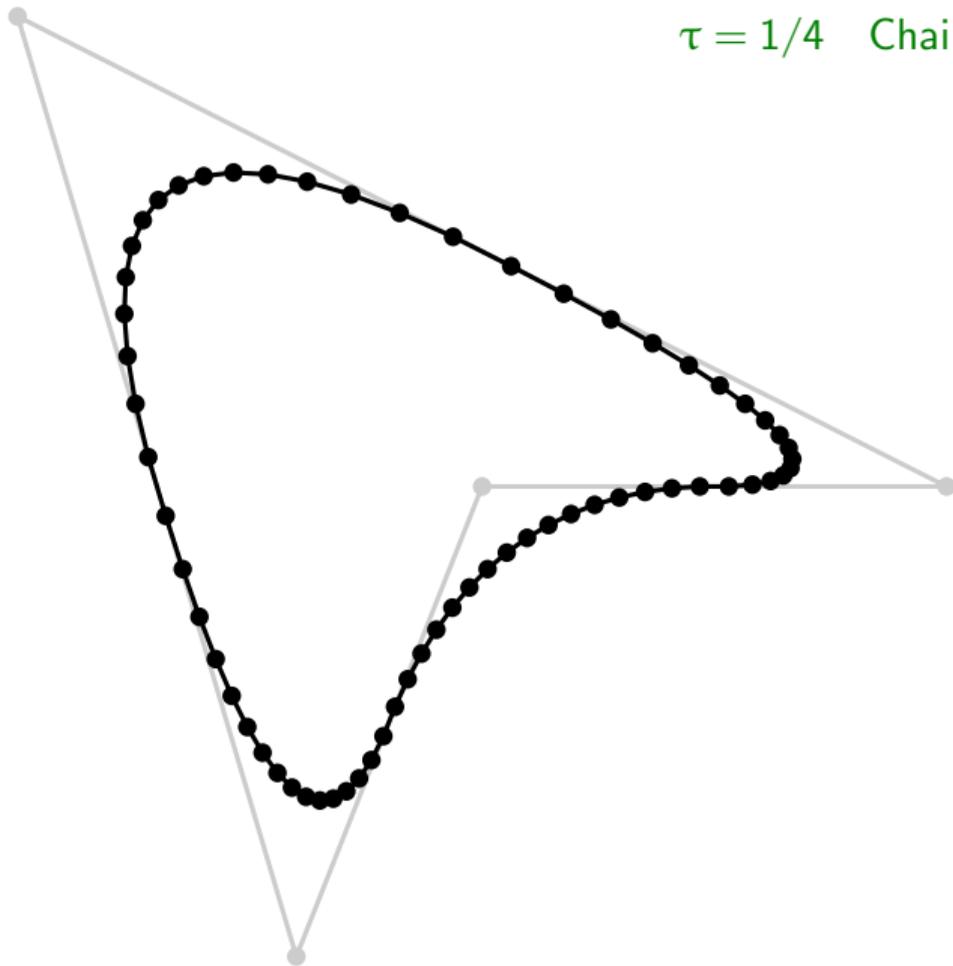
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



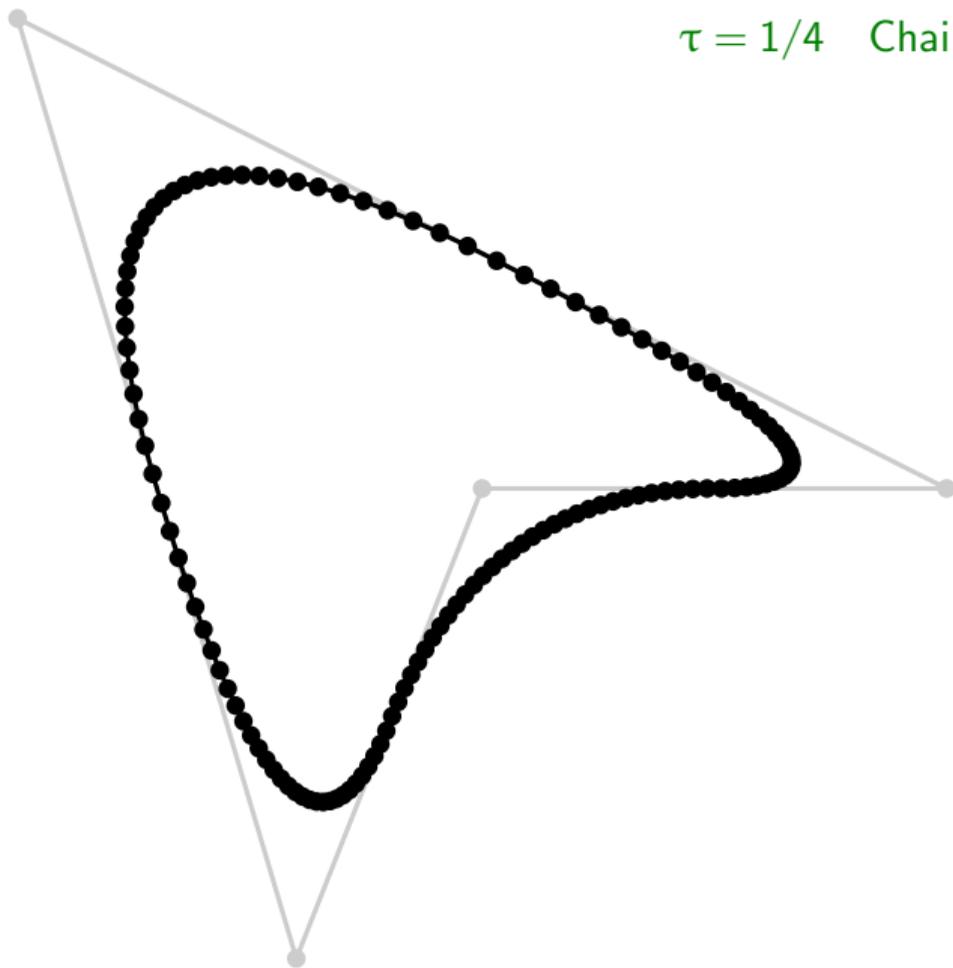
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



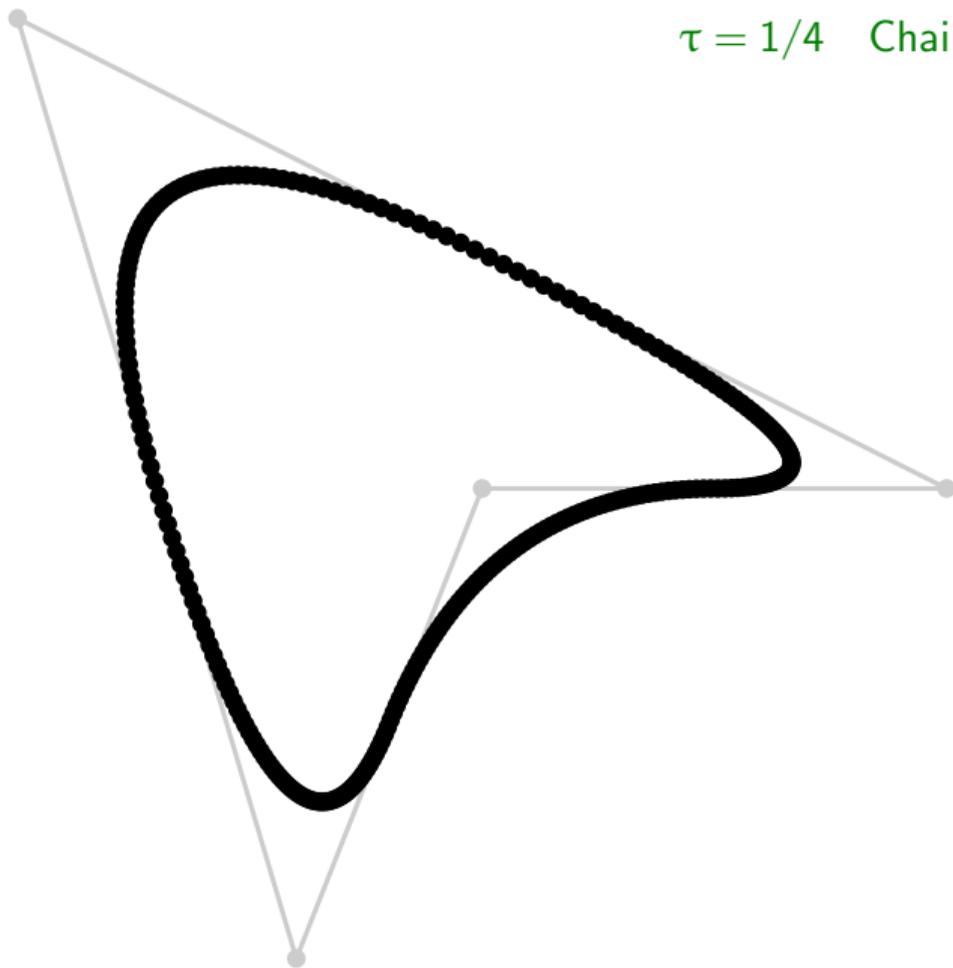
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



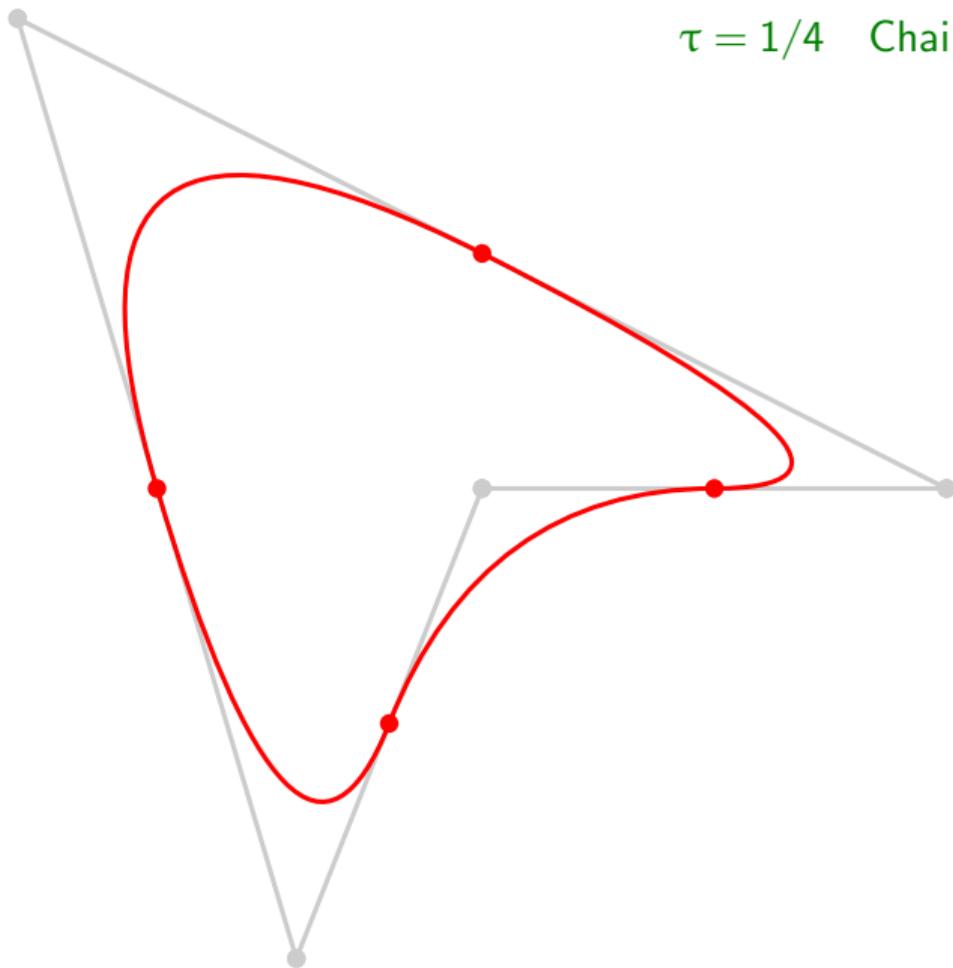
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



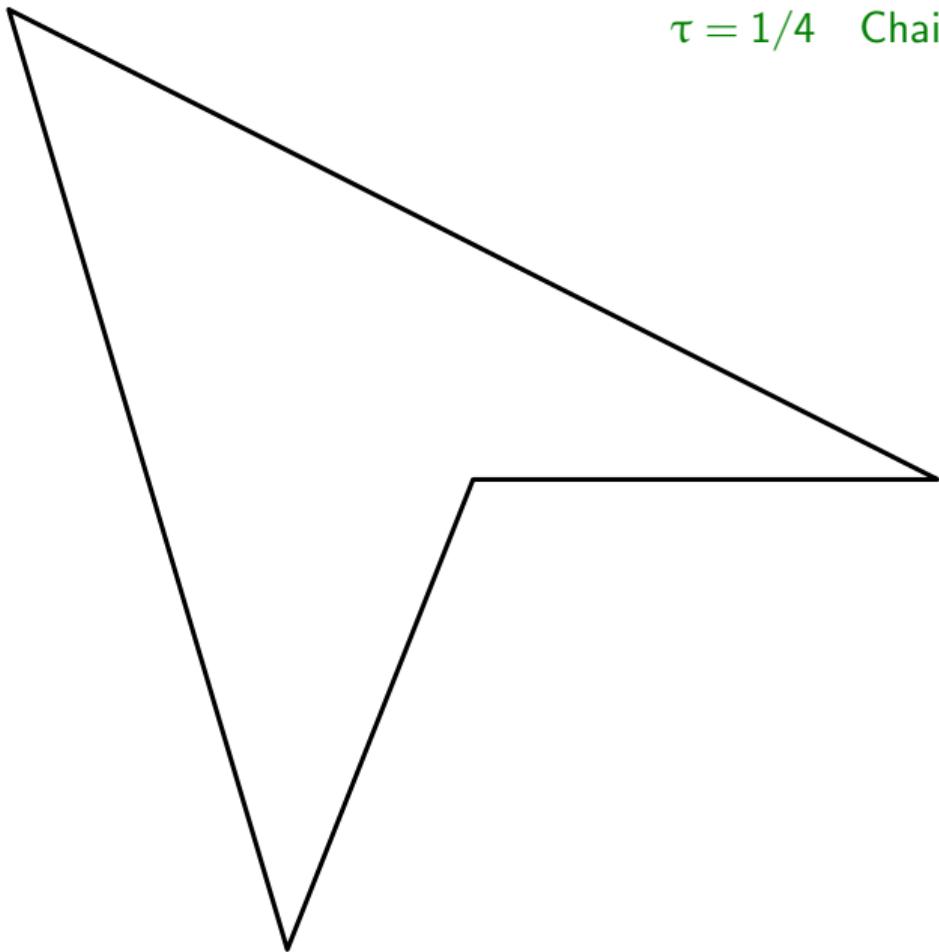
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



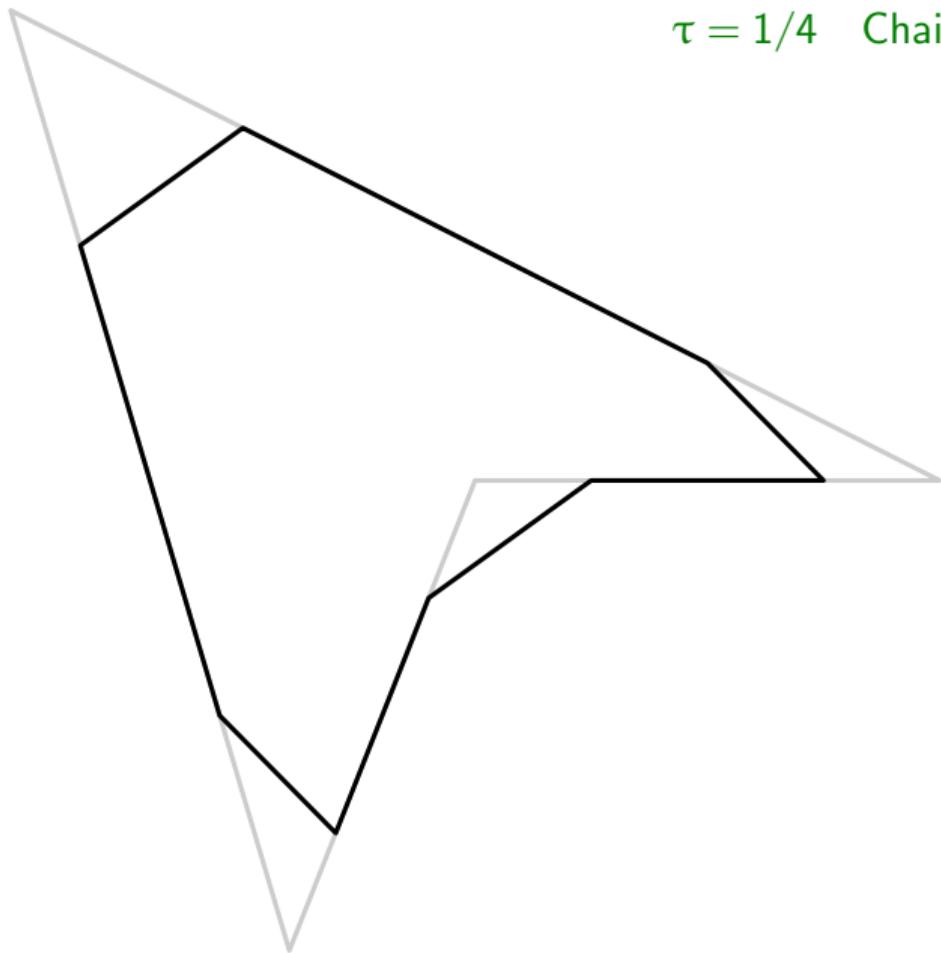
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



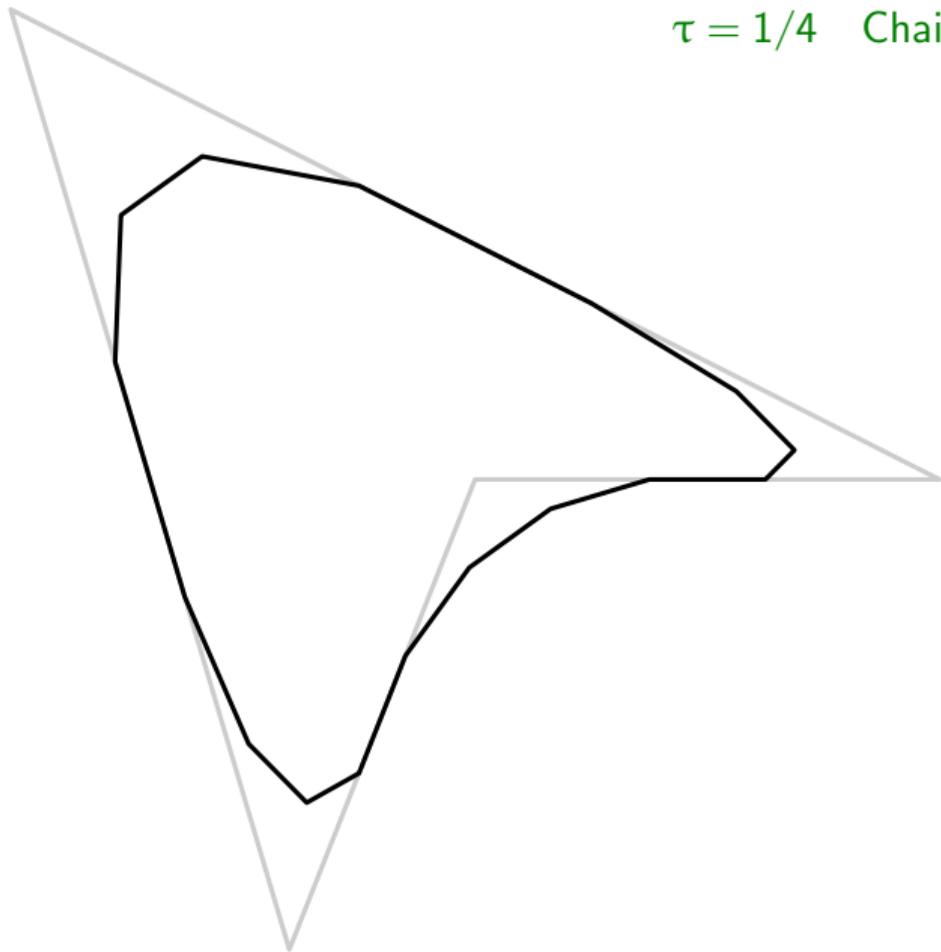
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



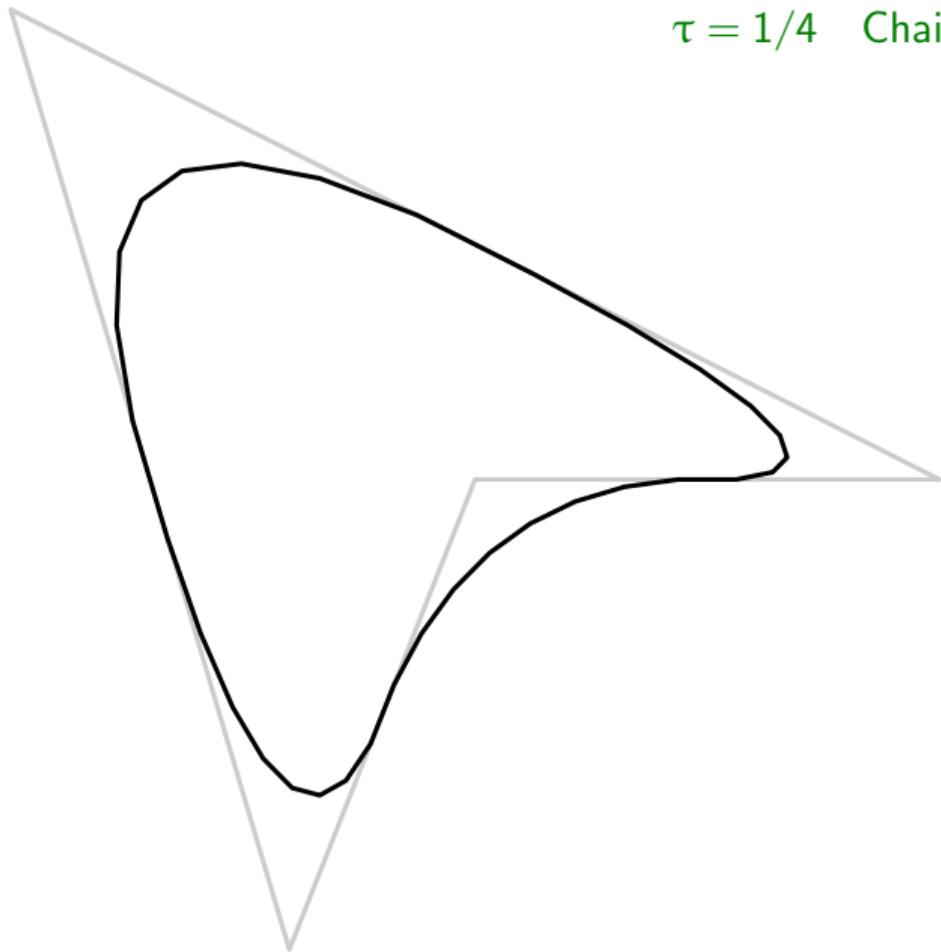
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



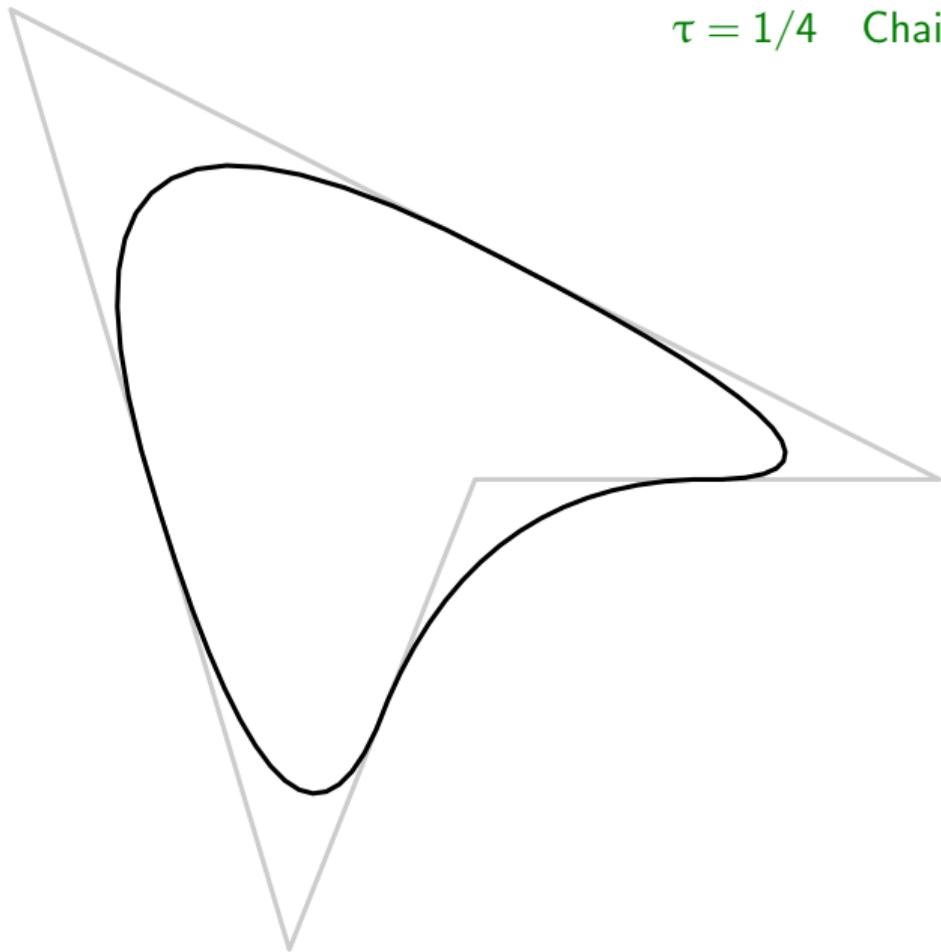
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



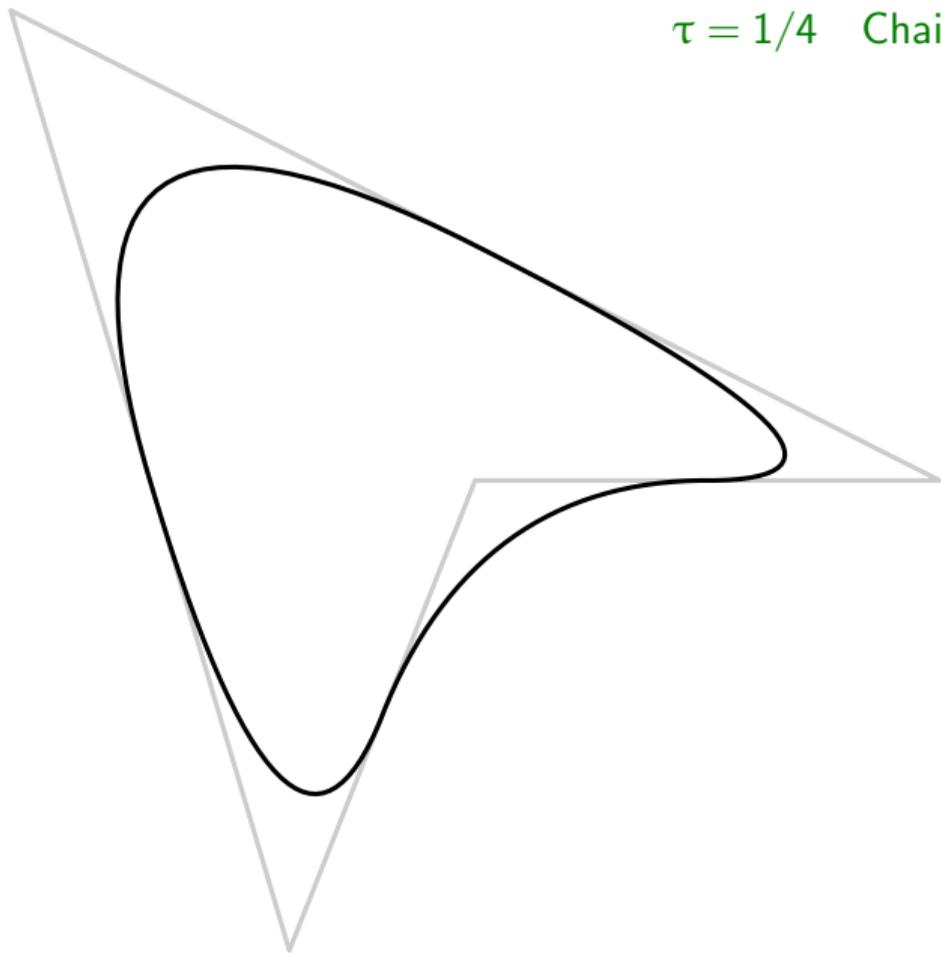
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



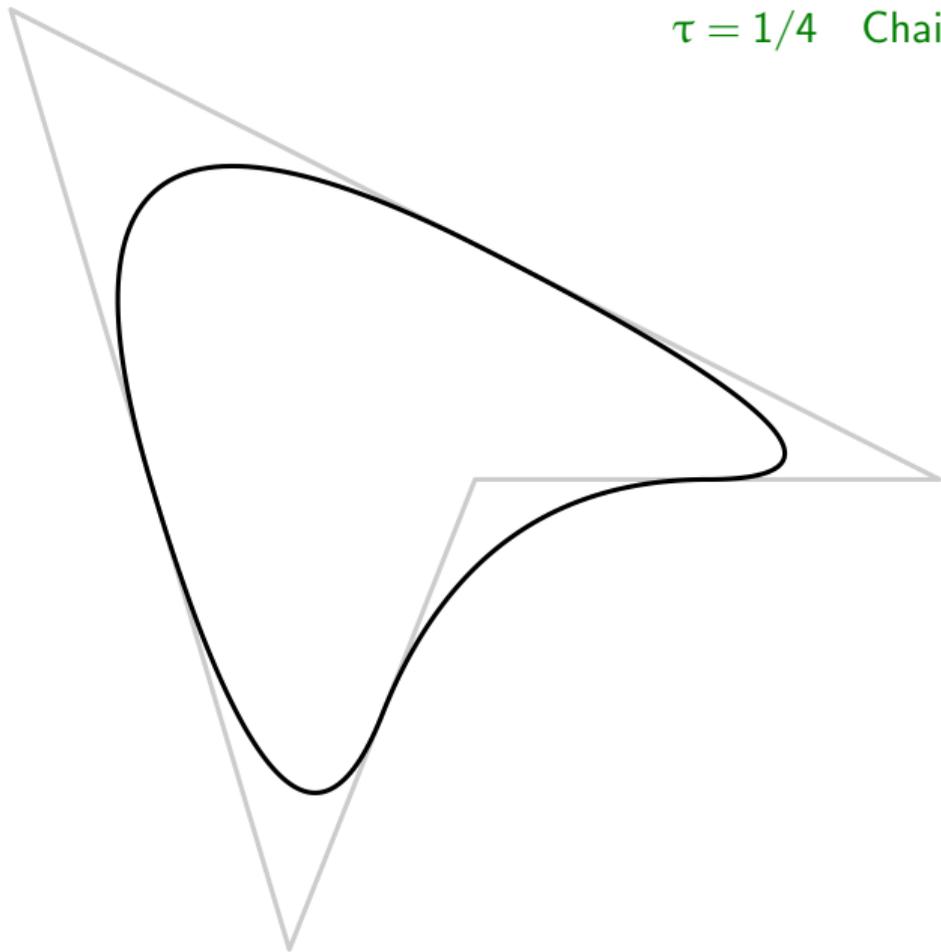
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



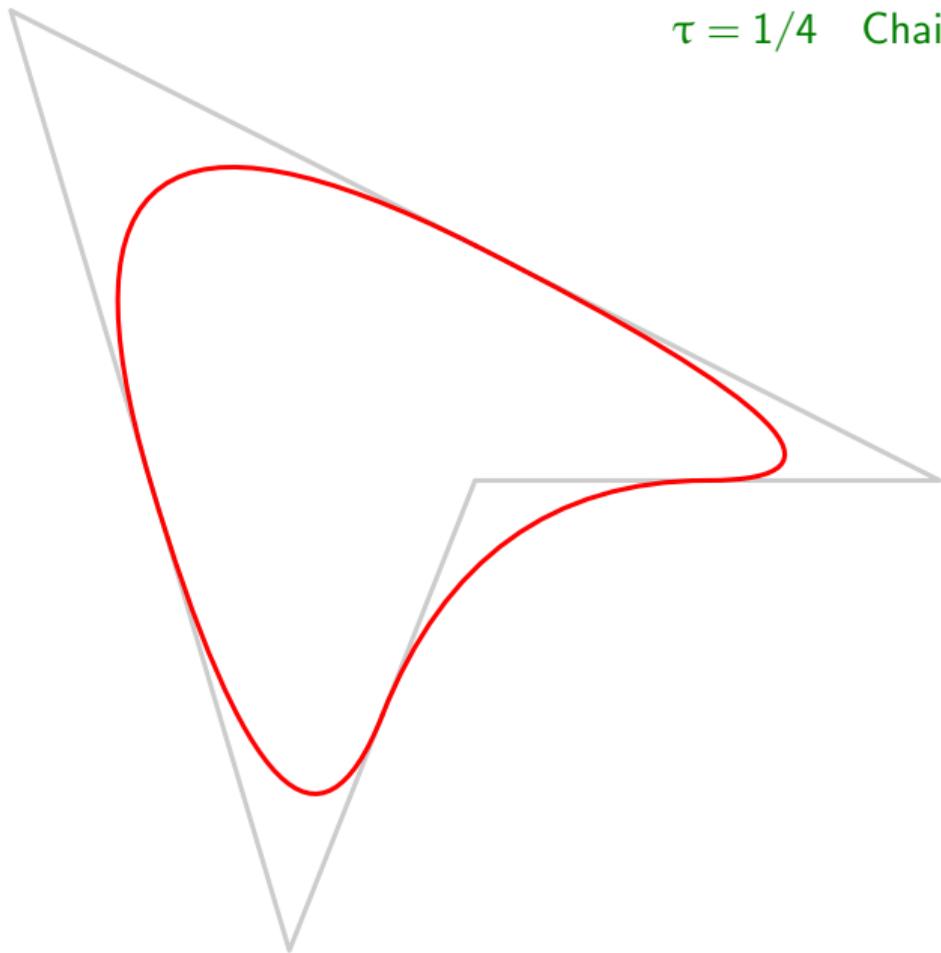
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



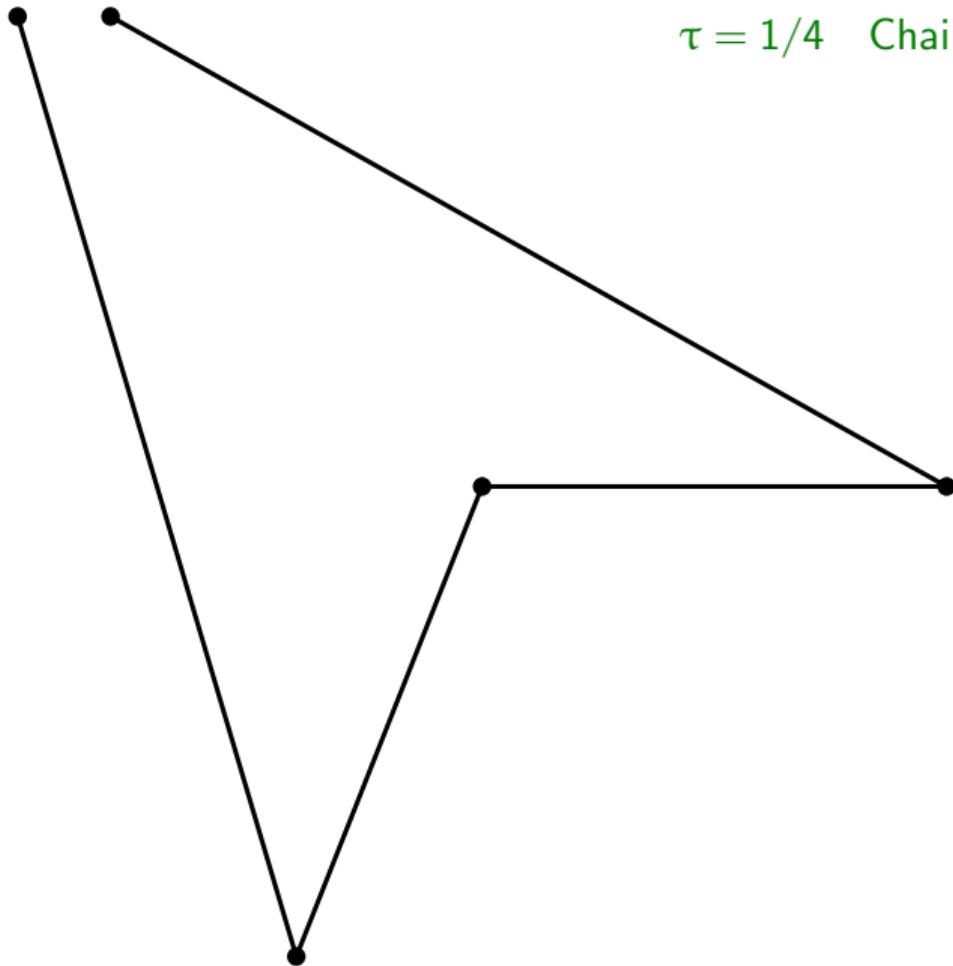
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



# Curvas de subdivisão

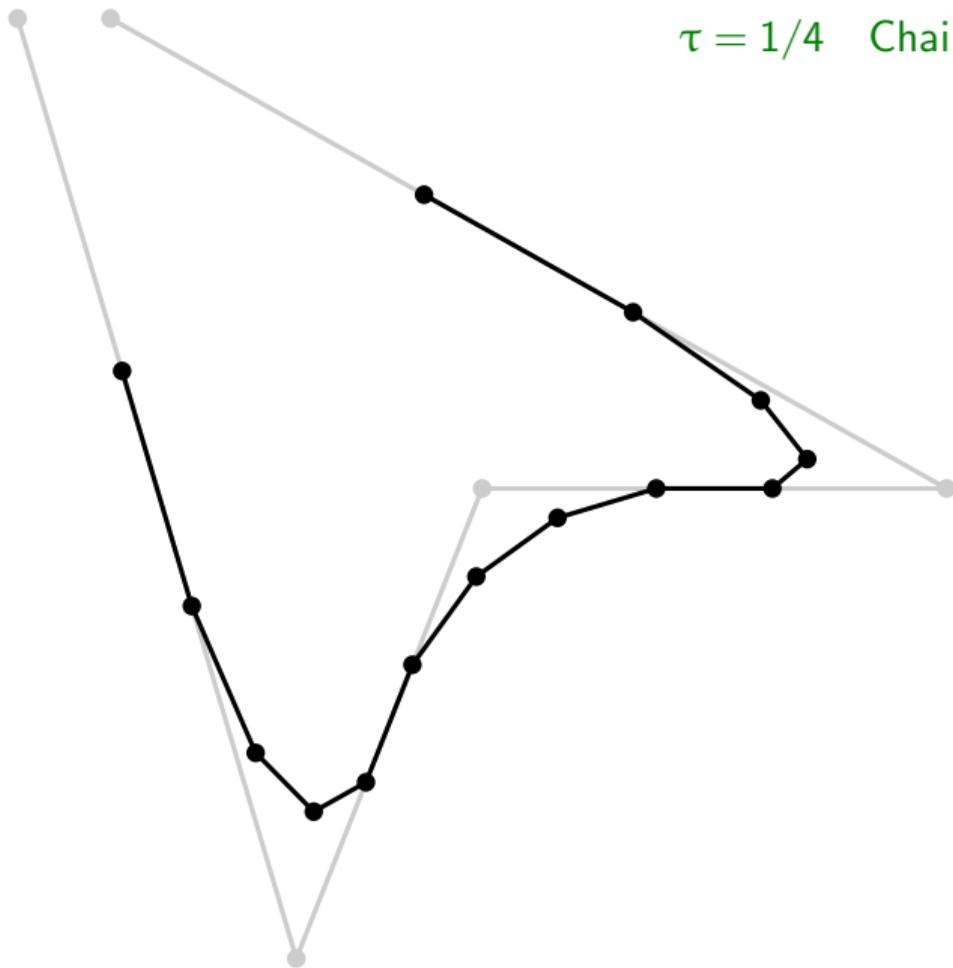
$\tau = 1/4$  Chaikin (1974)





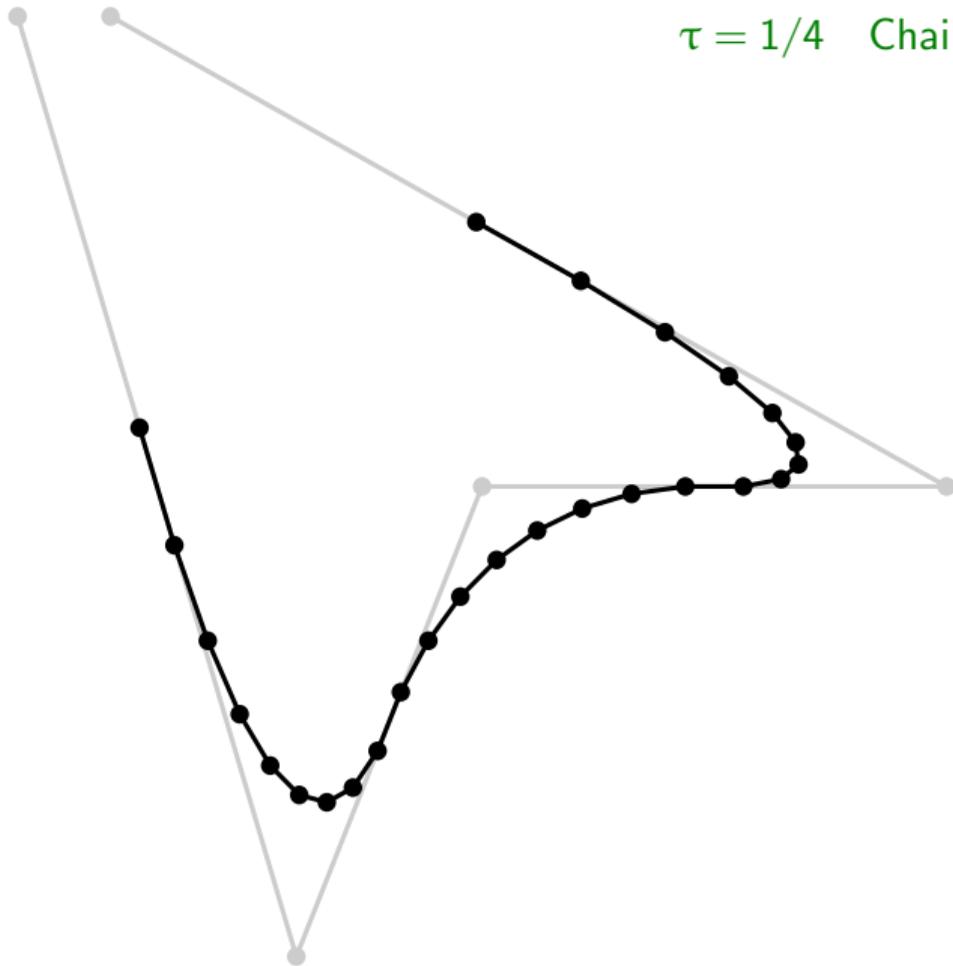
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



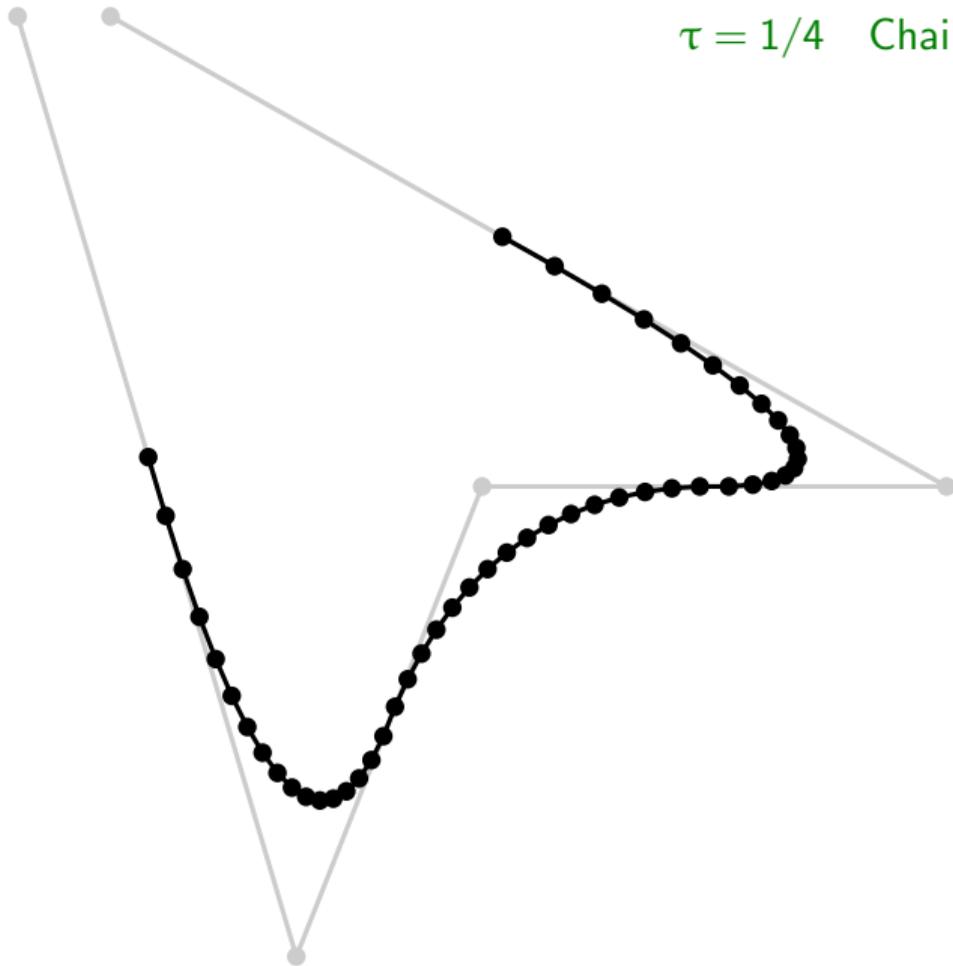
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



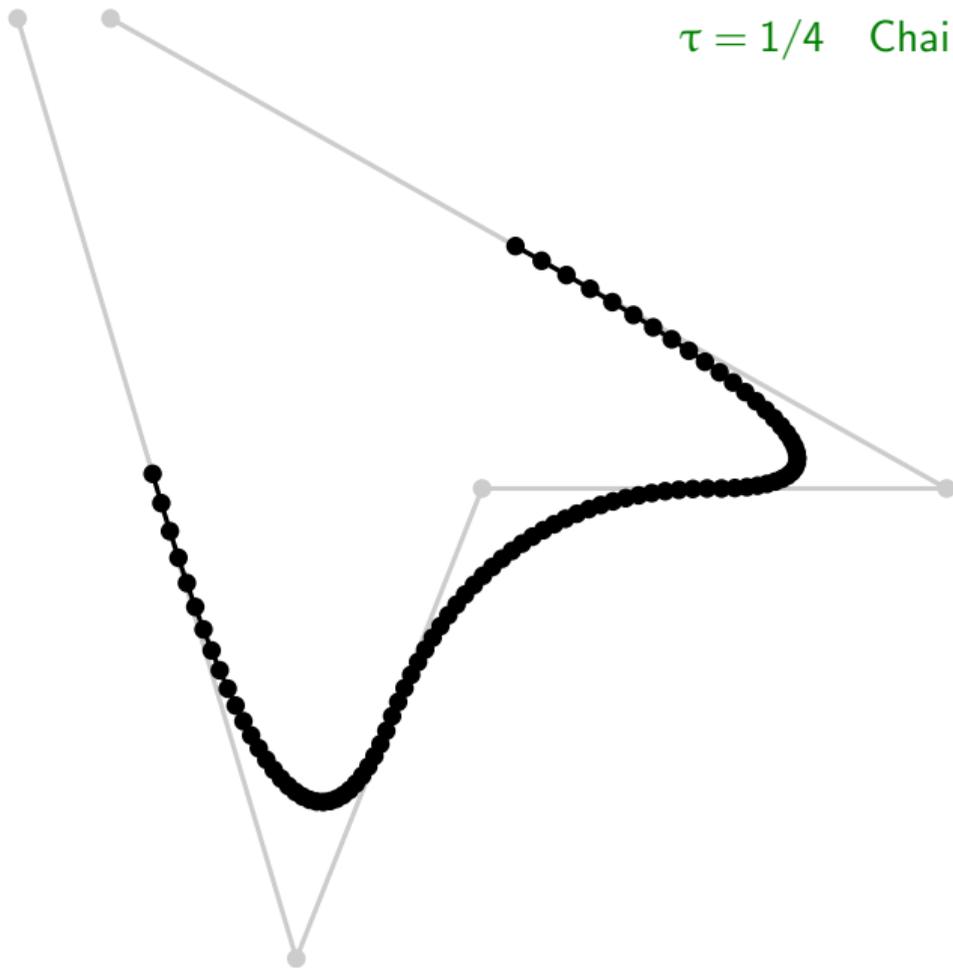
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



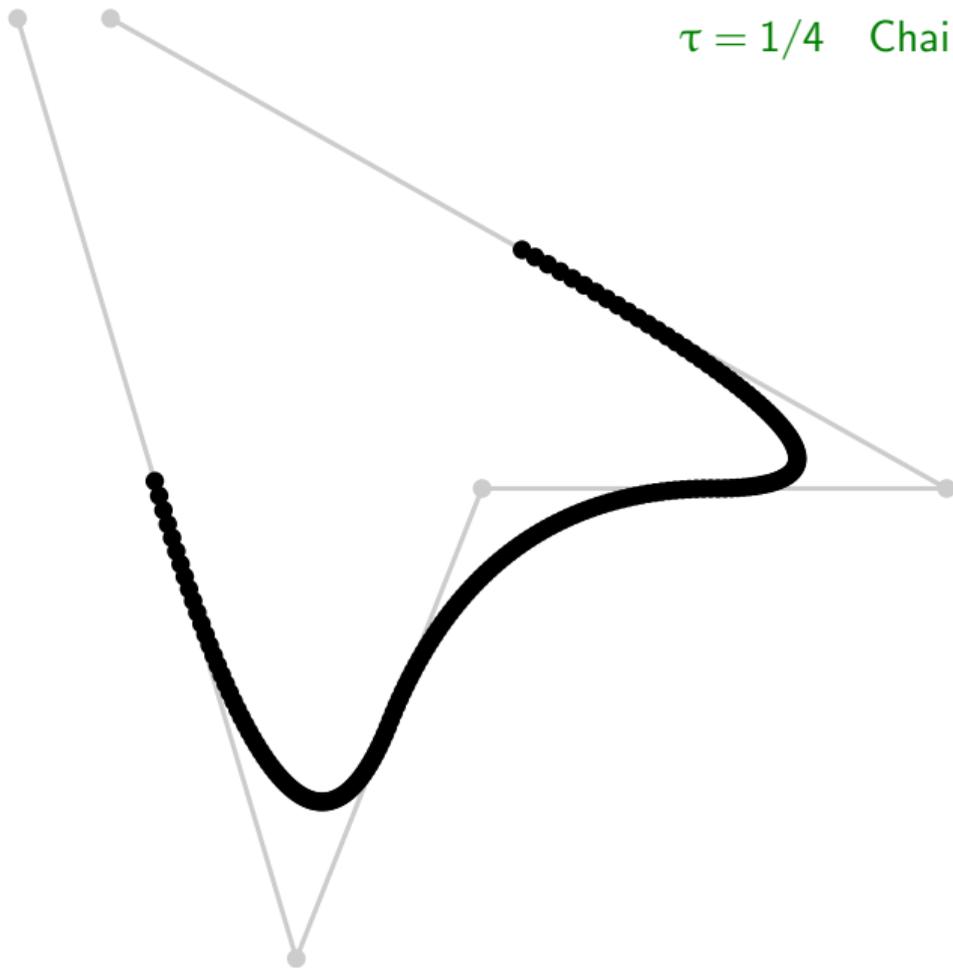
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



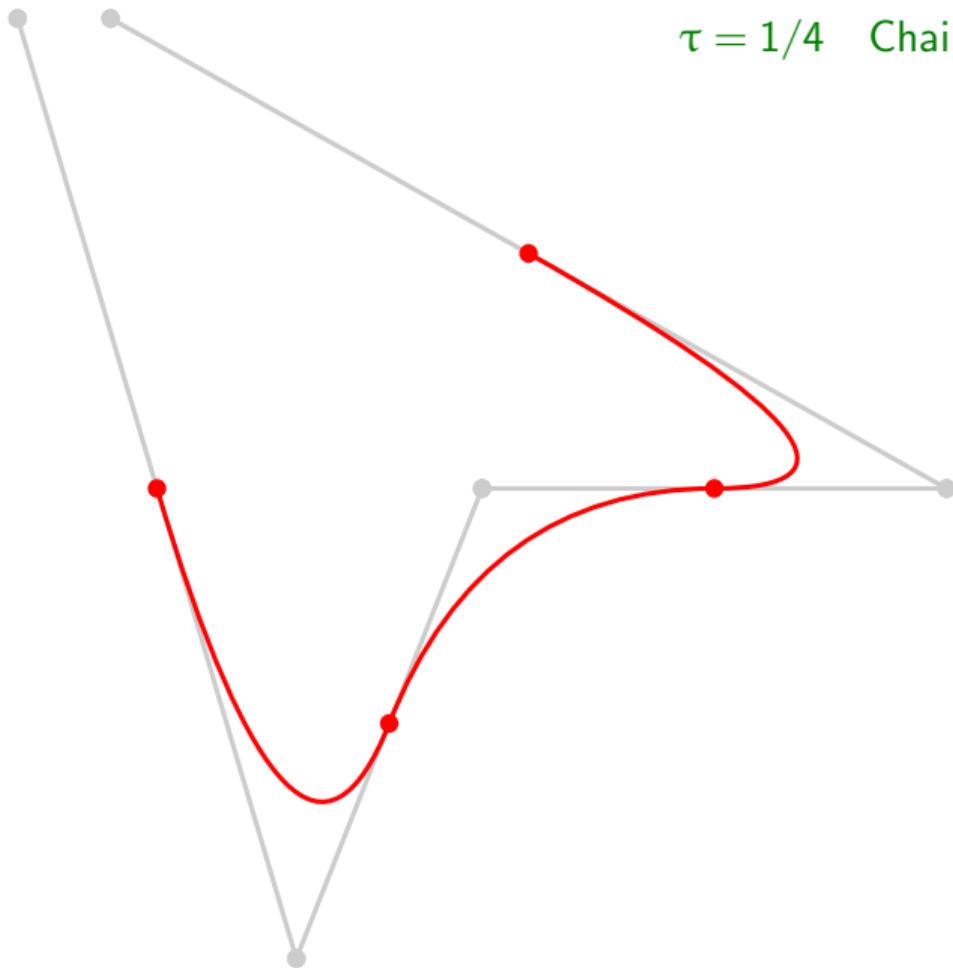
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



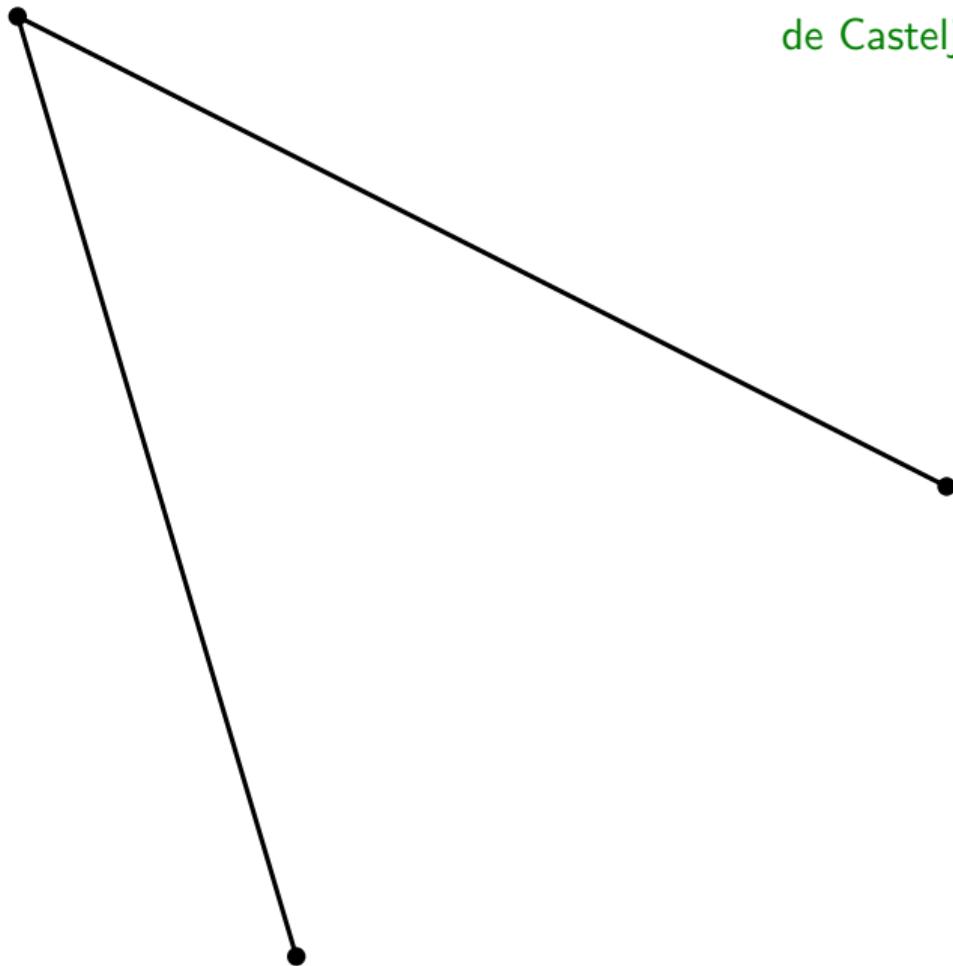
# Curvas de subdivisão

$\tau = 1/4$  Chaikin (1974)



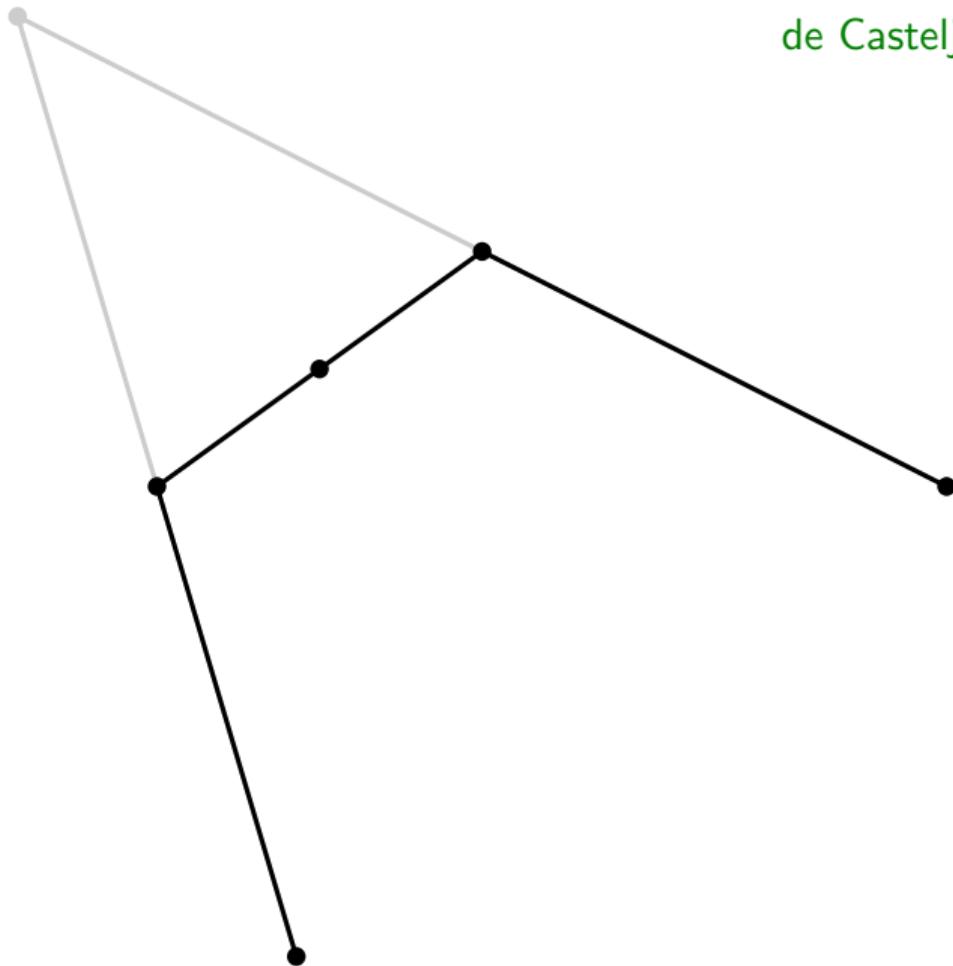
Curvas de subdivisão

de Casteljau (1963)



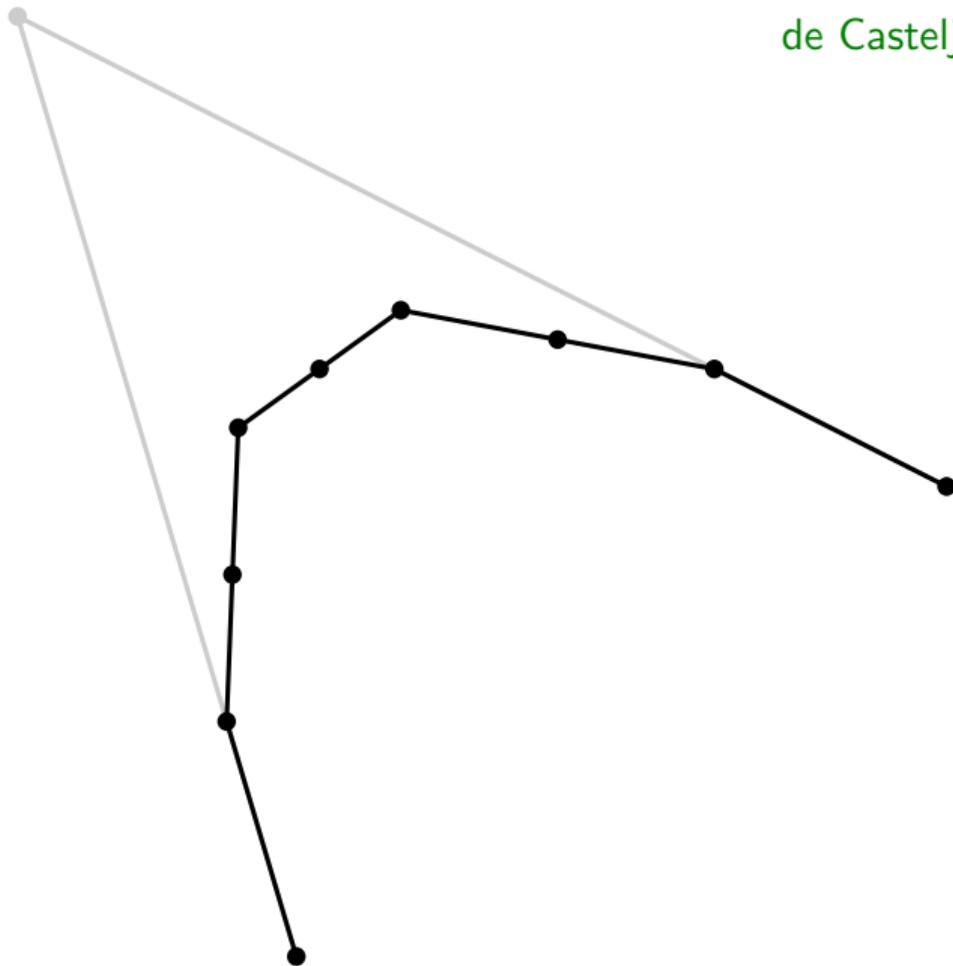
Curvas de subdivisão

de Casteljau (1963)



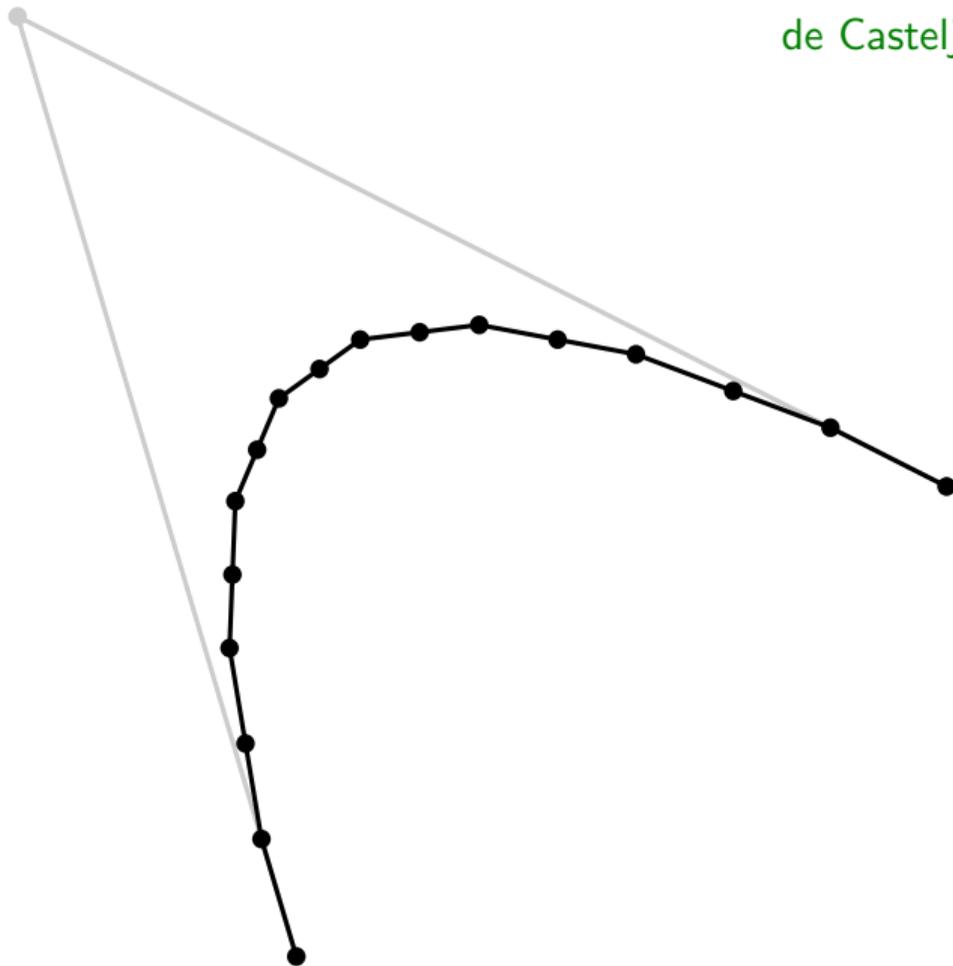
Curvas de subdivisão

de Casteljau (1963)



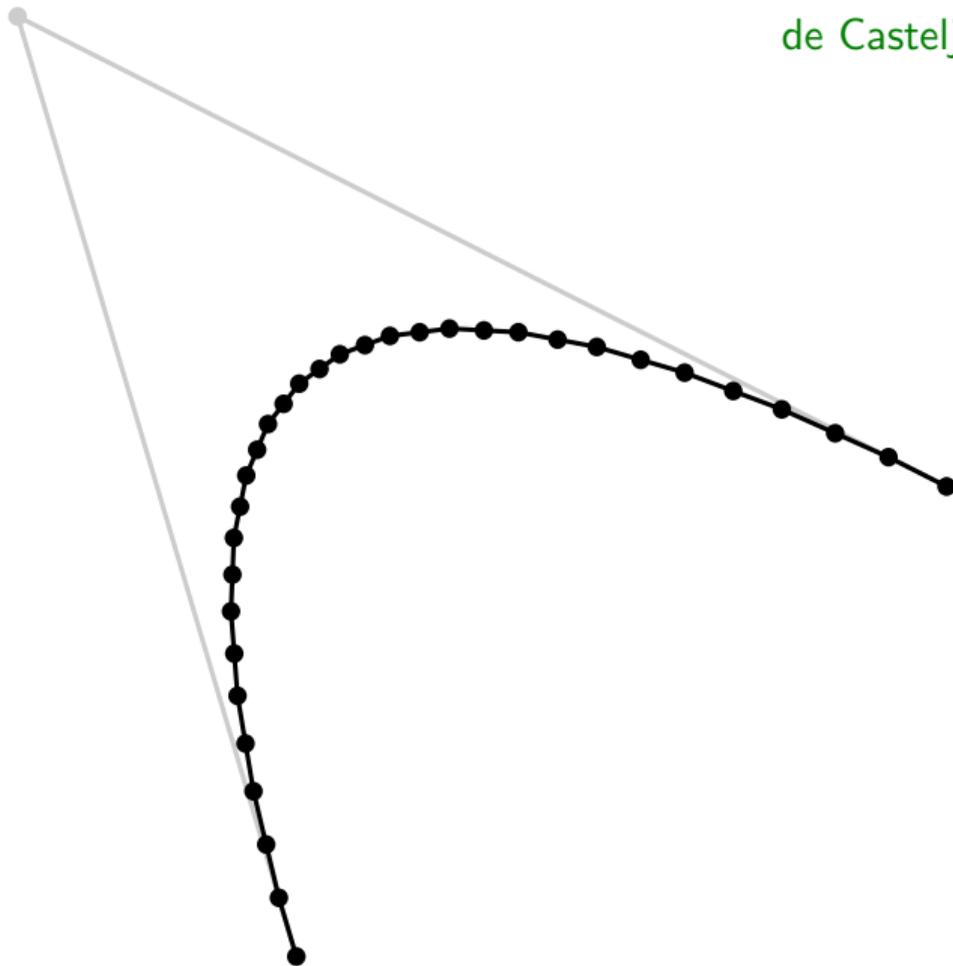
Curvas de subdivisão

de Casteljau (1963)



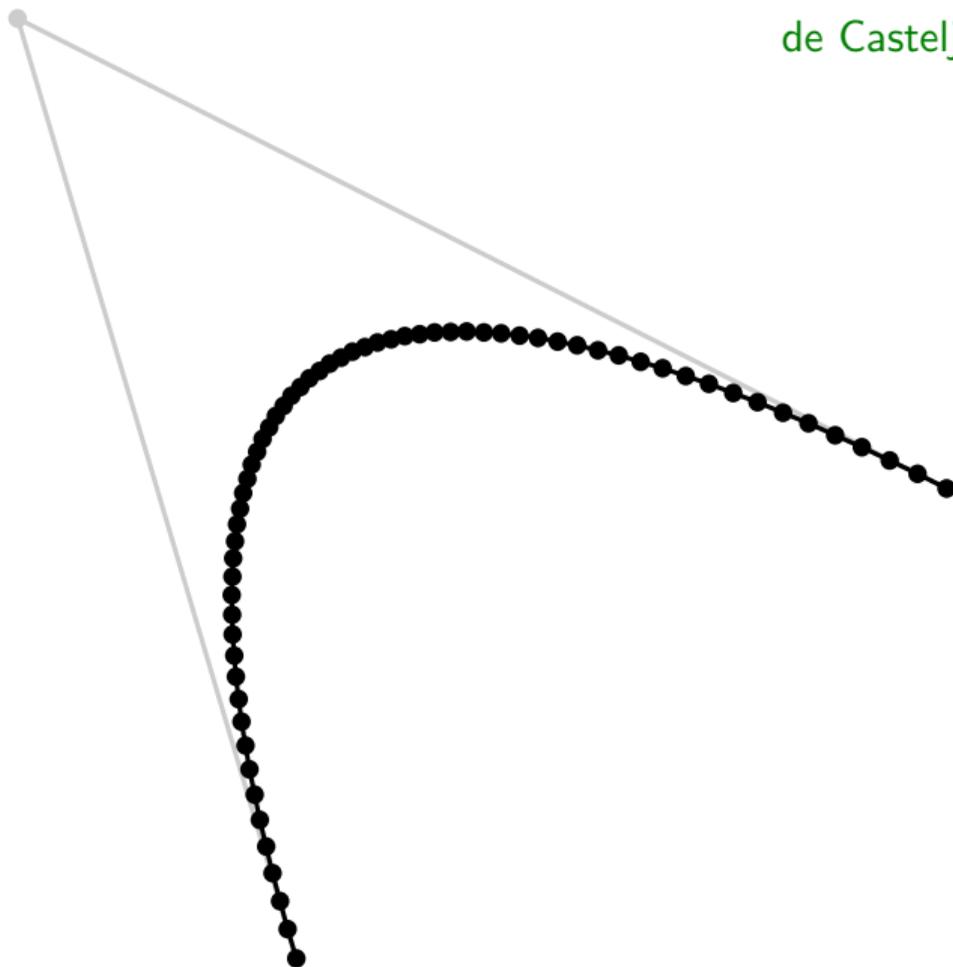
Curvas de subdivisão

de Casteljau (1963)



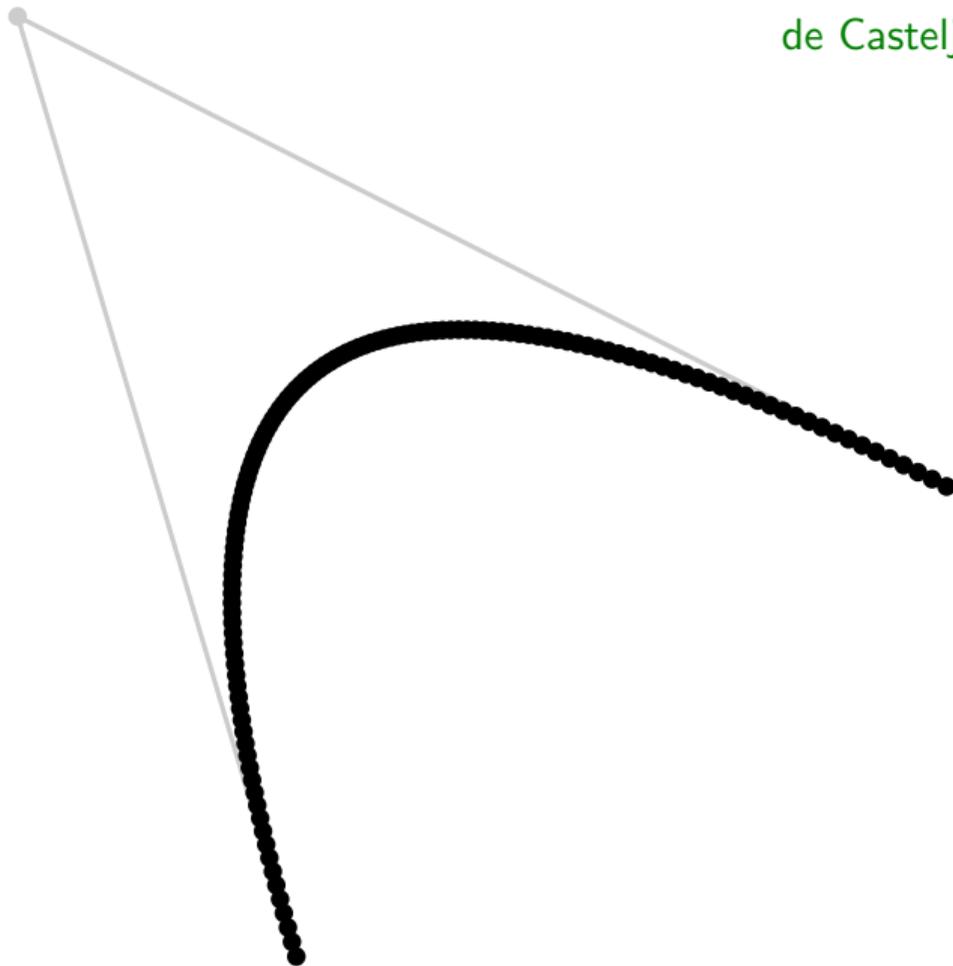
Curvas de subdivisão

de Casteljau (1963)



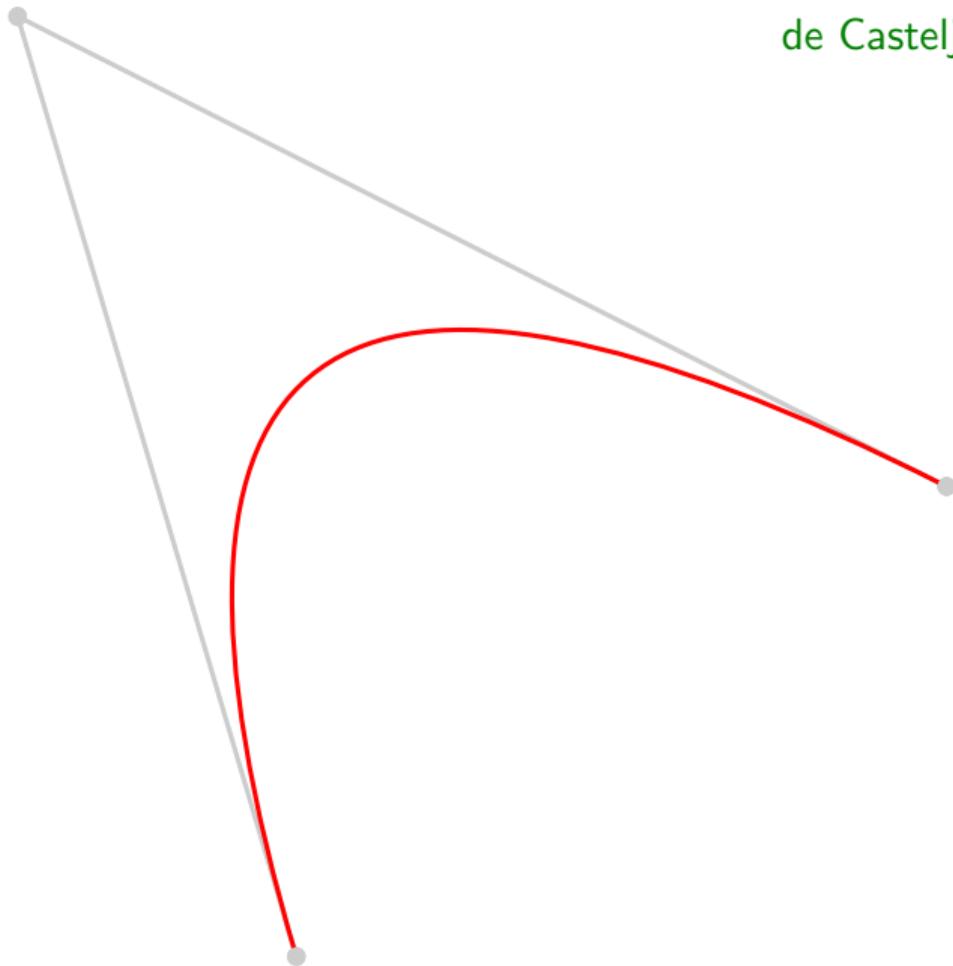
Curvas de subdivisão

de Casteljau (1963)



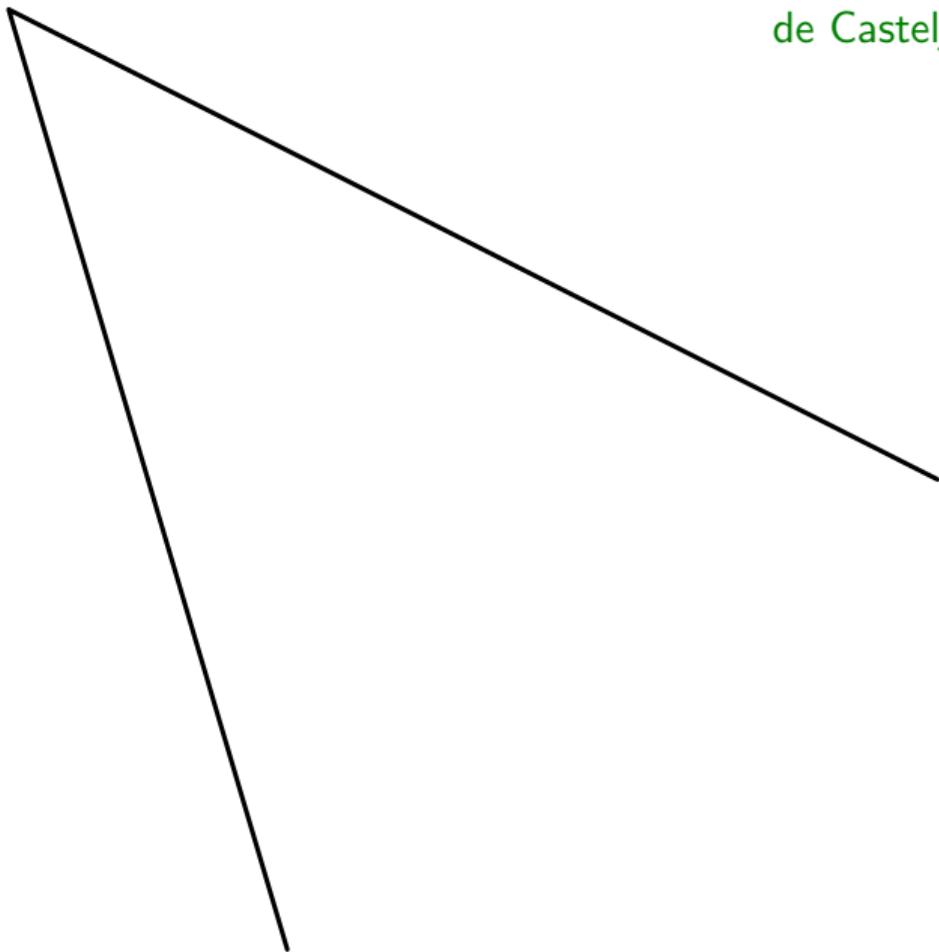
Curvas de subdivisão

de Casteljau (1963)



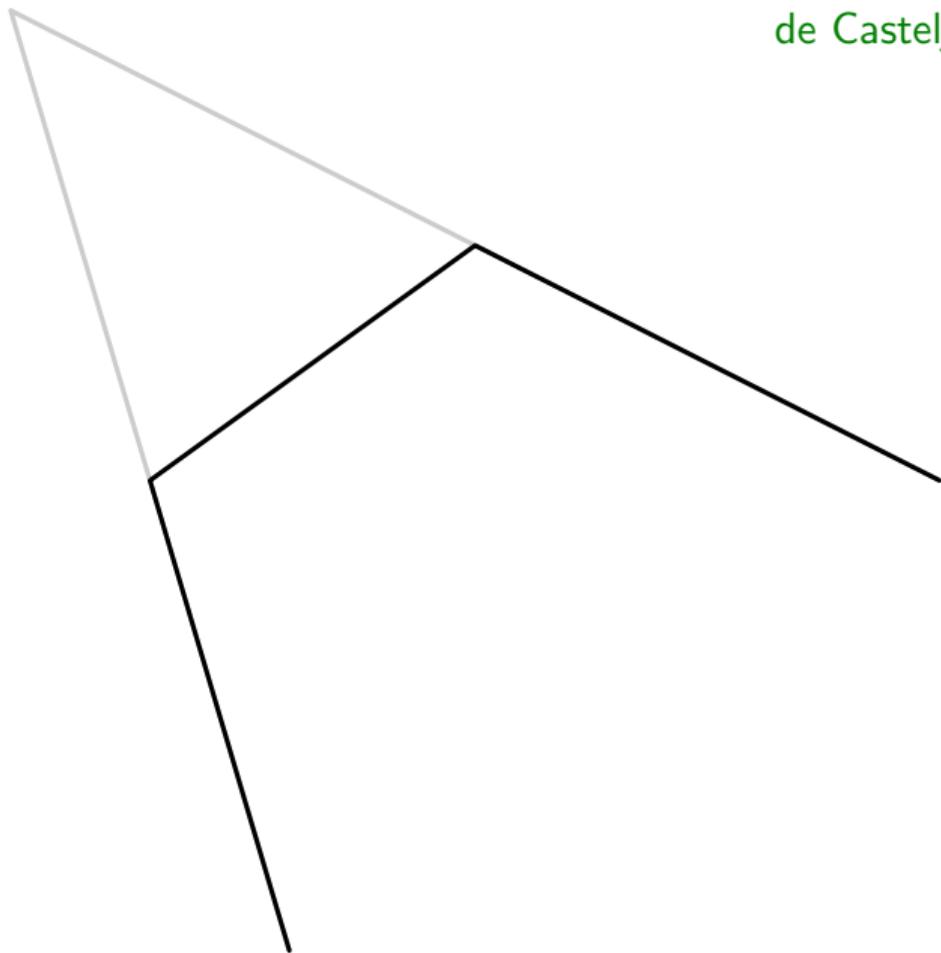
Curvas de subdivisão

de Casteljau (1963)



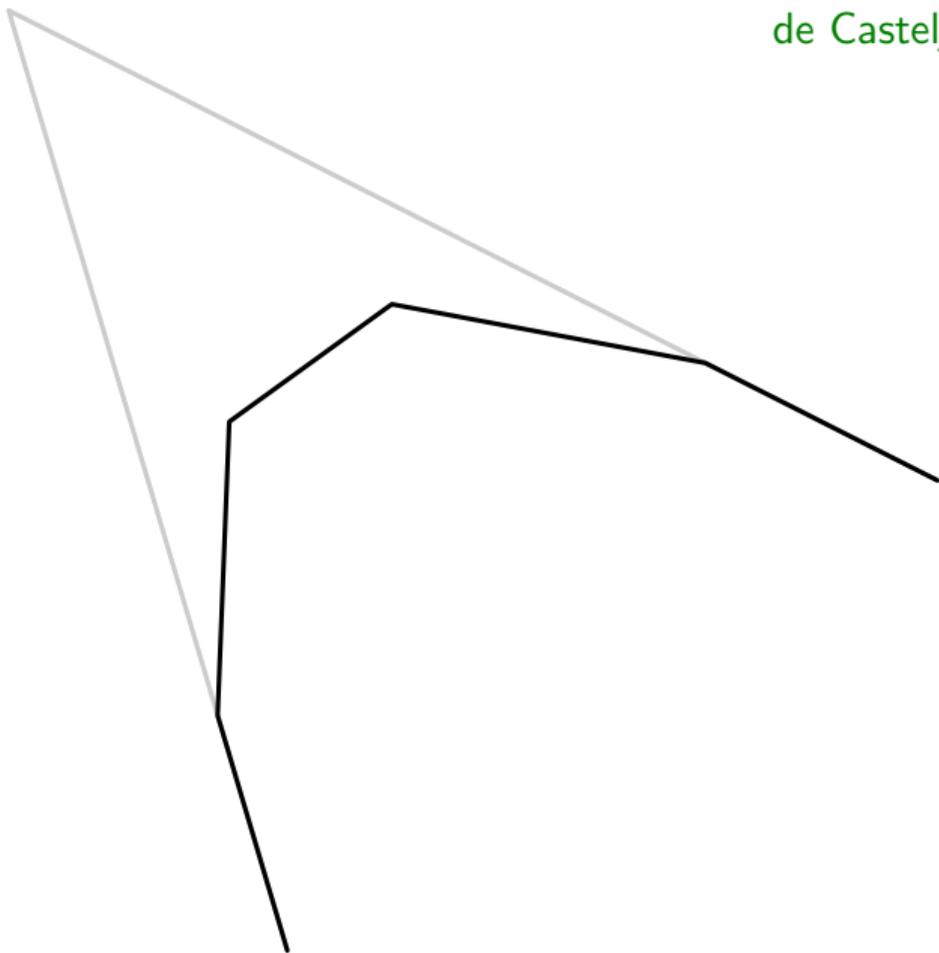
Curvas de subdivisão

de Casteljau (1963)



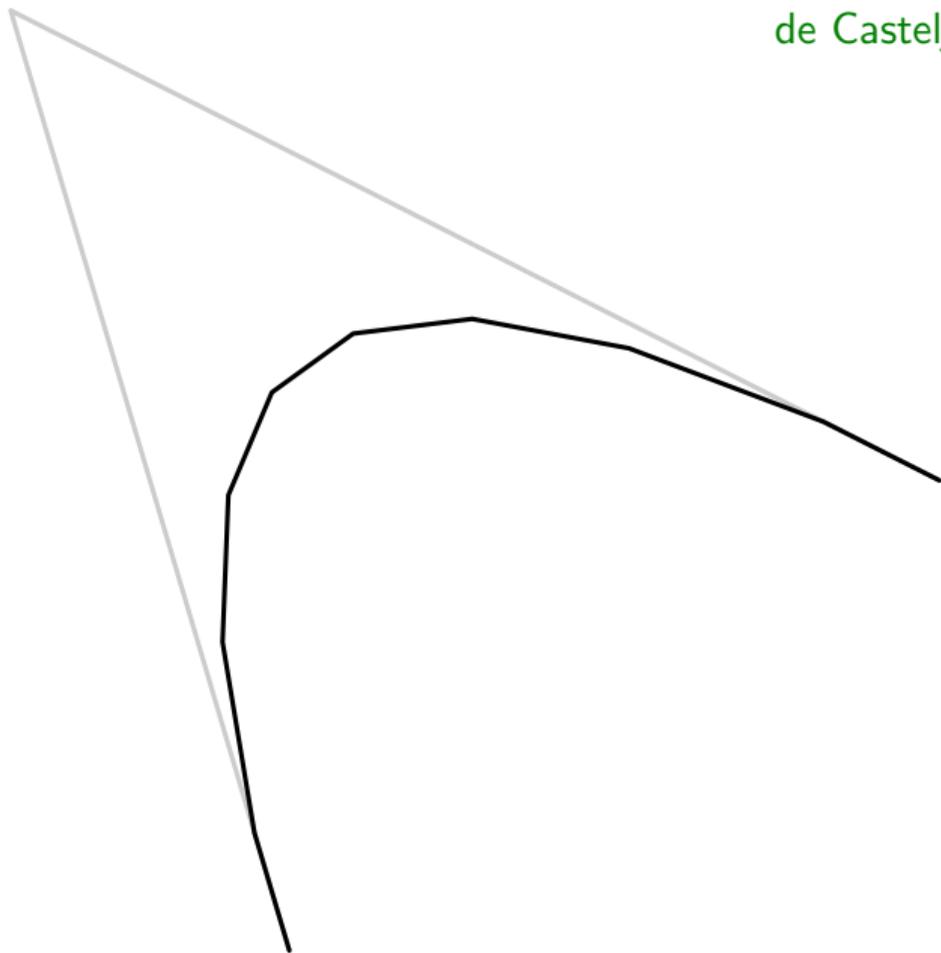
Curvas de subdivisão

de Casteljau (1963)



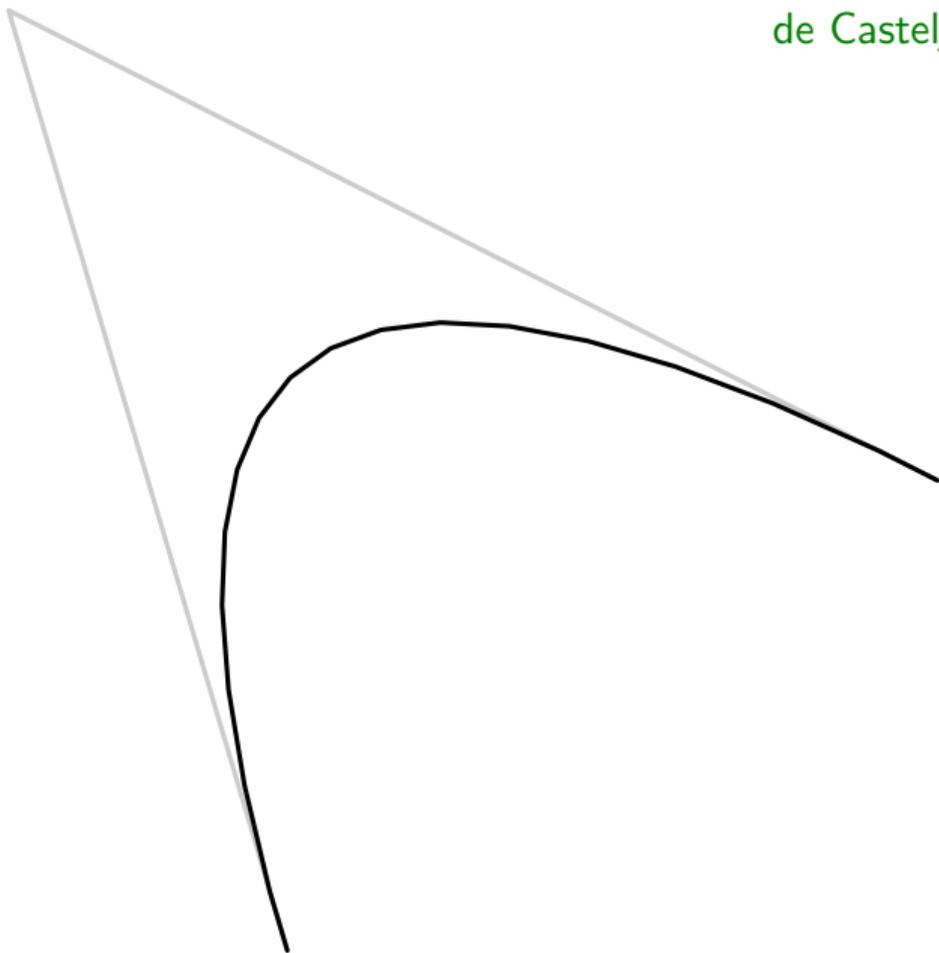
Curvas de subdivisão

de Casteljau (1963)



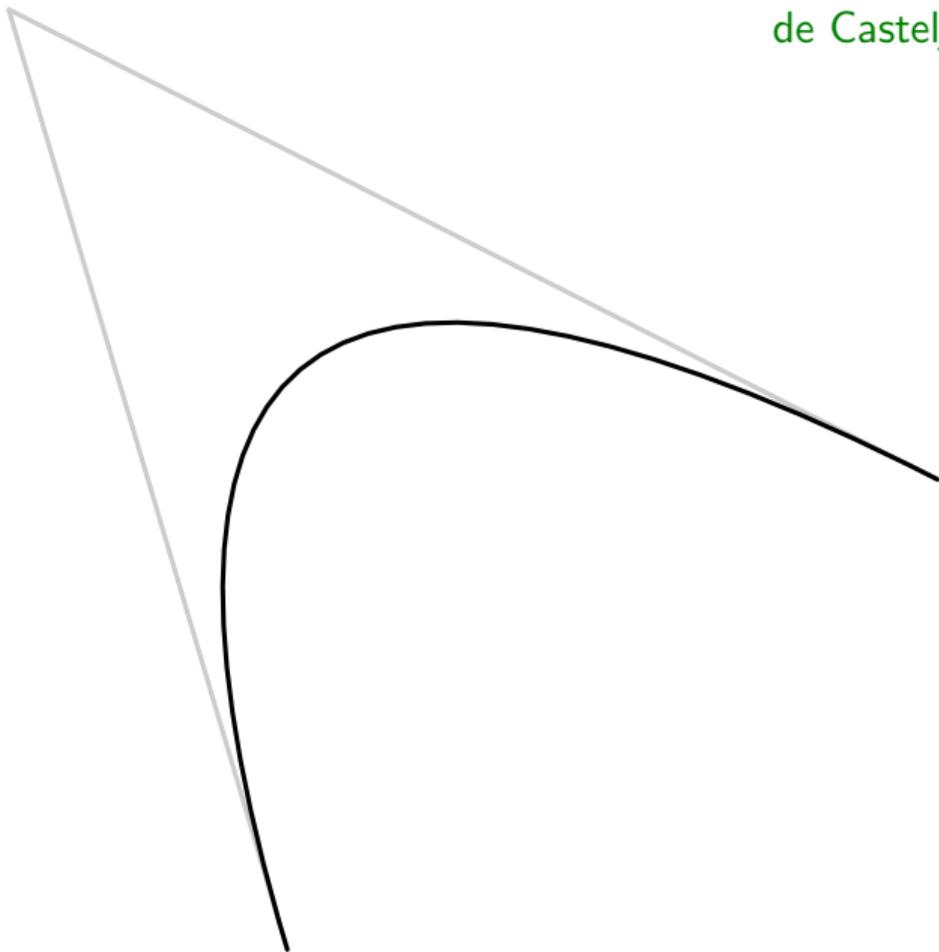
Curvas de subdivisão

de Casteljau (1963)



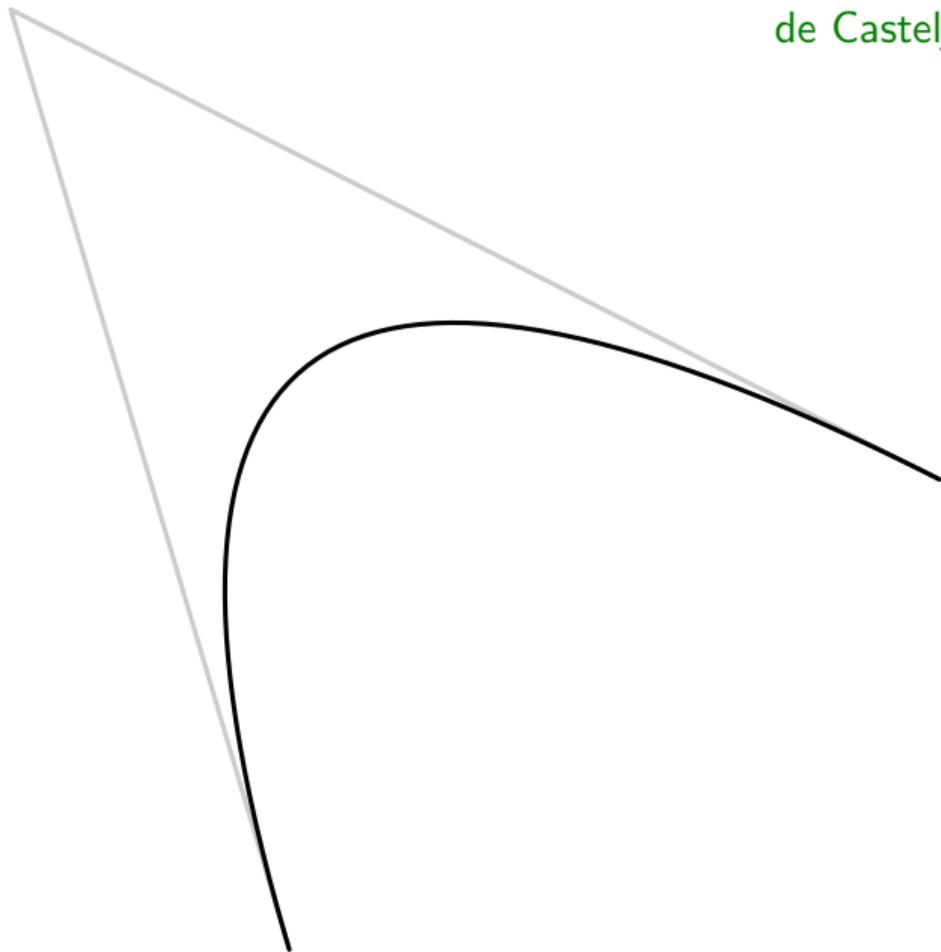
Curvas de subdivisão

de Casteljau (1963)



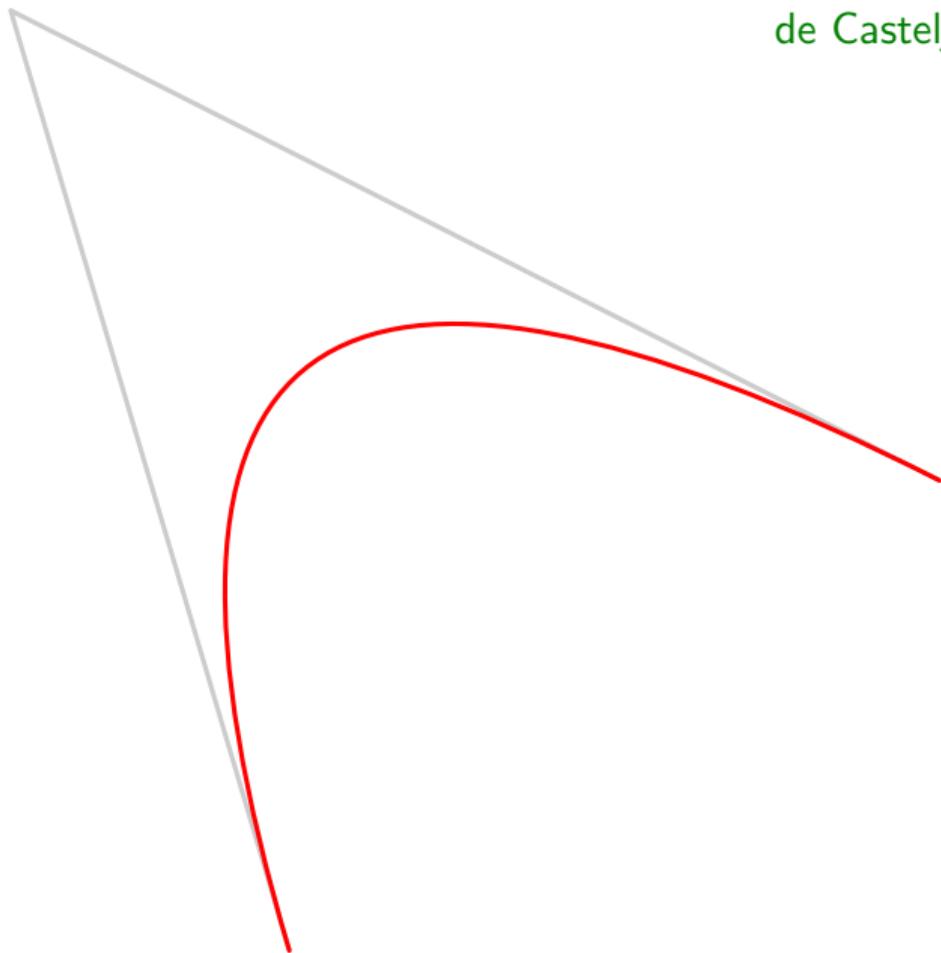
Curvas de subdivisão

de Casteljau (1963)



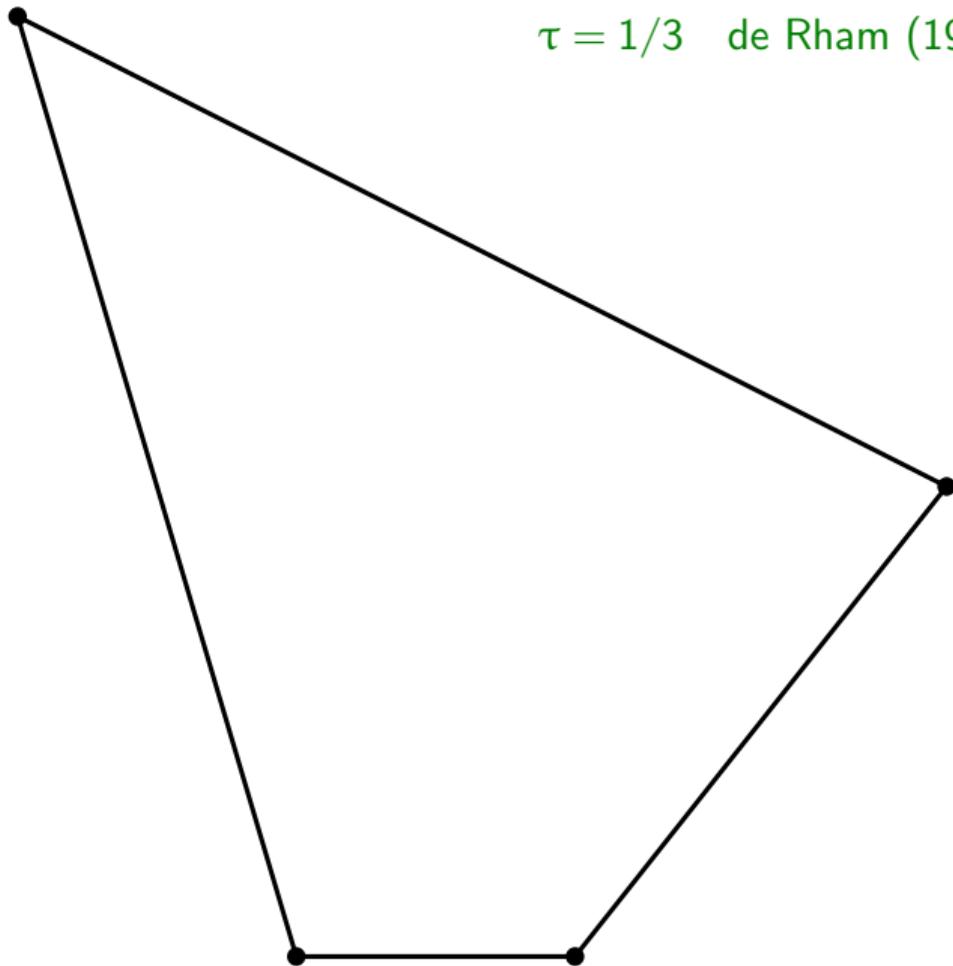
Curvas de subdivisão

de Casteljau (1963)



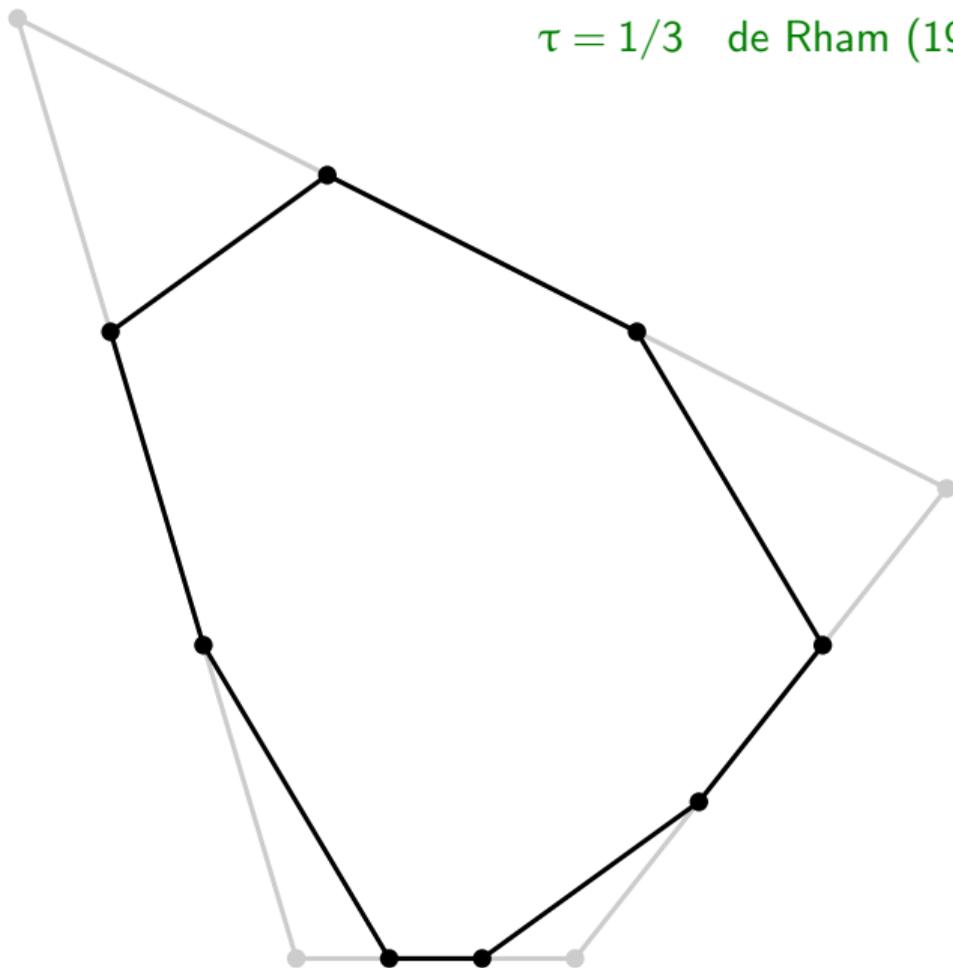
# Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



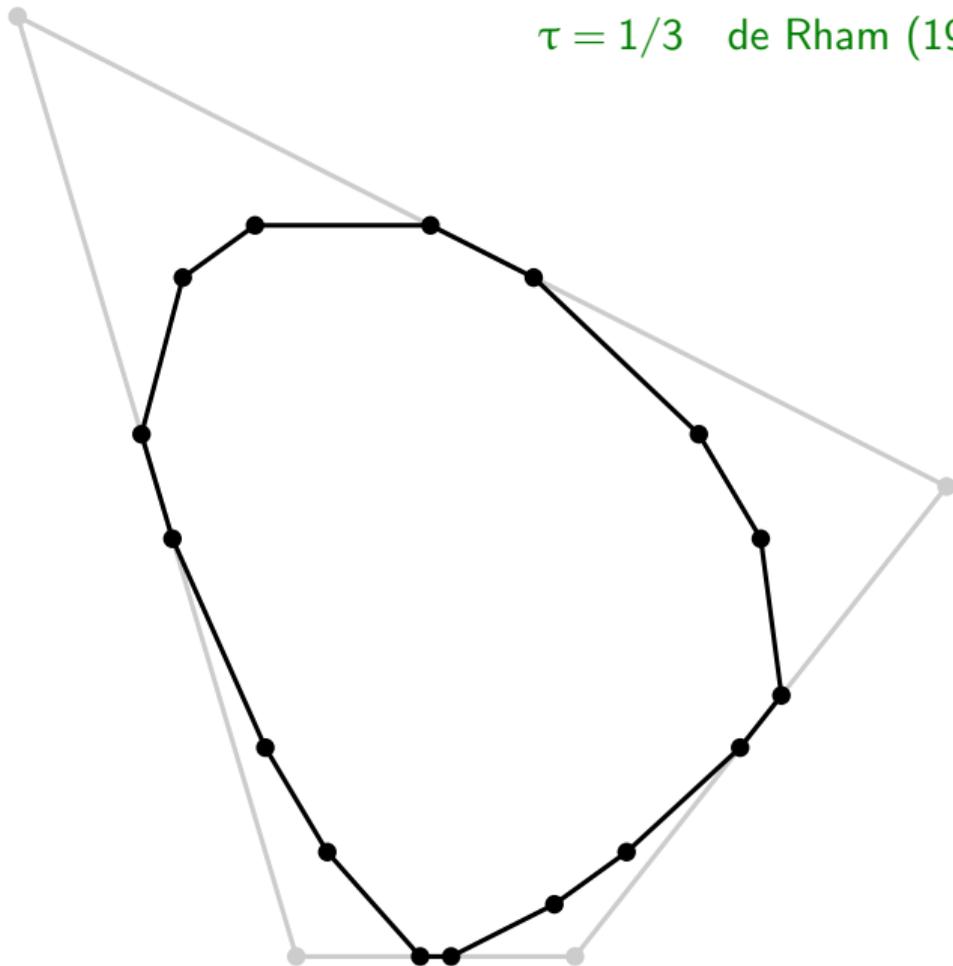
# Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



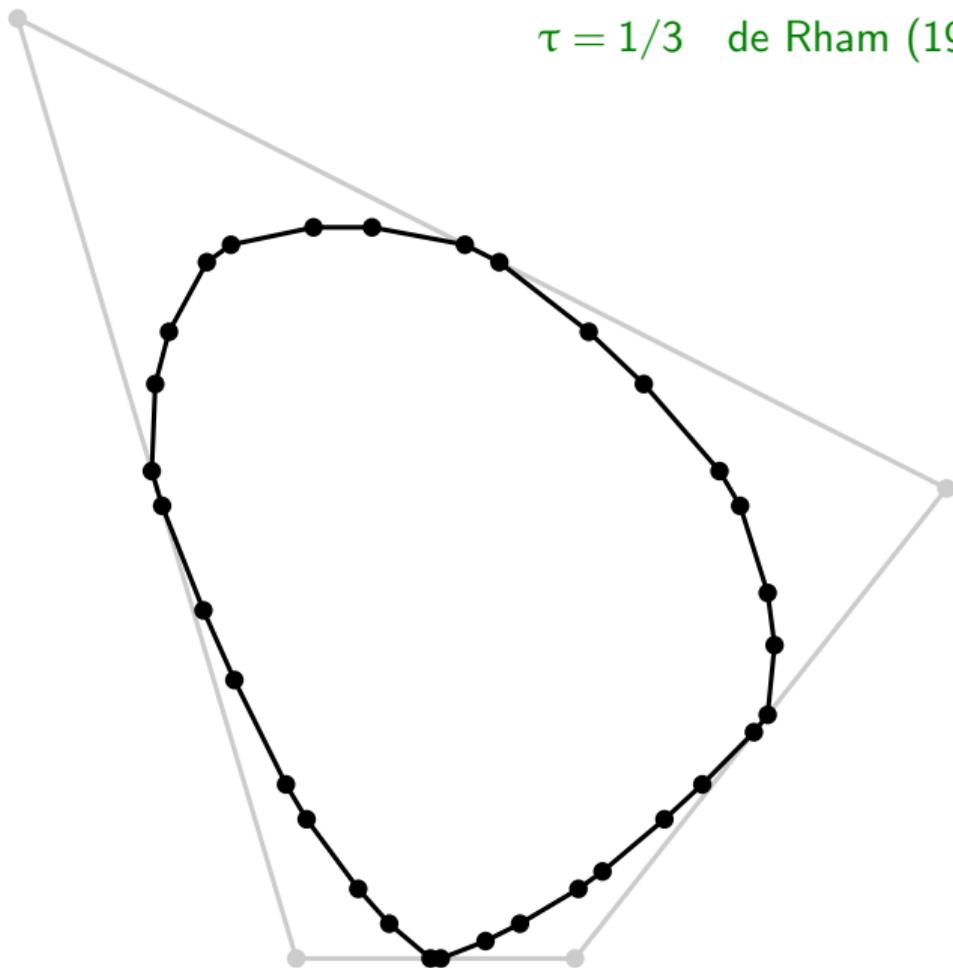
# Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



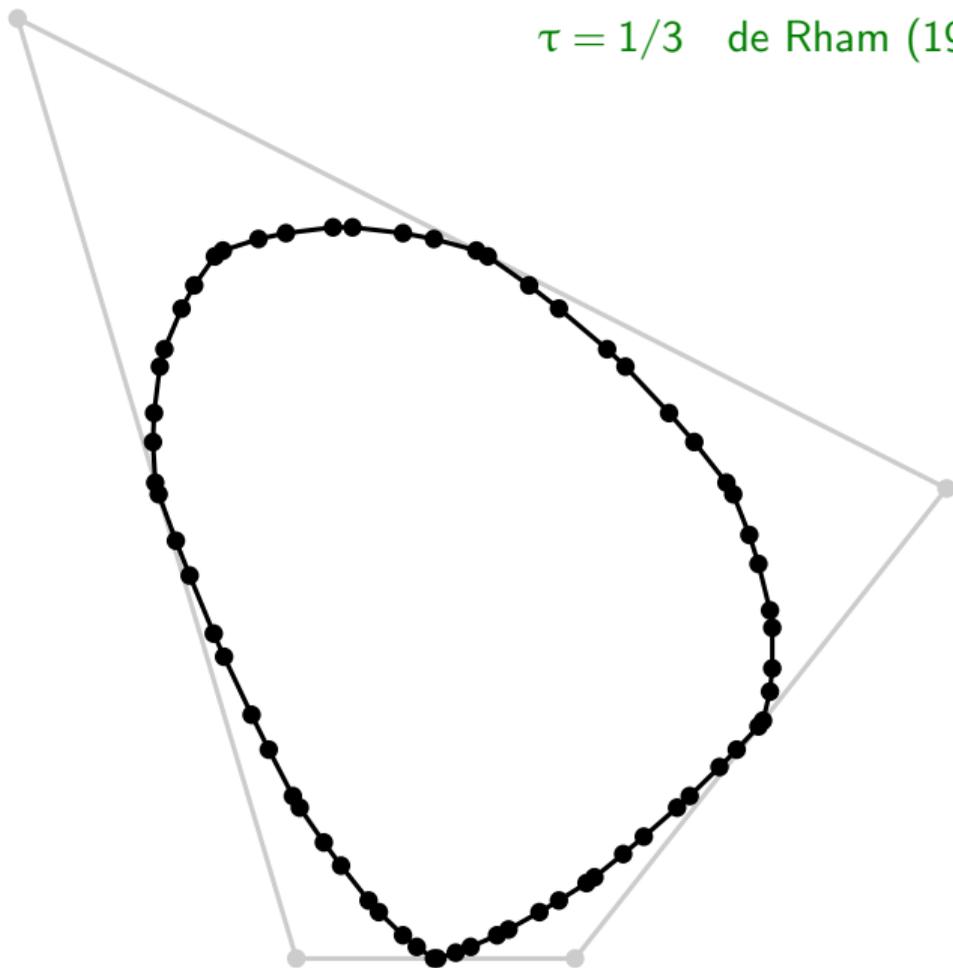
# Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



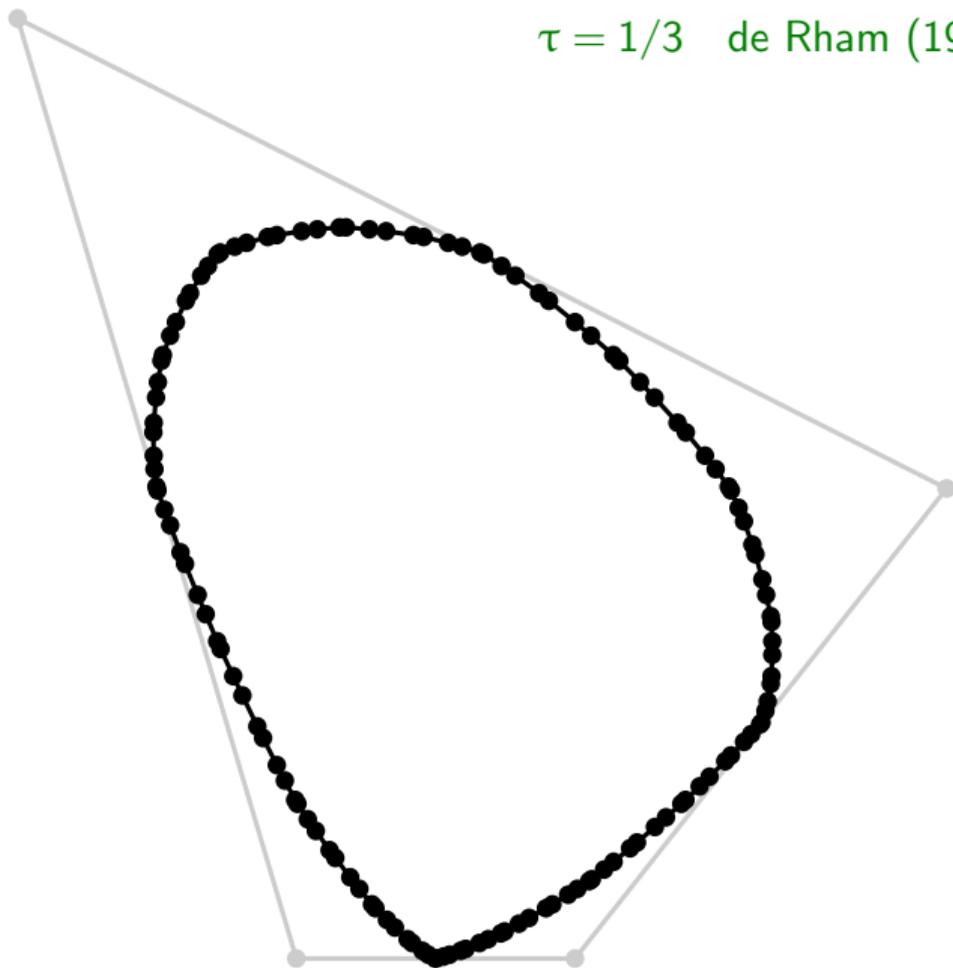
# Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



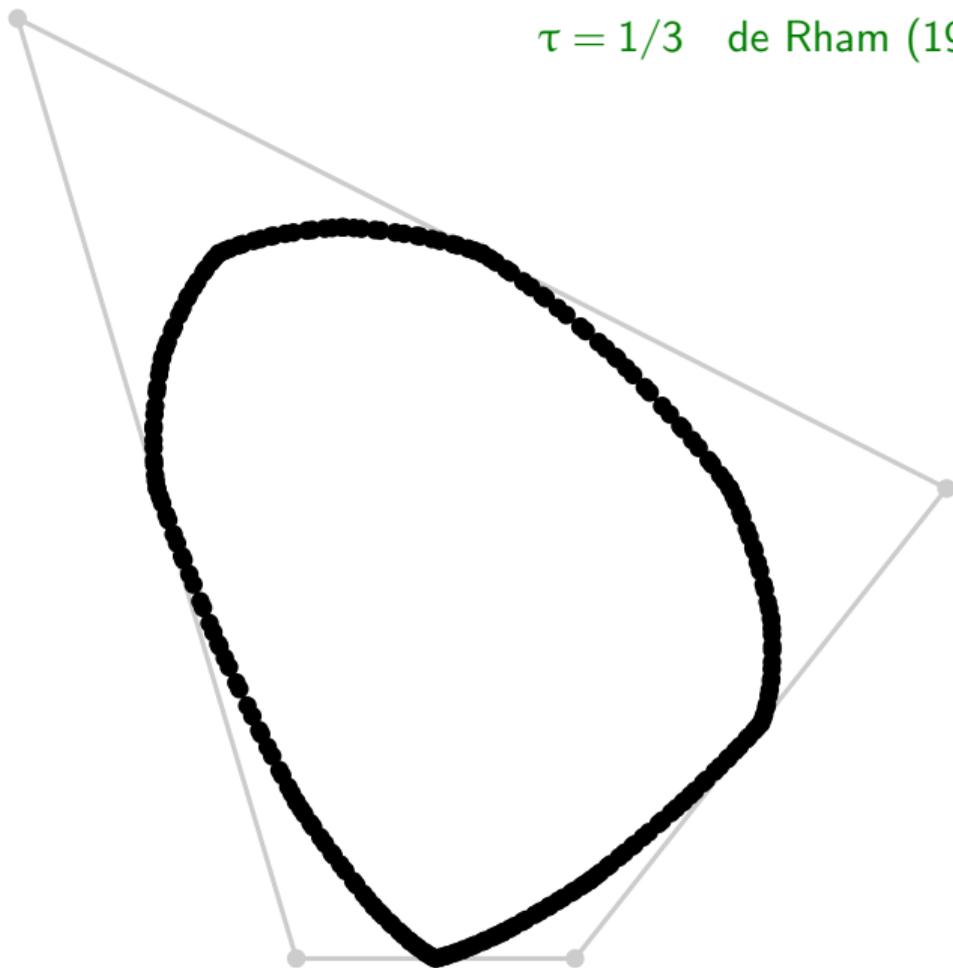
# Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



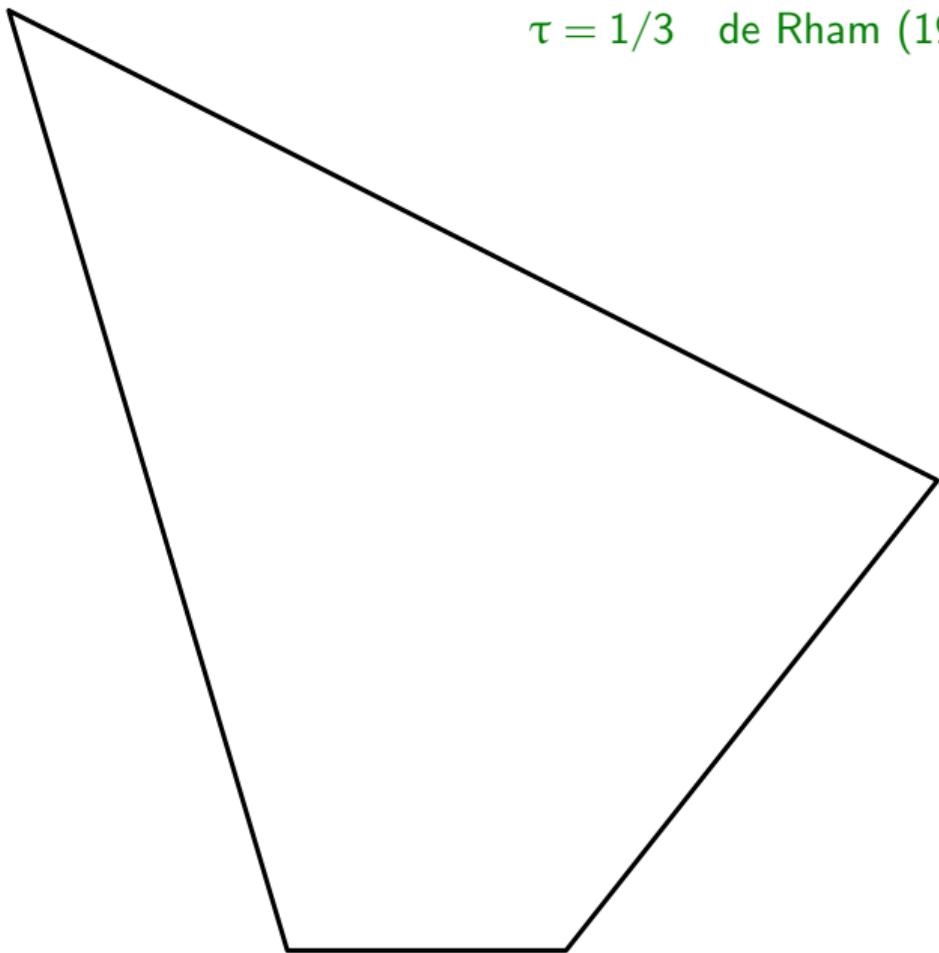
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$\tau = 1/3$  de Rham (1947, 1956)



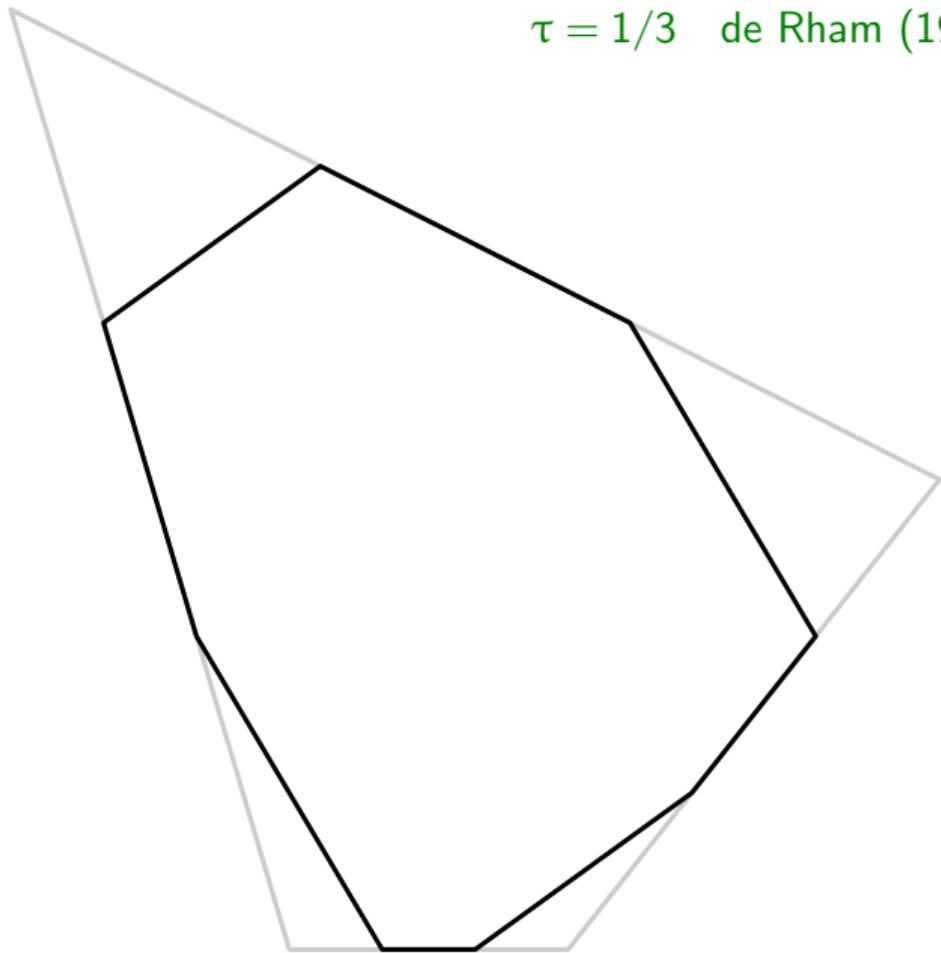
## Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



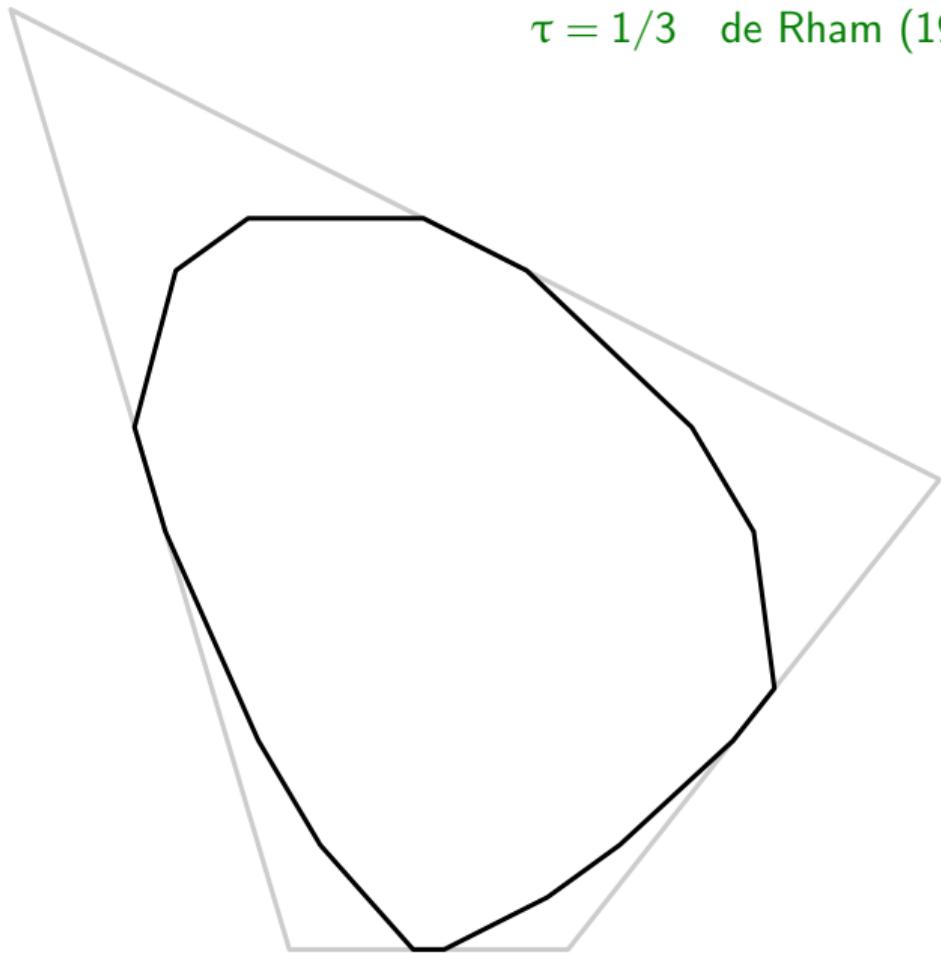
## Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



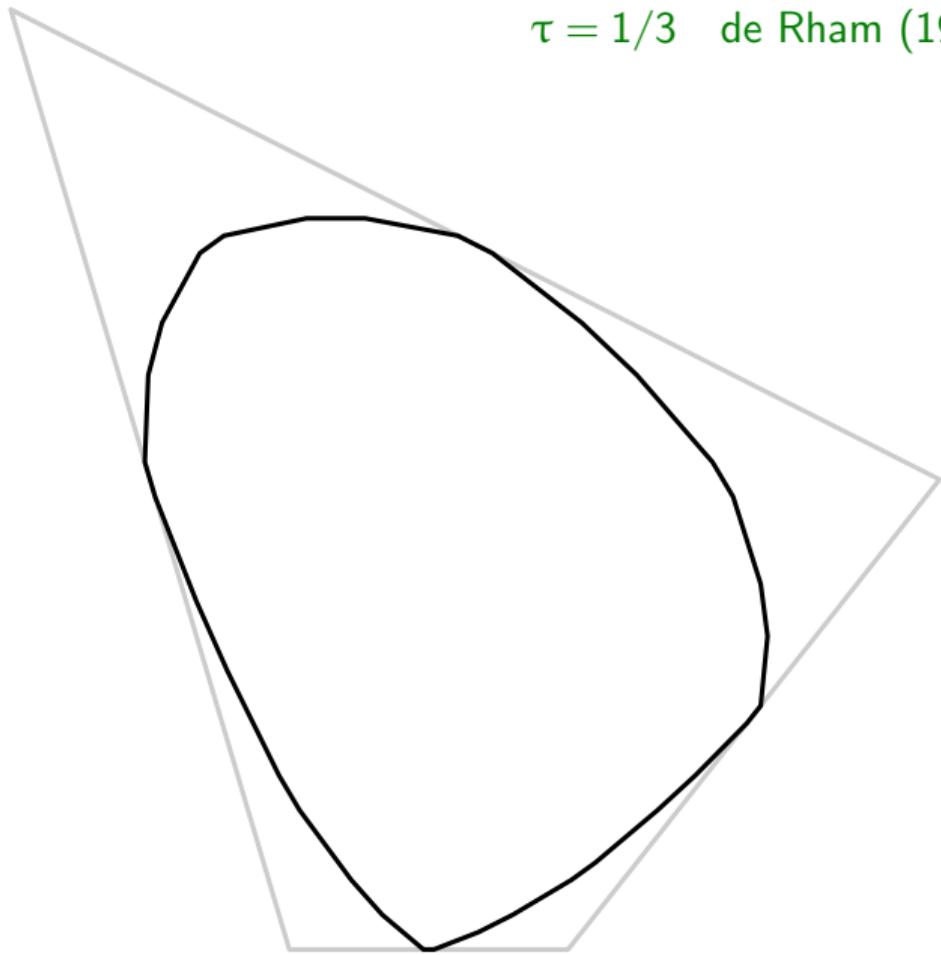
# Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



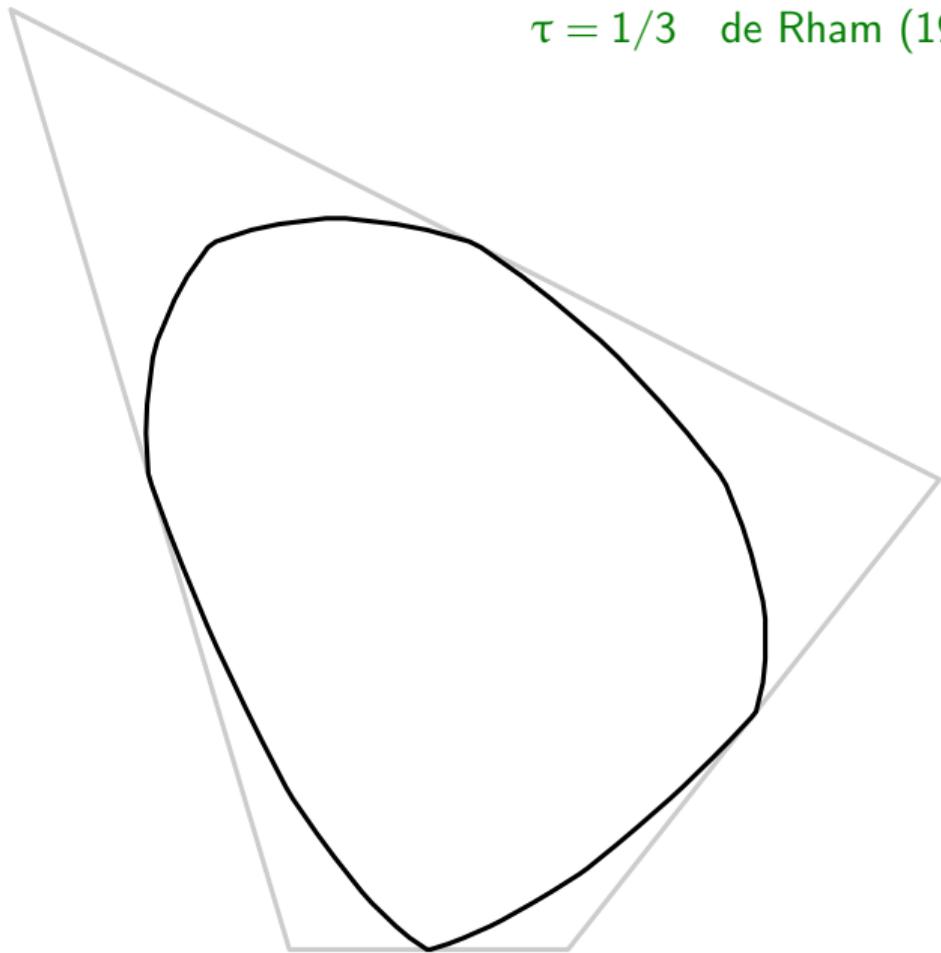
## Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



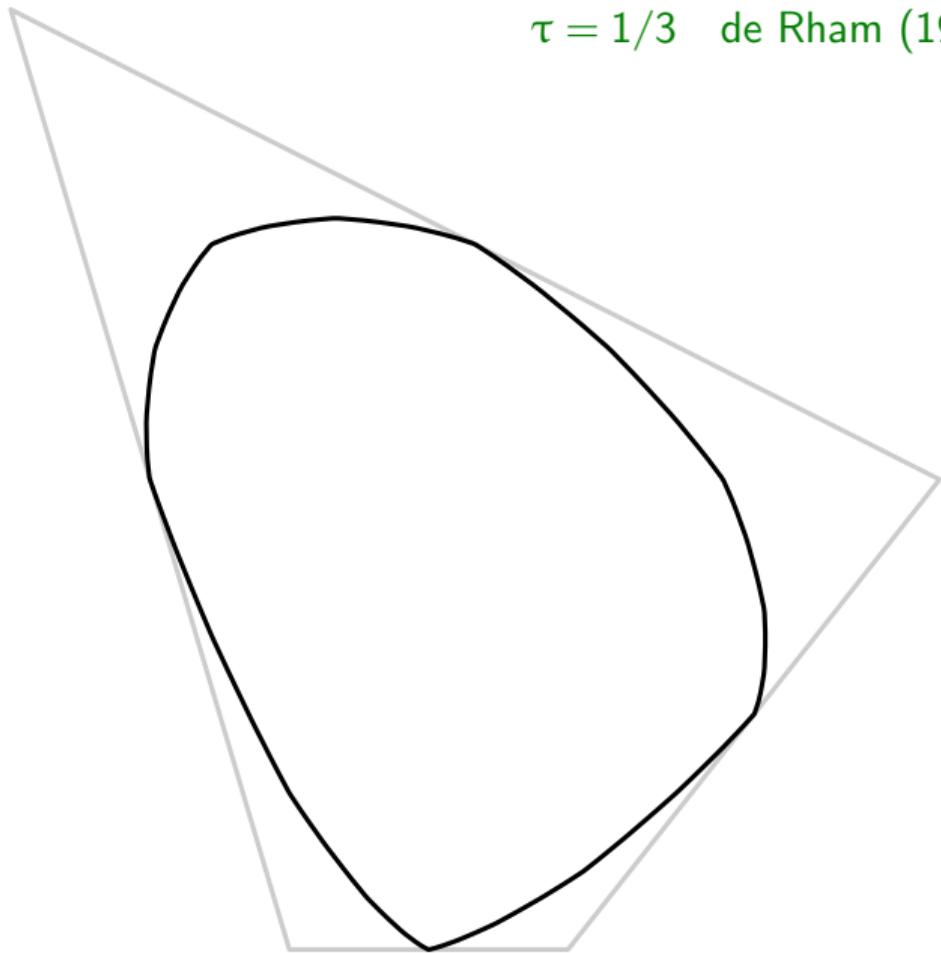
## Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



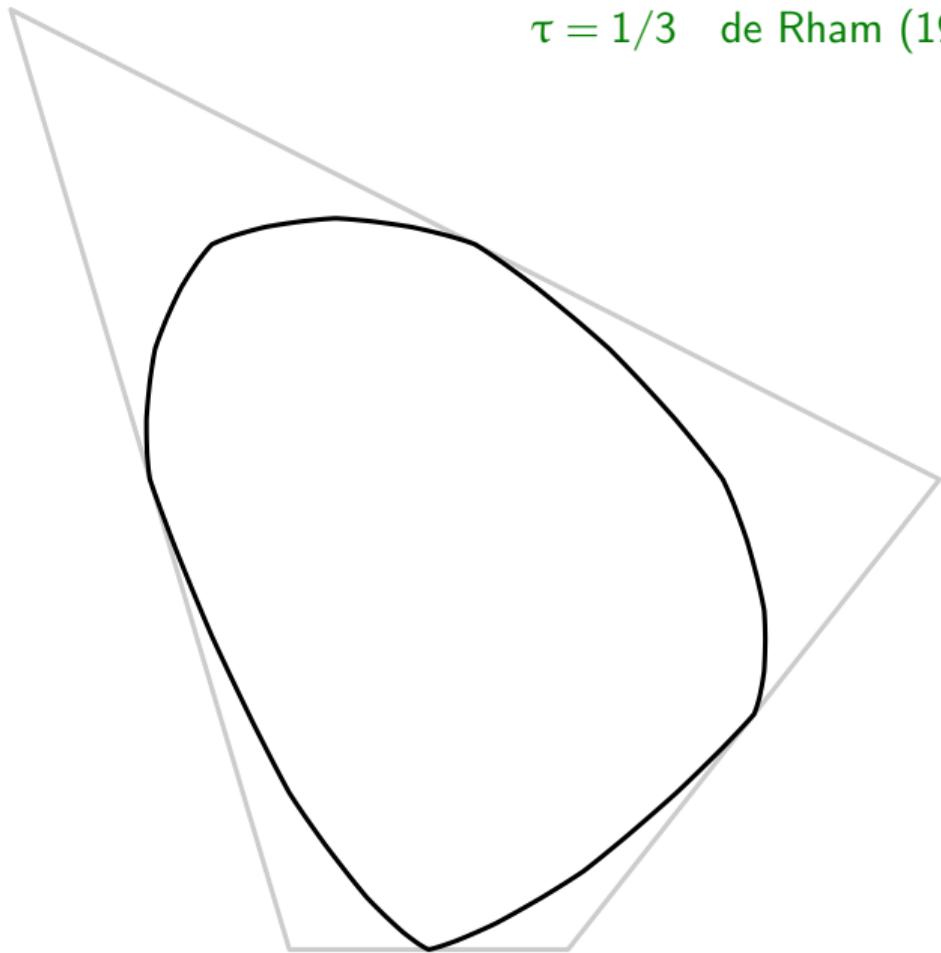
## Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



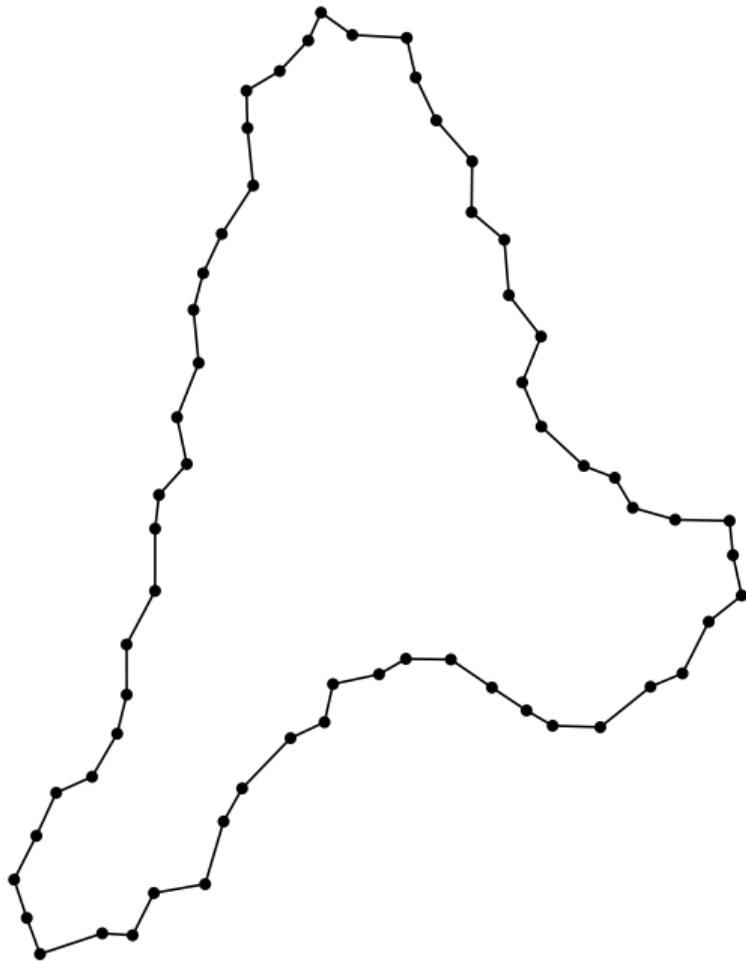
## Curvas de subdivisão

$\tau = 1/3$  de Rham (1947, 1956)



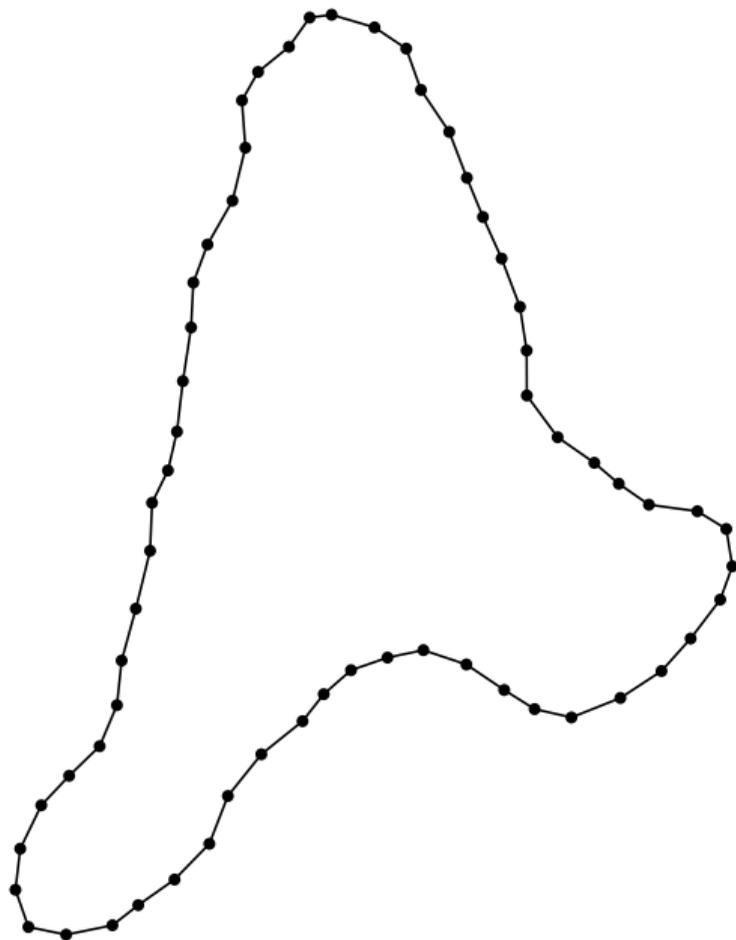
# Curvas de subdivisão

$$\tau = 1/2$$



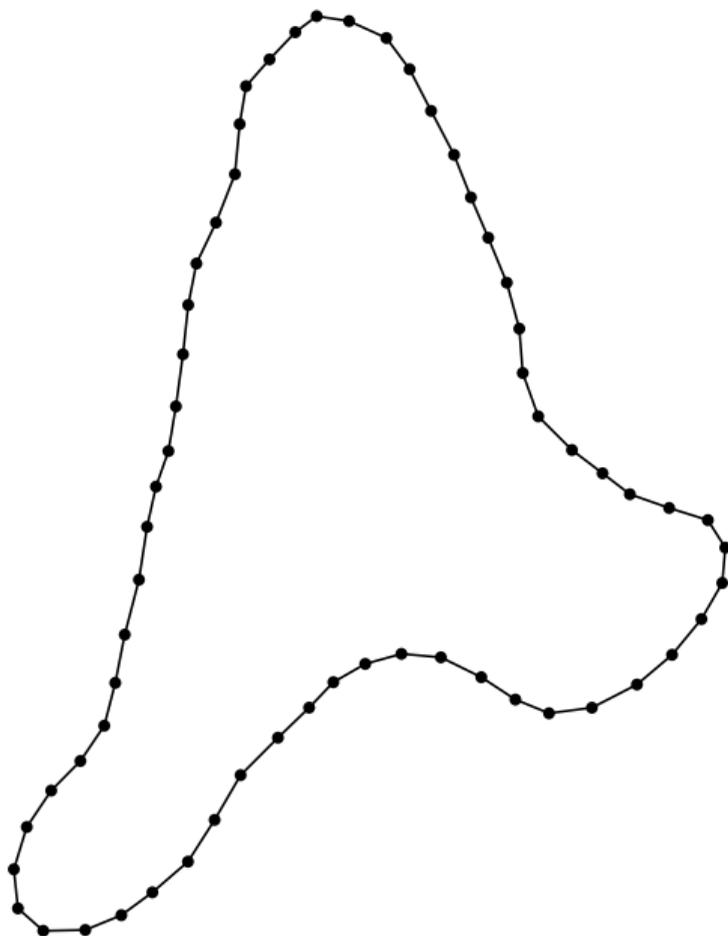
# Curvas de subdivisão

$$\tau = 1/2$$



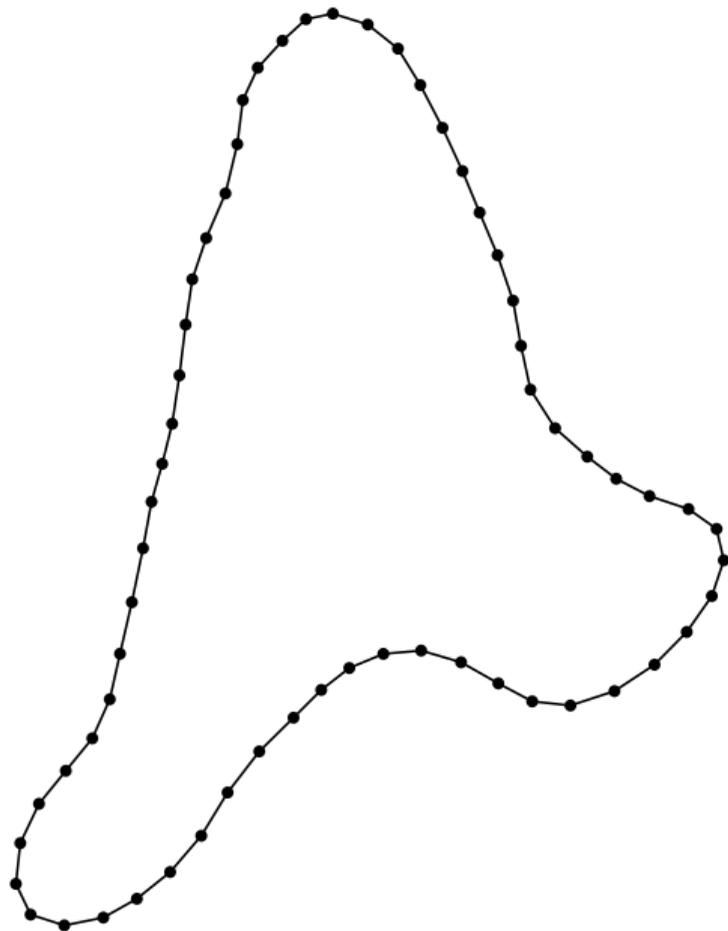
# Curvas de subdivisão

$$\tau = 1/2$$



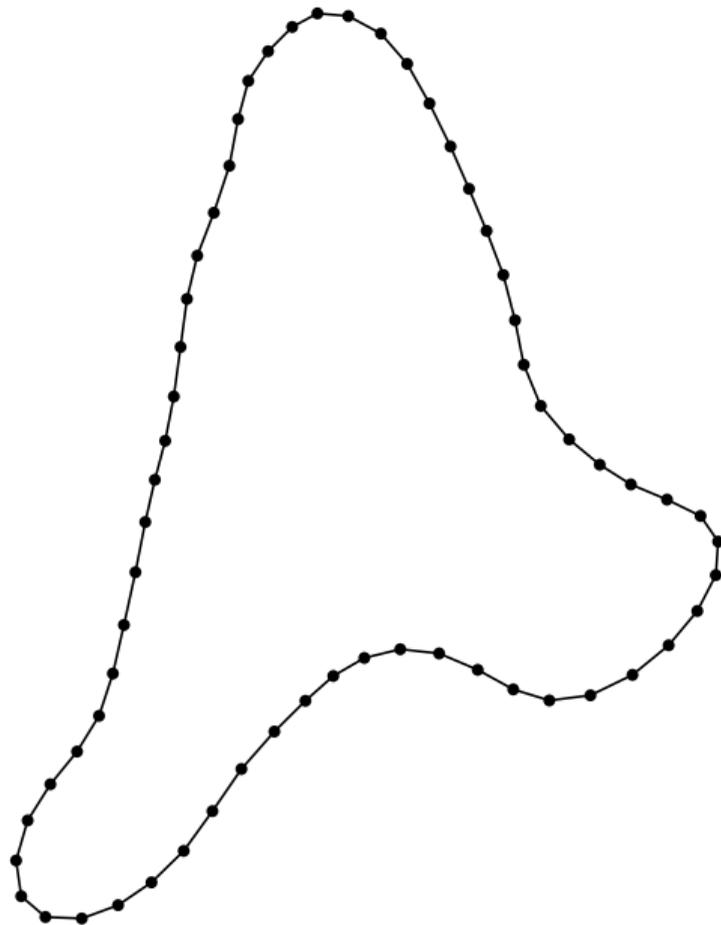
# Curvas de subdivisão

$$\tau = 1/2$$



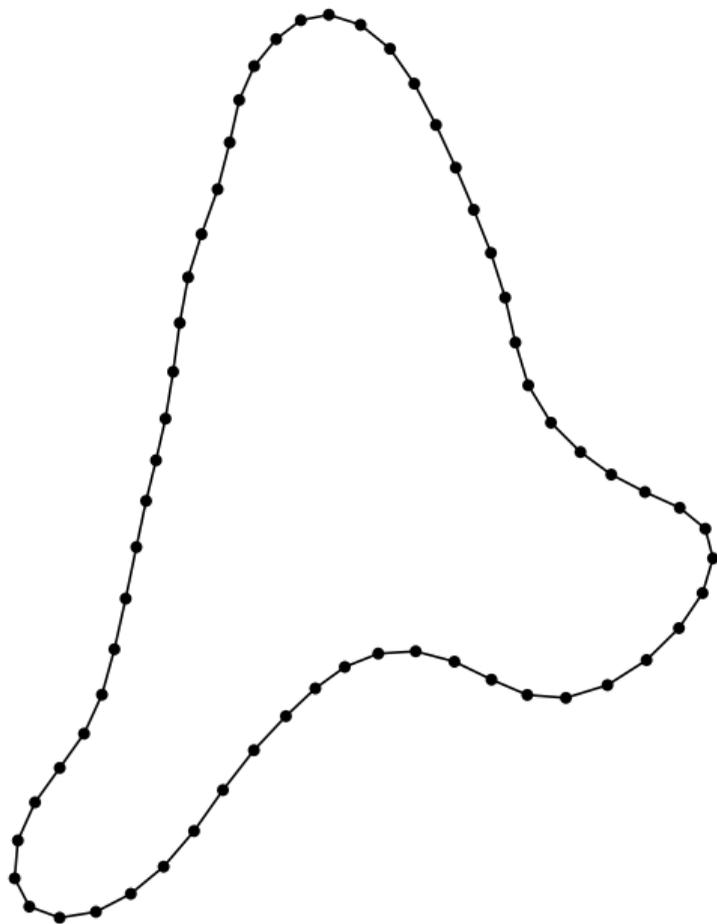
# Curvas de subdivisão

$$\tau = 1/2$$



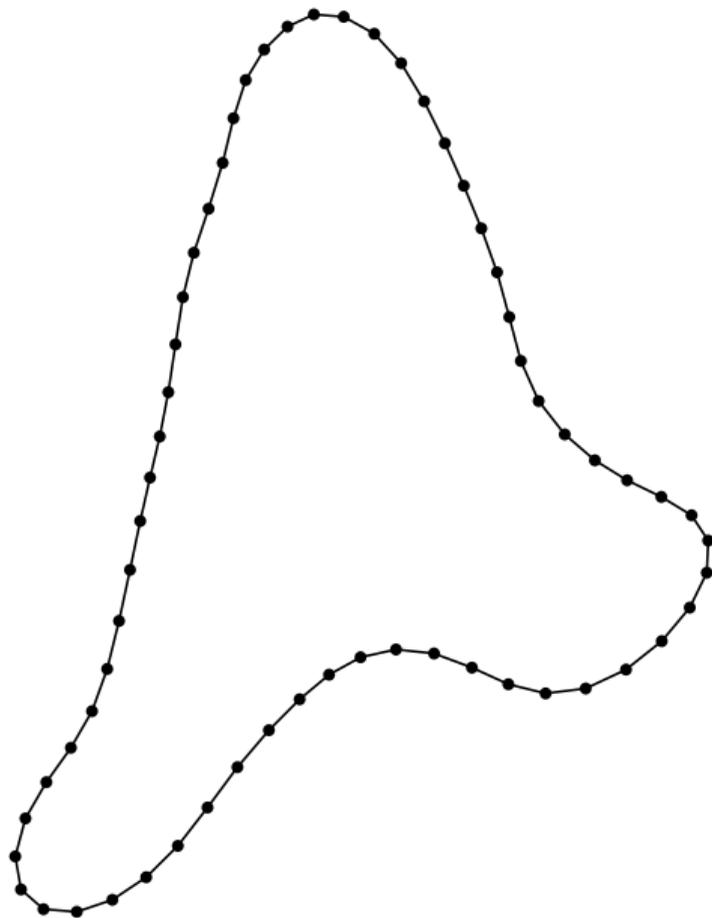
# Curvas de subdivisão

$$\tau = 1/2$$



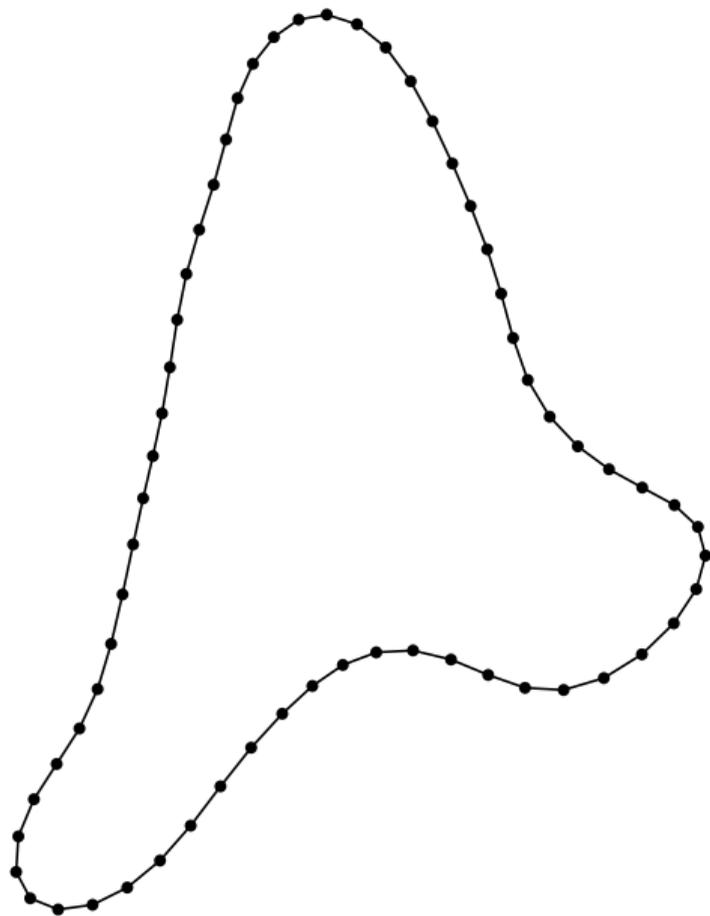
# Curvas de subdivisão

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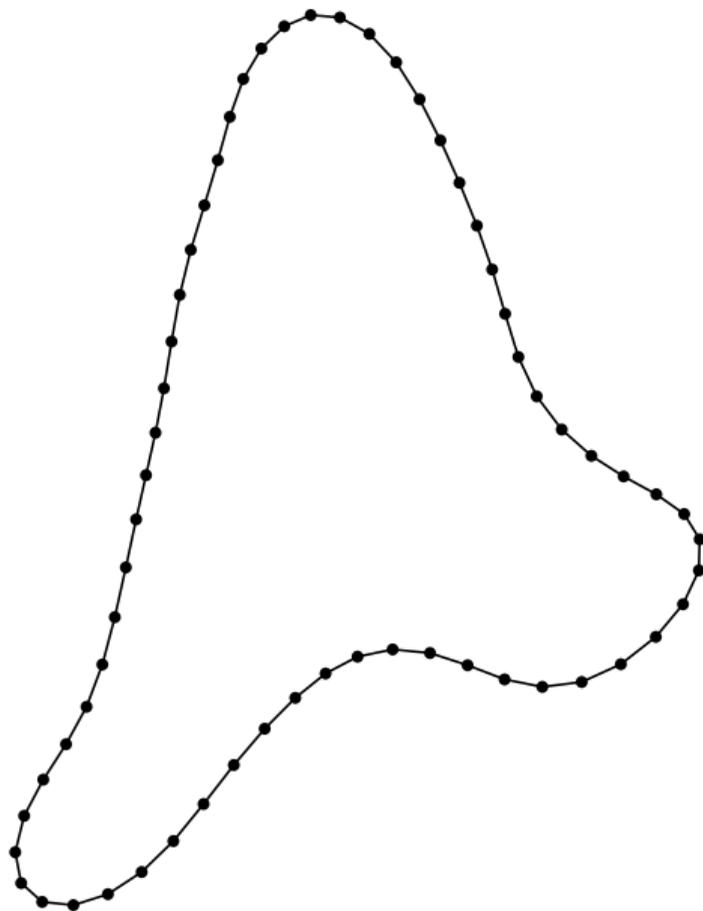
# Curvas de subdivisão

$$\tau = 1/2$$



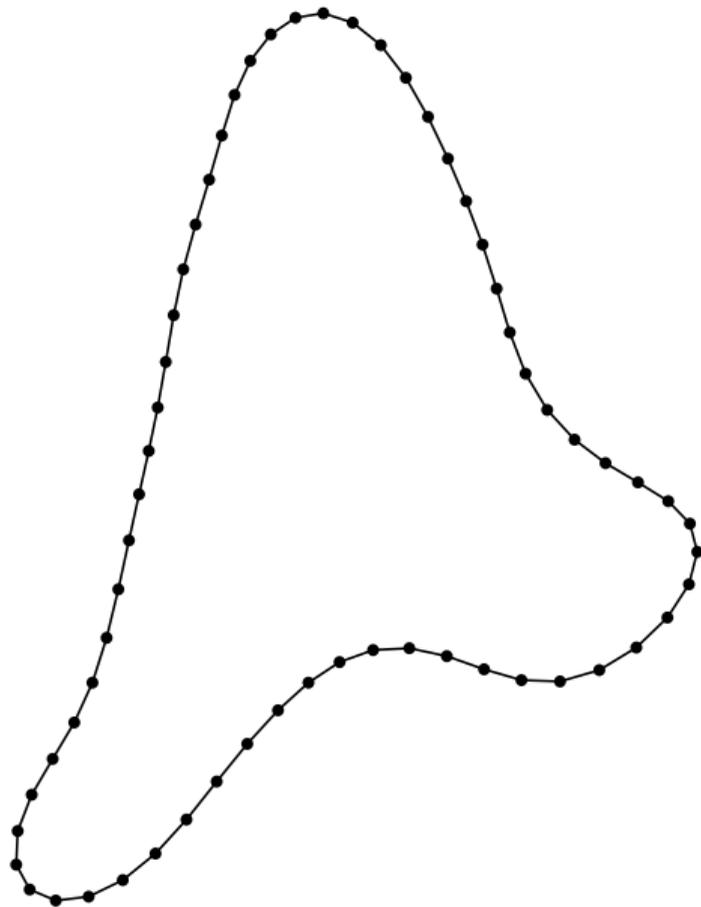
# Curvas de subdivisão

$$\tau = 1/2$$



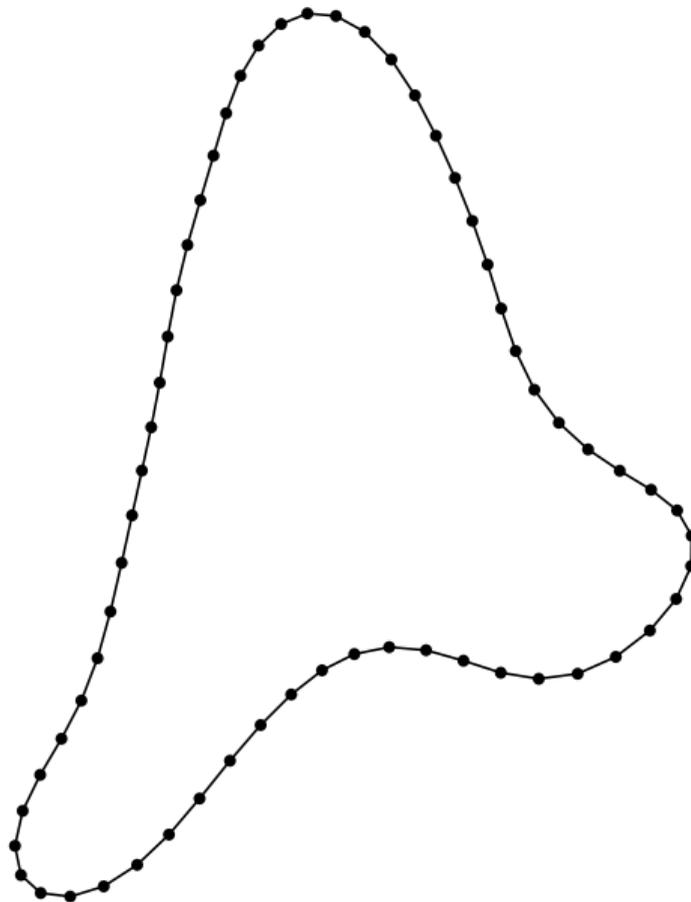
# Curvas de subdivisão

$$\tau = 1/2$$



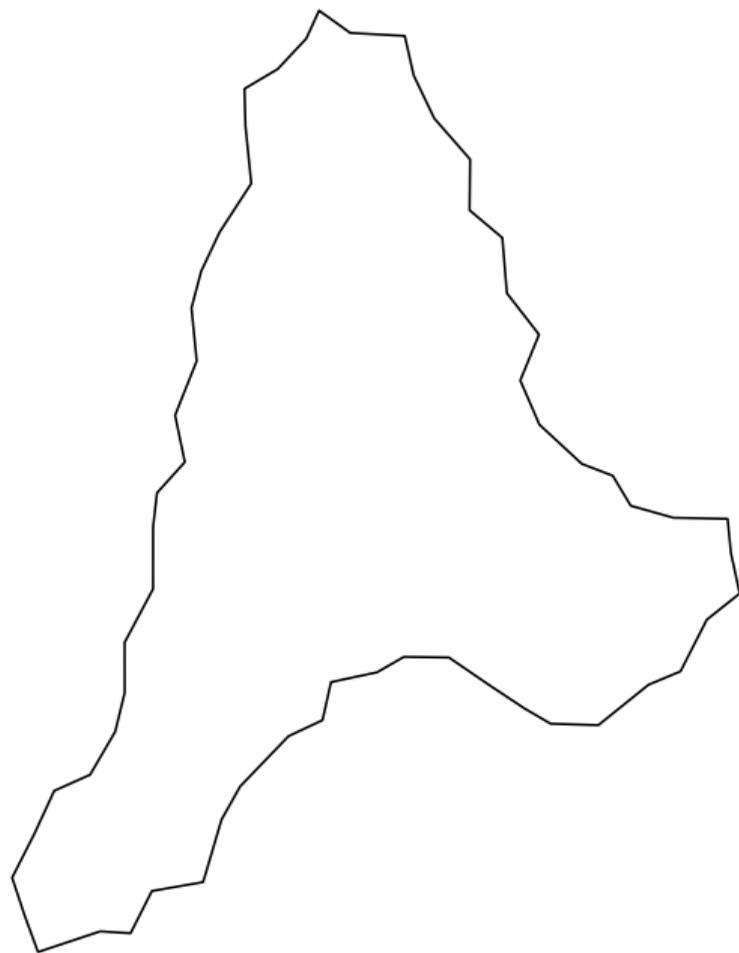
# Curvas de subdivisão

●  $\tau = 1/2$



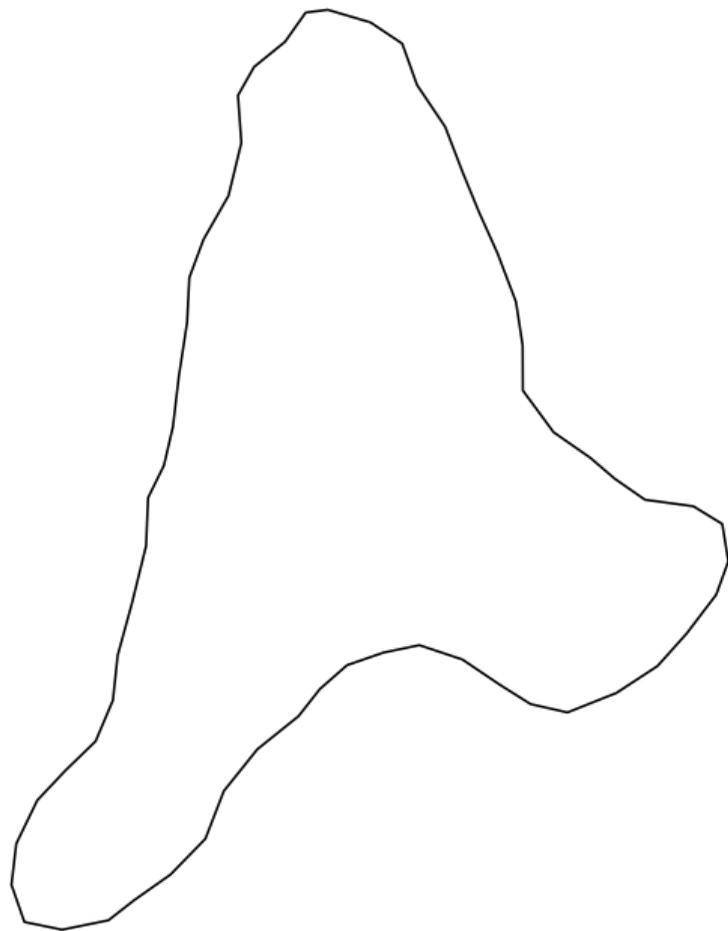
Curvas de subdivisão

$\tau = 1/2$



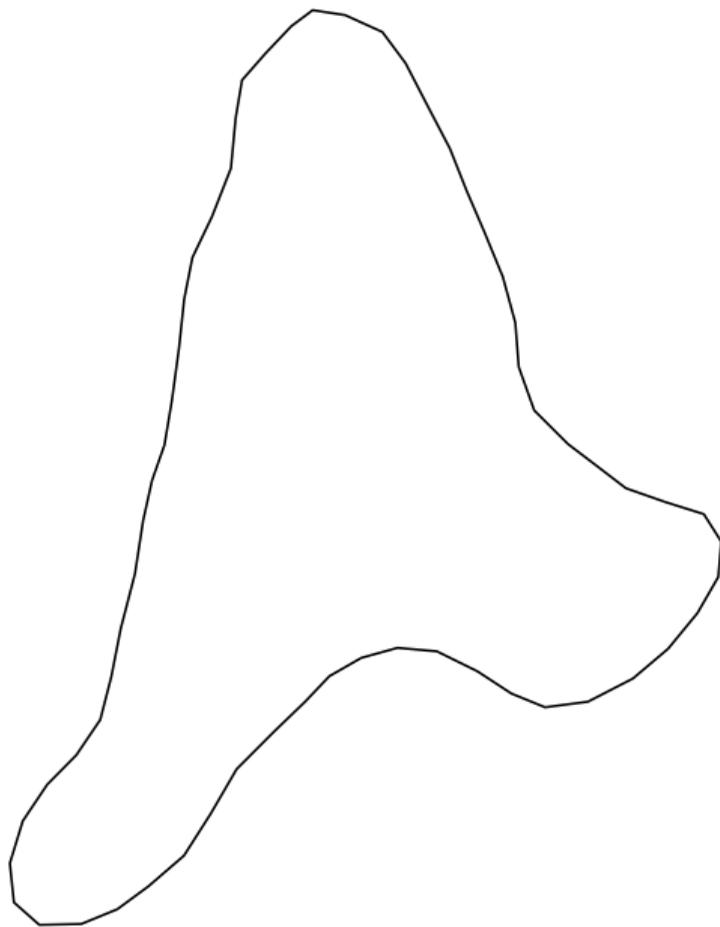
Curvas de subdivisão

$\tau = 1/2$



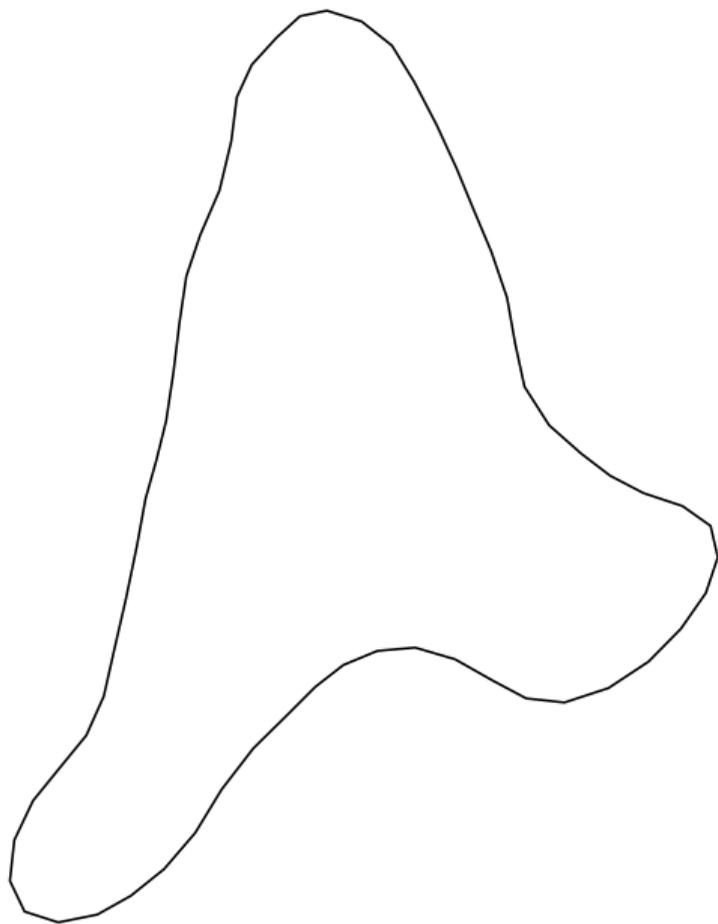
Curvas de subdivisão

$\tau = 1/2$



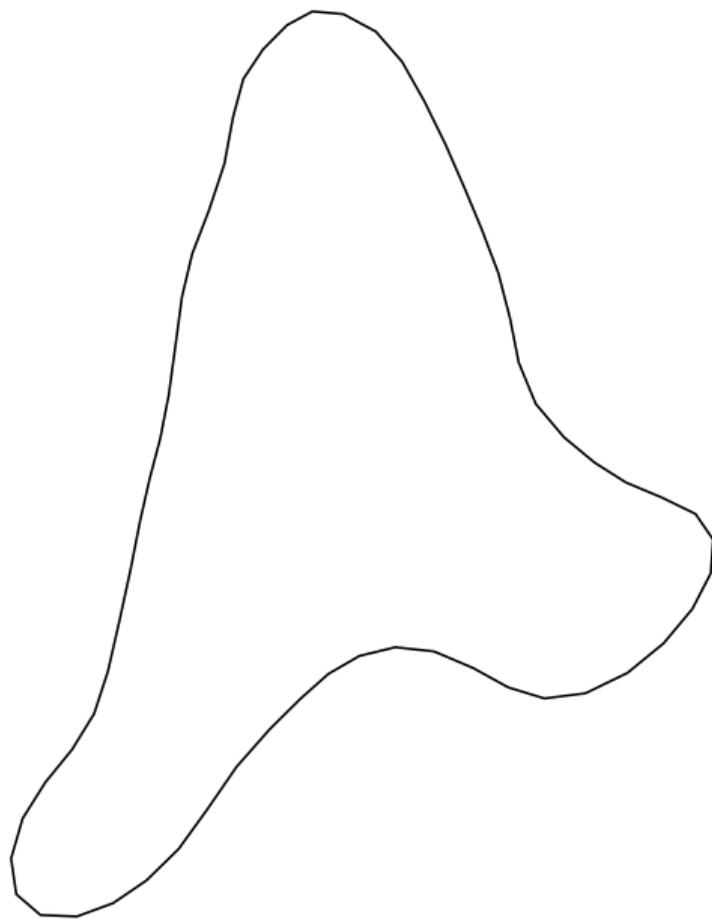
Curvas de subdivisão

$\tau = 1/2$



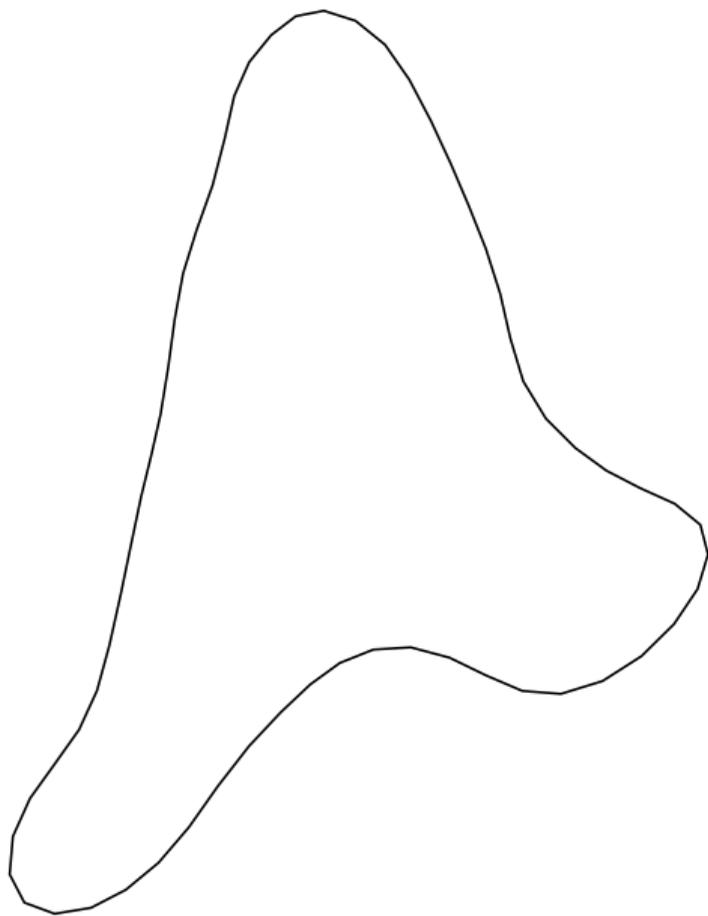
Curvas de subdivisão

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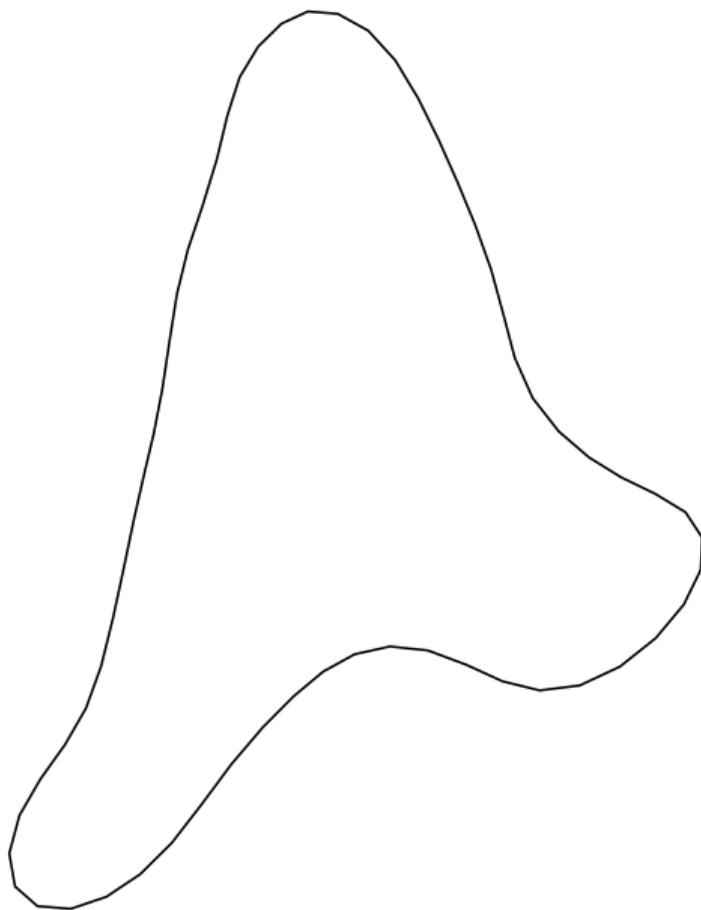
Curvas de subdivisão

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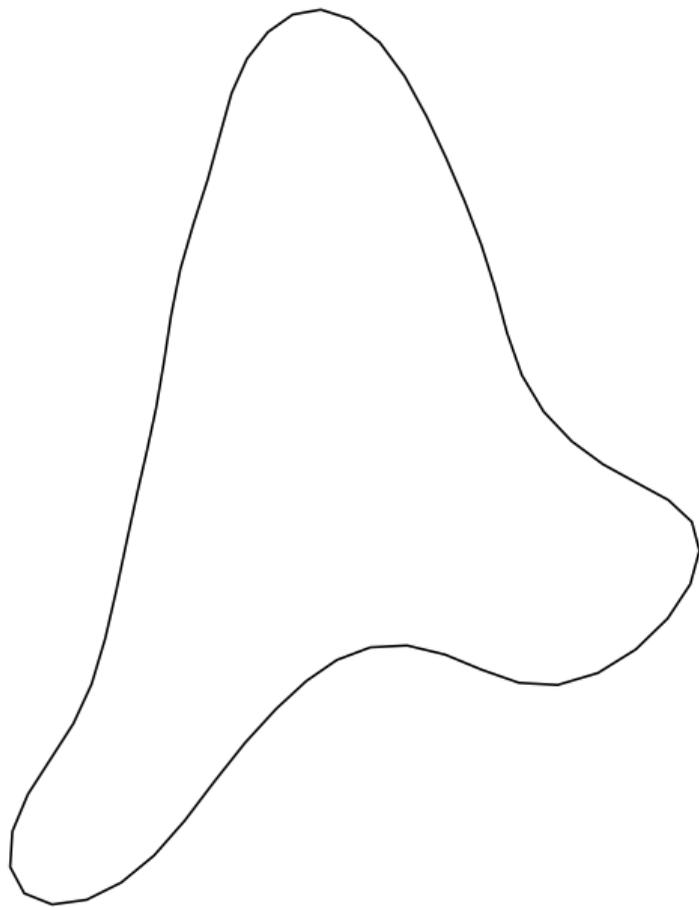
Curvas de subdivisão

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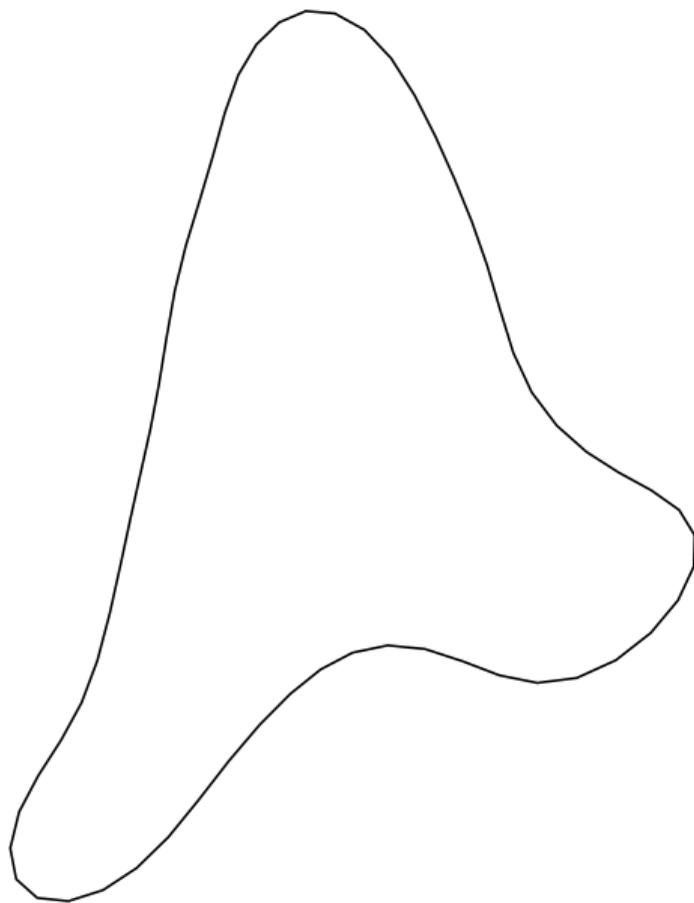
Curvas de subdivisão

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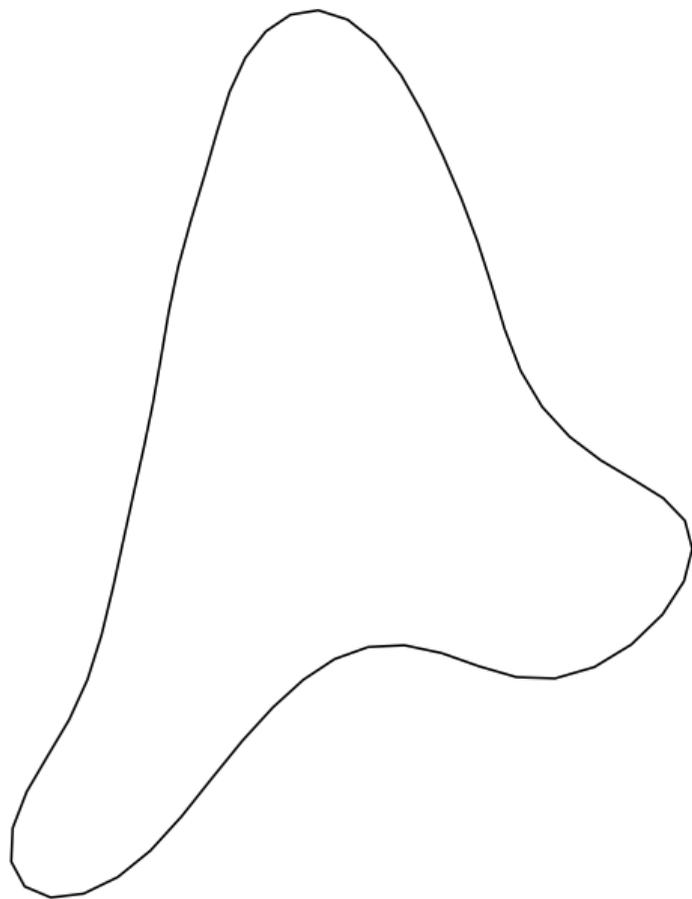
Curvas de subdivisão

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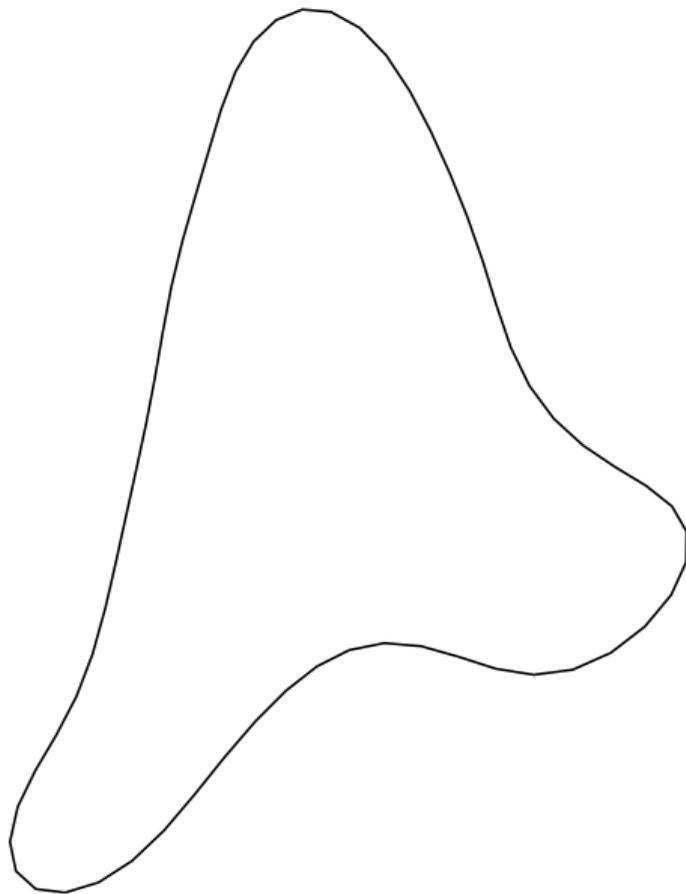
Curvas de subdivisão

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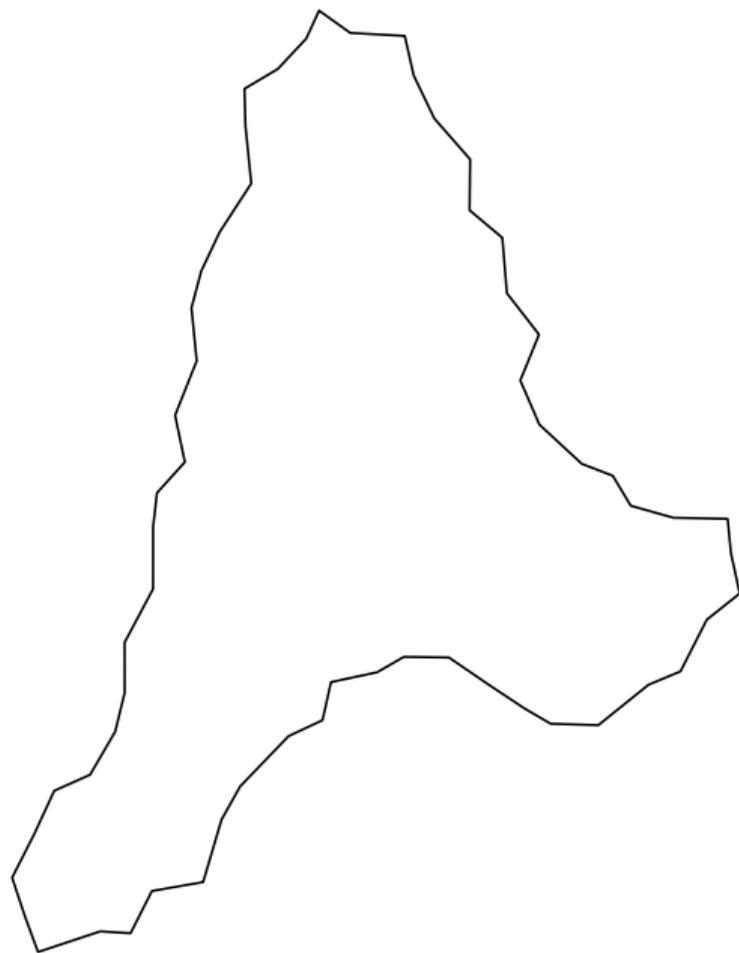
## Curvas de subdivisão

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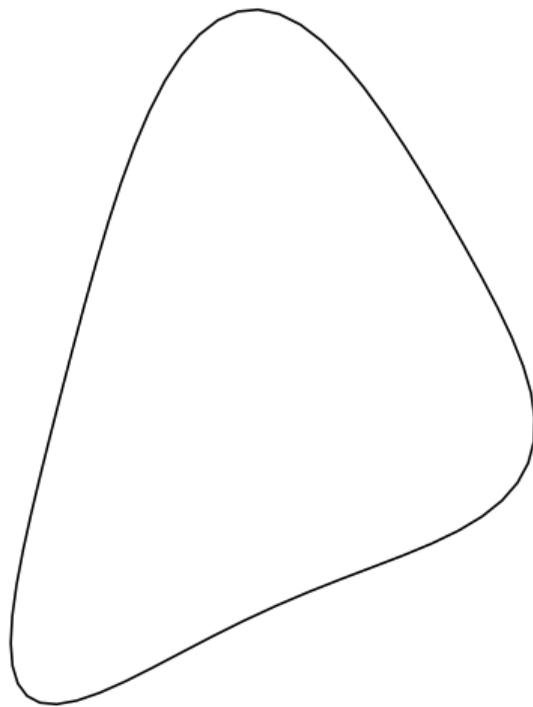
Curvas de subdivisão

$\tau = 1/2$



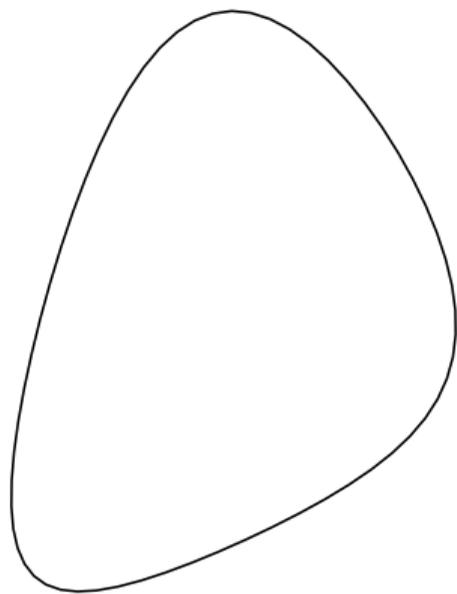
## Curvas de subdivisão

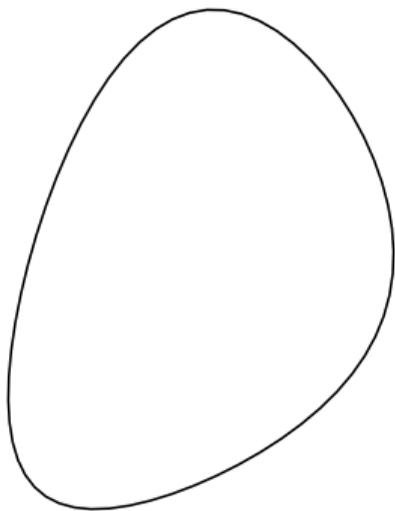
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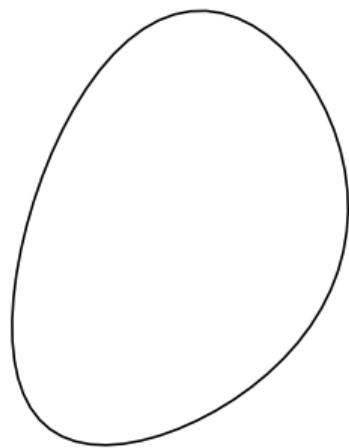


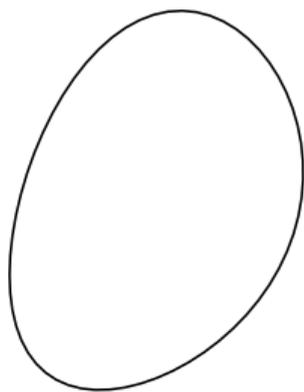
## Curvas de subdivisão

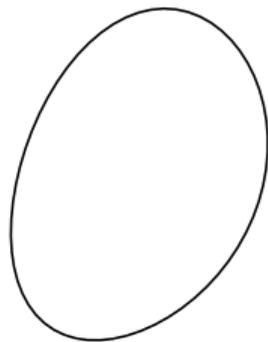
$$\tau = 1/2$$

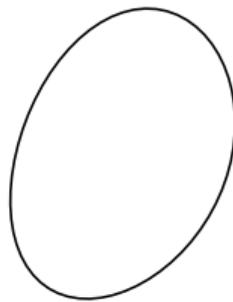


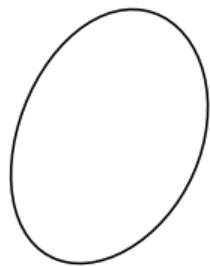


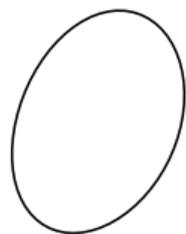






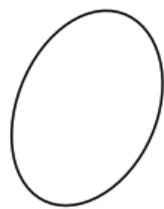






## Curvas de subdivisão

- $\tau = 1/2$

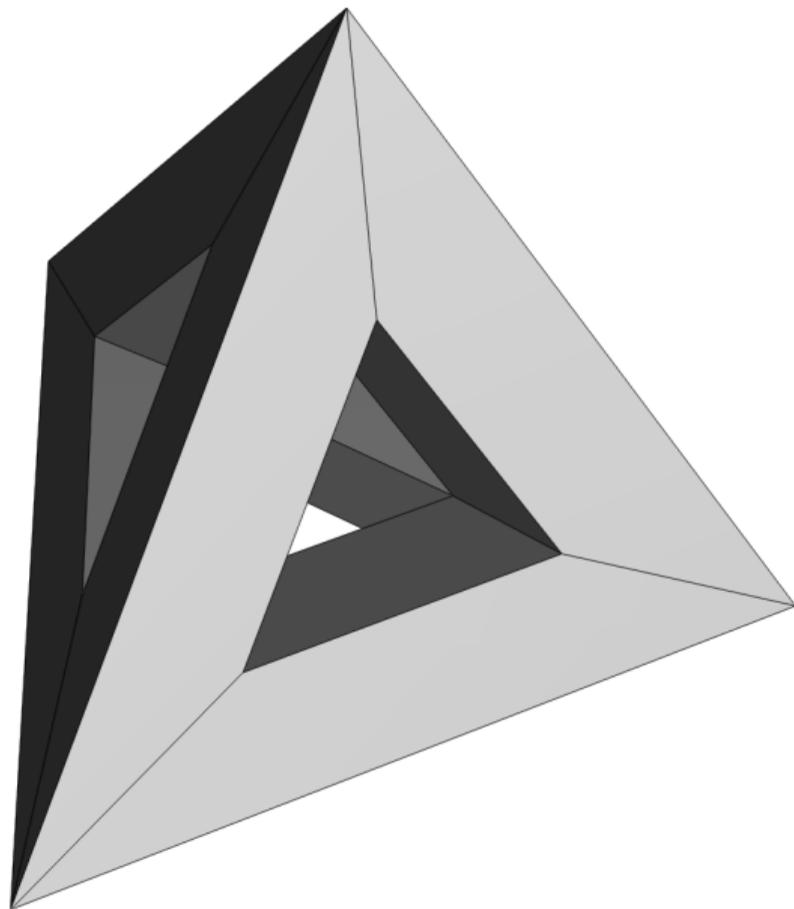


## Superfícies de subdivisão

- Catmull–Clark (1978) quadriláteros
- Doo–Sabin (1978) generaliza Chaikin
- Loop (1987) triângulos
- Stam (1998) avaliação exata
- Kobbelt (2000)  $\sqrt{3}$  triângulos
- Velho–Zorin, Velho–Gomes (2001) 4-8

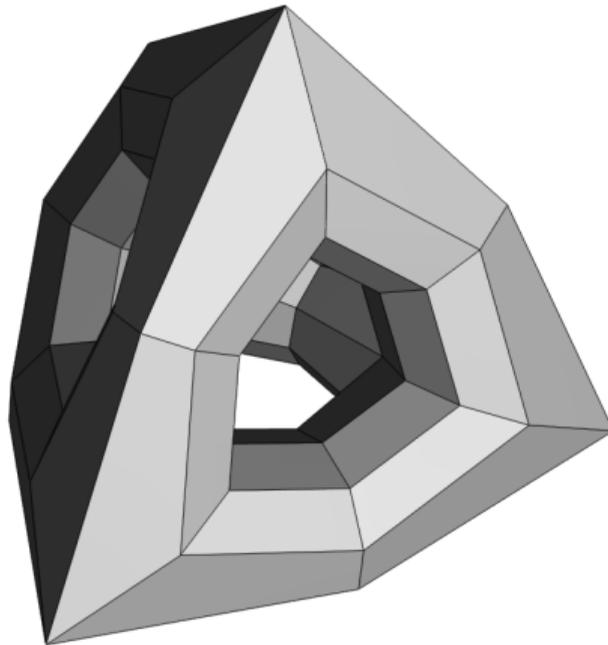
# Superfícies de subdivisão

Catmull-Clark (1978)



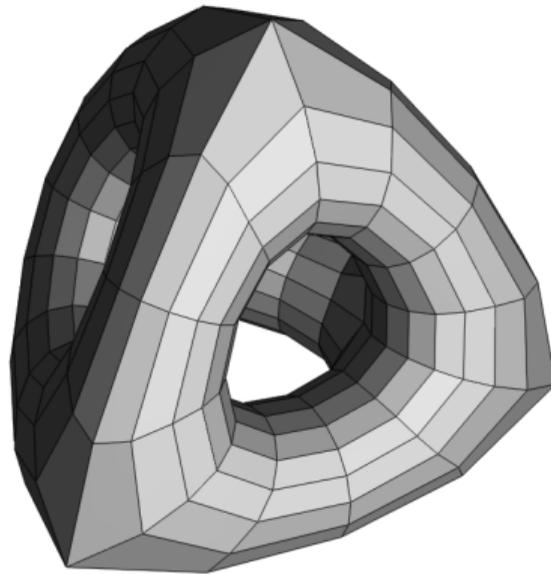
# Superfícies de subdivisão

Catmull-Clark (1978)



# Superfícies de subdivisão

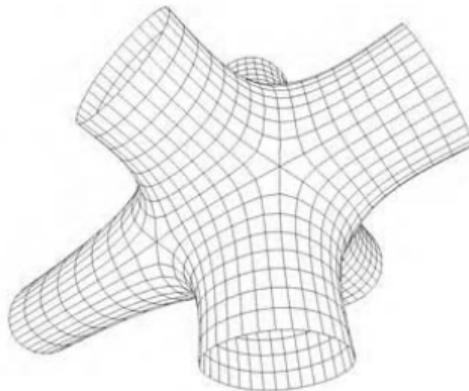
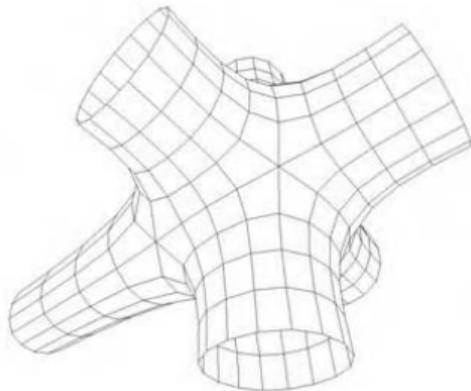
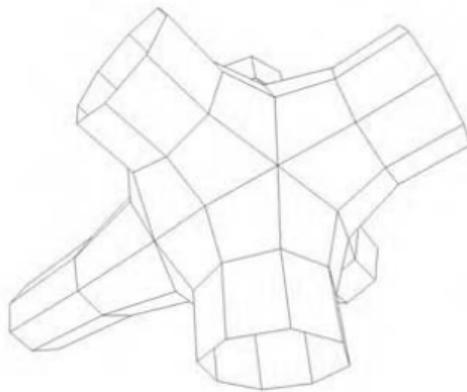
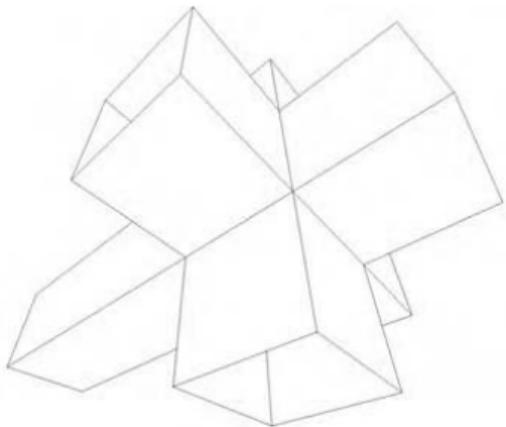
Catmull-Clark (1978)

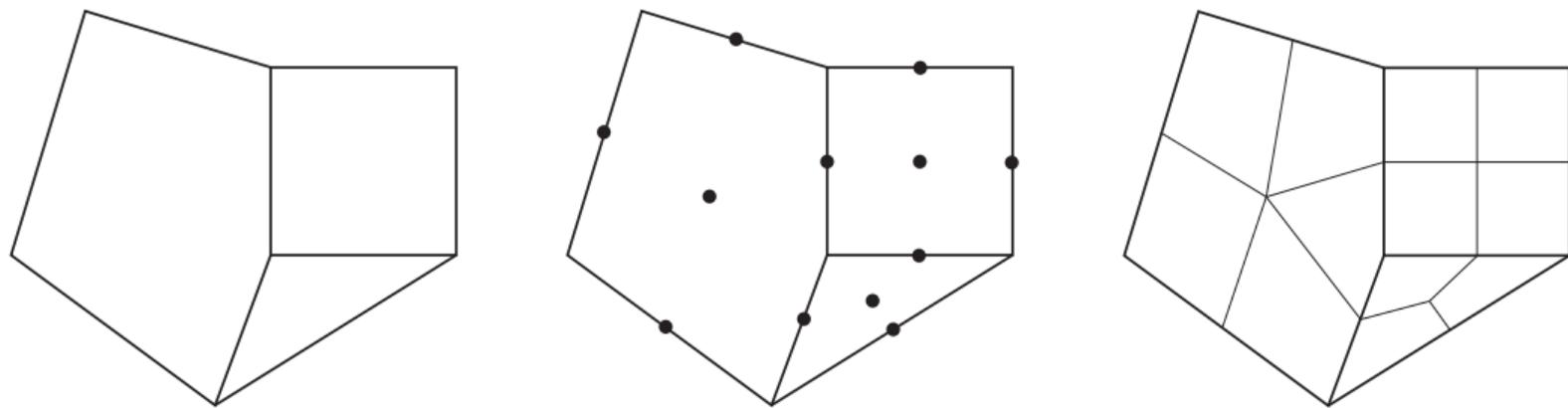




# Superfícies de subdivisão

Catmull-Clark (1978)

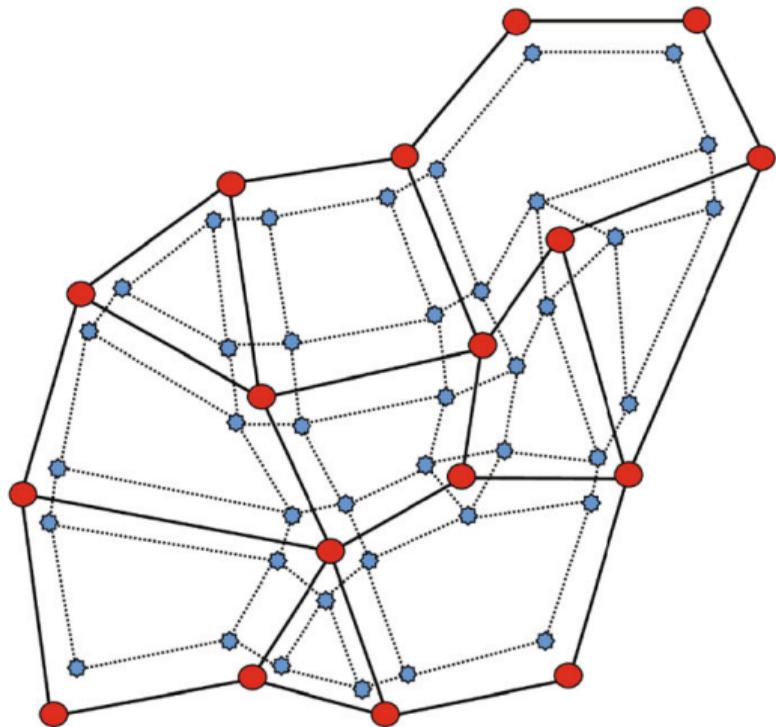
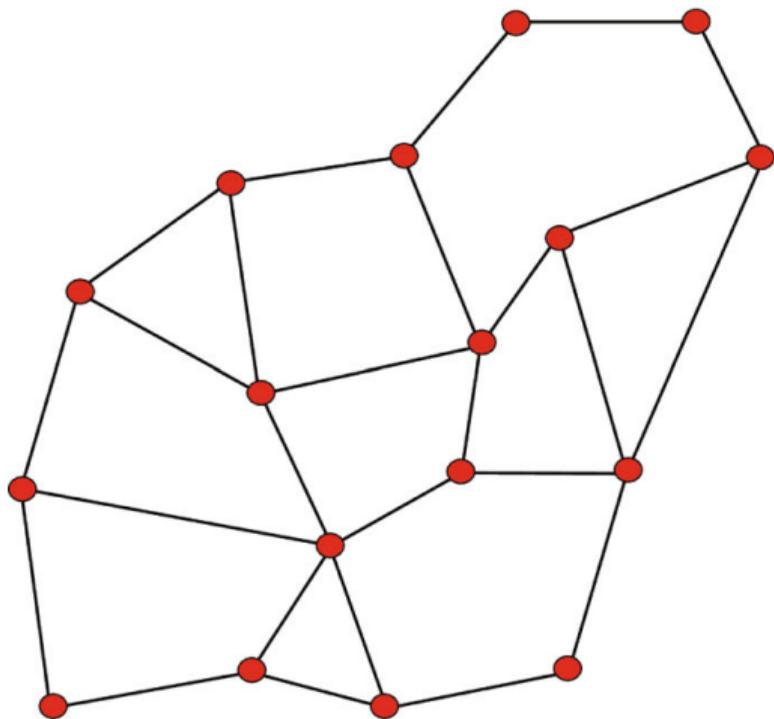




**Figure 13.47.** The basic idea of Catmull-Clark subdivision. Each polygon generates a new point, and each edge generates a new point. These are then connected as shown to the right. Weighting of the original points is not shown here.

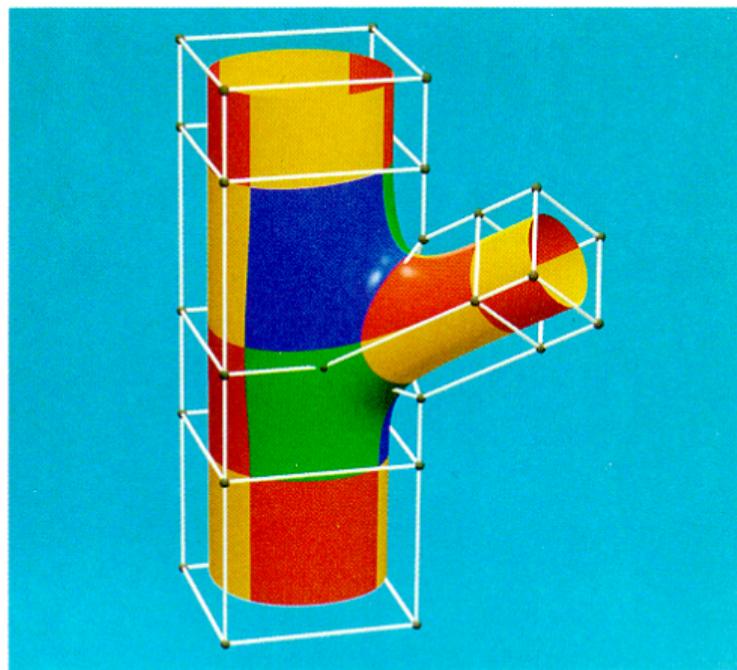
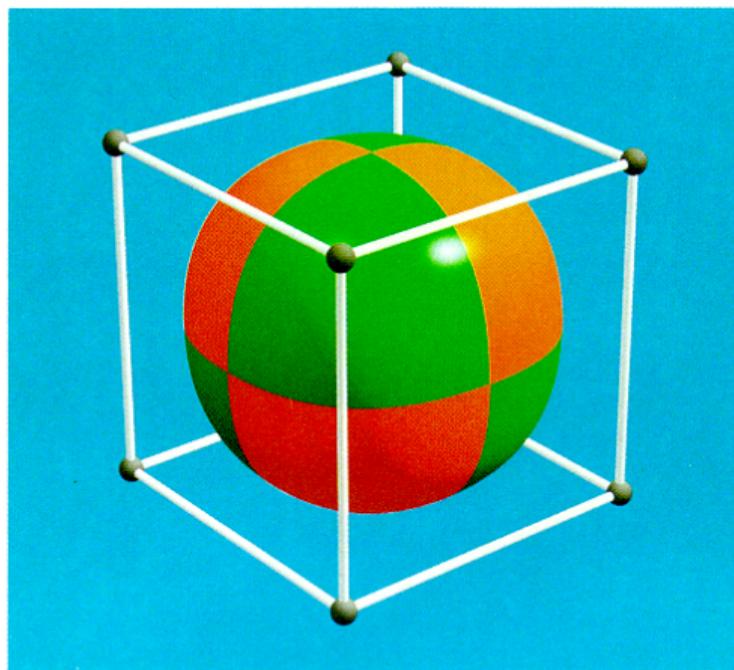
# Superfícies de subdivisão

Doo-Sabin (1978)



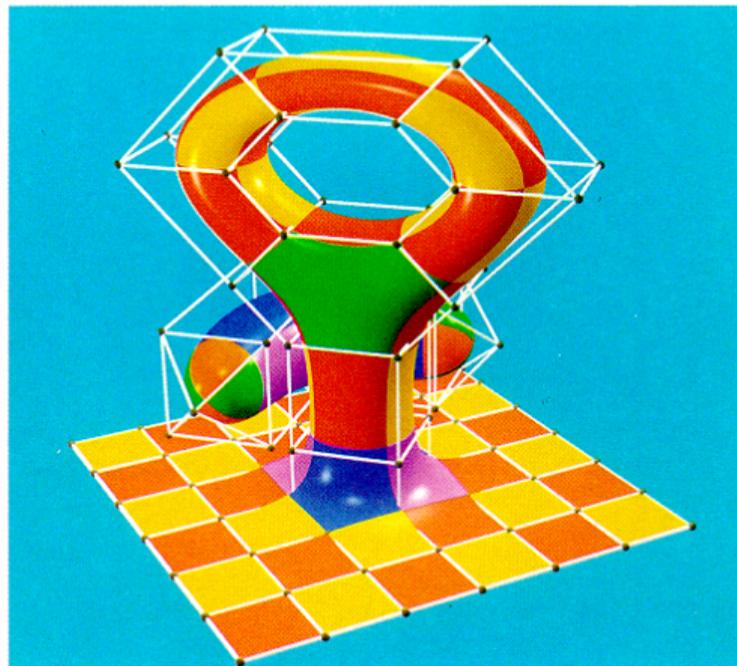
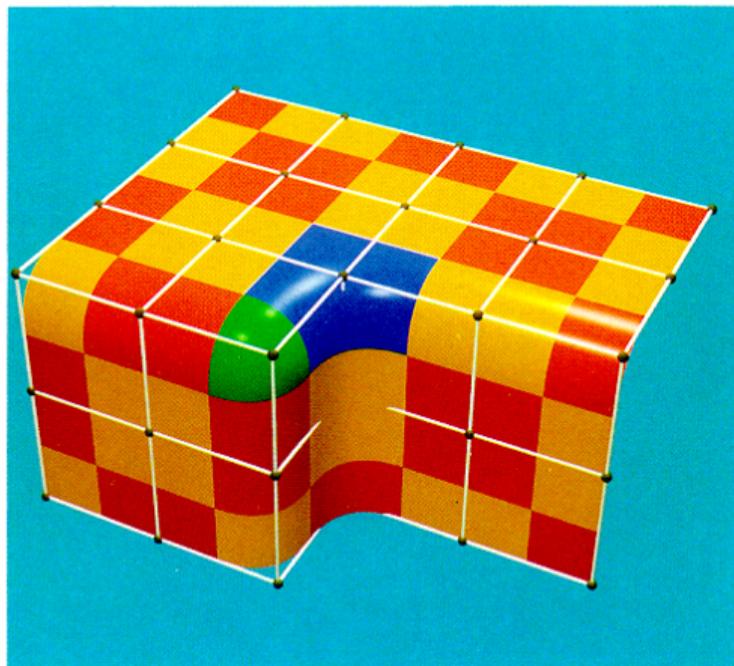
## Superfícies de subdivisão

Loop-DeRose (1990)



## Superfícies de subdivisão

Loop-DeRose (1990)



# Temas de Computação Visual

## Modelagem Geométrica

Luiz Henrique de Figueiredo

