Omnidirectional Images and Moebius Transformations

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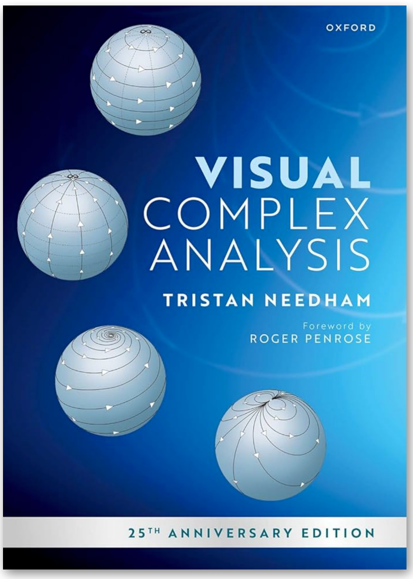
IMPA

Outline

- Mathematical Fundamentals
  - Complex Projective Geometry
  - Moebius Tranformations
- Applications
  - Wide Field of View
  - Cinema 360

Moebius Transformations

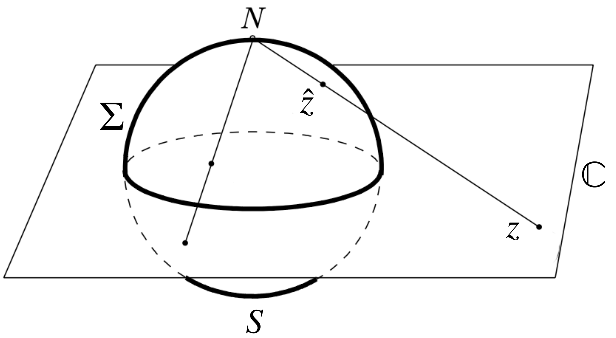
The Book



- Visual Complex Analysis
  - Tristan Needham

Stereographic Projection

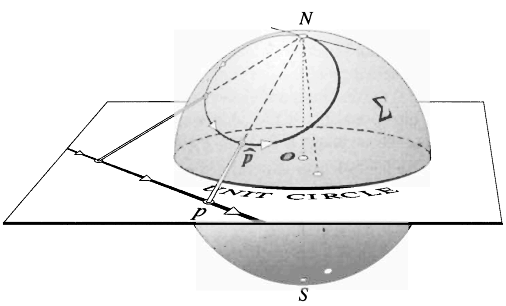
- Riemann Sphere Σ and Complex Plane $\mathbb{C}$



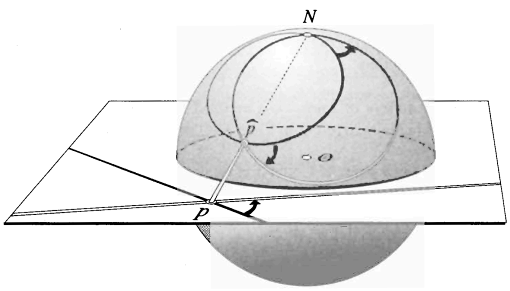
$$\hat{z} = (\theta, \phi) \mapsto z = \cot(\phi/2)e^{i\theta}$$

Properties of Stereographic Projection

- Preserves circles and angles
  - Conformal



The stereographic image of a line in the plane is a circle on Σ passing through $N = \infty$.

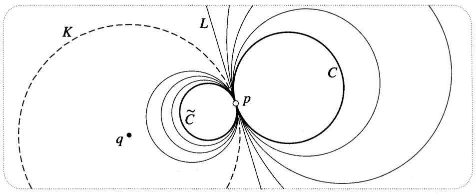


The magnitude of the angle of intersection between the circles is the same at their two intersection points, p and N

Mapping of Stereographic Projection

- The Unity Circle C and the Riemann Sphere Σ

(i) the interior of the unit circle C is mapped to the southern hemisphere of Σ and in particular 0 is mapped to the south pole, S;
(ii) each point on the unit circle C is mapped to itself, now viewed as lying on the equator of Σ;
(iii) the exterior of the unit circle C is mapped to the northern hemisphere of Σ except that $N = \infty$ is not the image of any finite point in the plane.



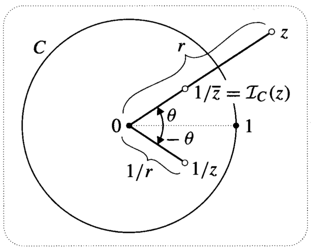
(iv) *Inversion of $\mathbb{C}$ in the unit circle induces a reflection of the Riemann sphere in its equatorial plane, $\mathbb{C}$.*

Inversion

The image of $z = r\,e^{i\theta}$ under complex inversion is $1/(r\,e^{i\theta}) = (1/r)\,e^{-i\theta}$

- Two-Stage Decomposition
  i) Send $z = re^{i\theta}$ to $(1/r)e^{i\theta} = (1/\bar{z})$

  ii) Apply complex conjugation
  i.e., reflection on the real line
  $(1/\bar{z}) \mapsto \overline{(1/\bar{z})} = (1/z)$



- Inversion on the Unity Circle C
  - Interchanges the *interior* and *exterior* of C
  - Each point on C remain fixed
    i.e., C is mapped to itself

$$z \mapsto \mathcal{I}_C(z) = (1/\bar{z})$$

(Geometric) Inversion on C

Möbius Transformations

- Complex Map

$$M : \mathbb{C} \mapsto \mathbb{C}$$

- Definition:

$$M(z) = \frac{az + b}{cz + d} \qquad z \in \mathbb{C}$$

with

$$(ad - bc) \neq 0$$

Extended Complex Plane

- Point at Infinity ∞

$$\frac{1}{\infty} = 0 \qquad \frac{1}{0} = \infty$$

- Decomposition of $\hat{\mathbb{C}}$

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

Complex Projective Space

- Isomorphism

$$z \mapsto w = M(z) \quad \text{in} \quad \hat{\mathbb{C}}$$

induces

$$\hat{z} \mapsto \hat{w} \quad \text{in} \quad \Sigma$$

- Geometry and Algebra

Riemann Sphere $\Longleftrightarrow$ Complex Plane

Anatomy of *M*

- Decomposition into Sequence

$$m_4 \circ m_3 \circ m_2 \circ m_1(z)$$

$m_1(z) = z + \frac{d}{c}$	*translation*
$m_2(z) = \frac{1}{z}$	*inversion*
$m_3(z) = \frac{(bc-ad)}{c^2} z$	*scaling and rotation*
$m_4(z) = z + \frac{a}{c}$	*translation*

The Formula (two cases)

- General case $c \neq 0$:

$$M(z) = \frac{az+b}{cz+d} = T_2\big(S\big(I(T_1(z))\big)\big) = \underbrace{\frac{a}{c}}_{T_2} + \underbrace{\left(-\frac{ad-bc}{c^2}\right)}_{S} \cdot \underbrace{\frac{1}{z+\underbrace{\frac{d}{c}}_{T_1}}}_{I}$$

- $m_1 \equiv T_1$
- $m_2 \equiv I$
- $m_3 \equiv S$
- $m_4 \equiv T_2$

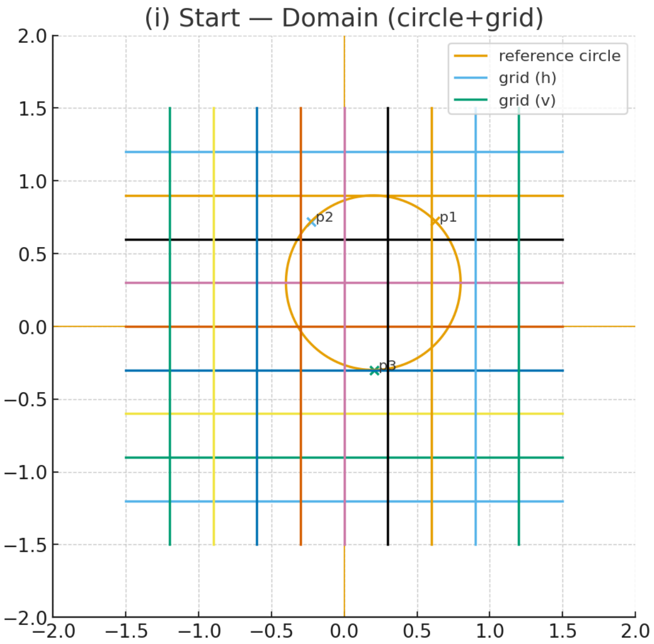
- Special case $c = 0$:

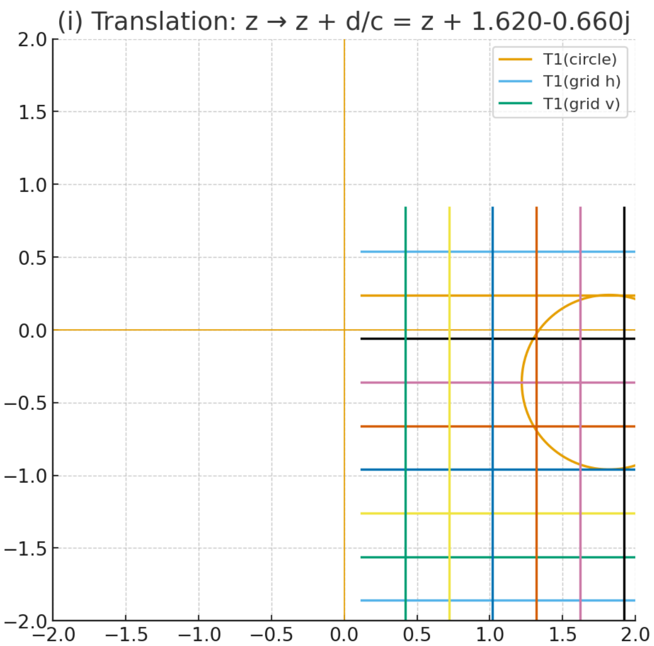
  $M(z) = (a/d)z + (b/d)$

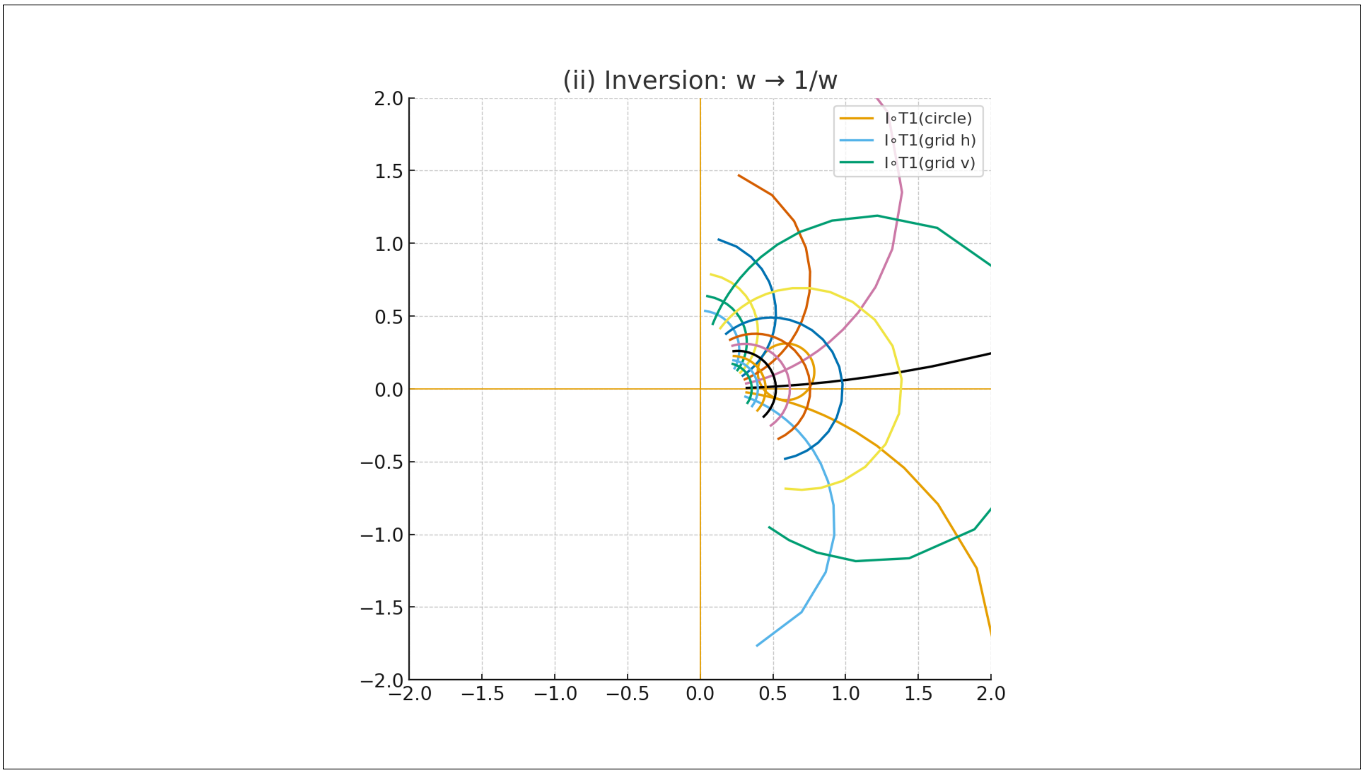
- M is affine (*steps reduce to scaling+rotation then translation*).

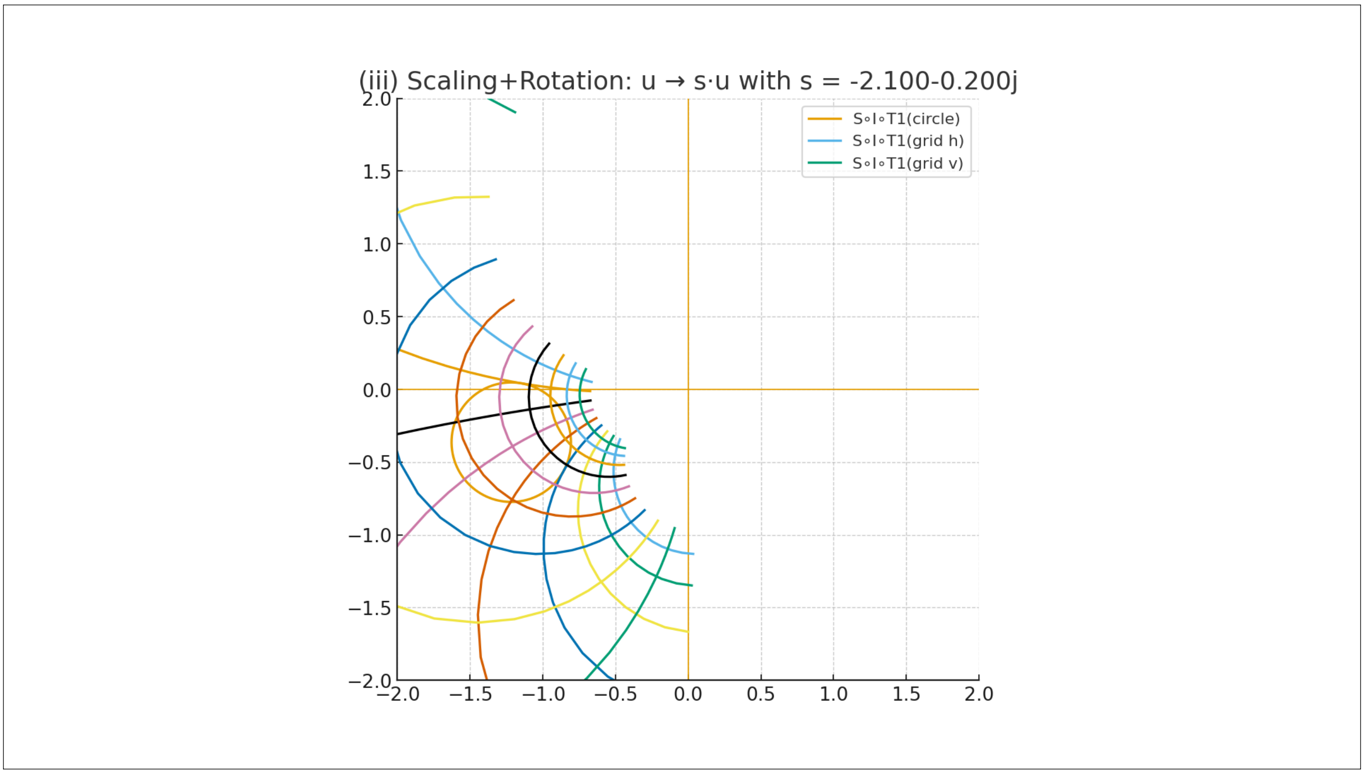
Visualizing the Steps

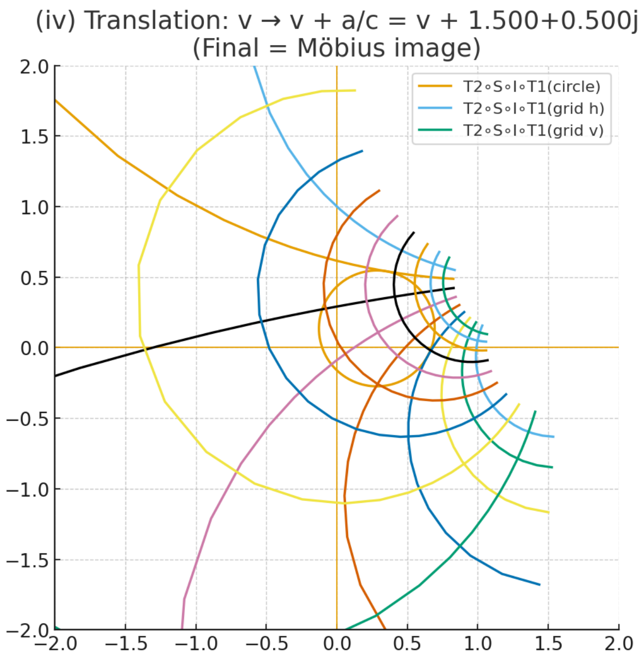
- **the geometry on the complex plane** (circle + small grid)
  - (i) Start — domain (reference circle + grid)
  - (i) After T_1: translation by d/c
  - (ii) After I: inversion $w \mapsto 1/w$
  - (iii) After S: complex scaling+rotation by $s = -(ad-bc)/c^2$
  - (iv) After T_2: final translation by a/c (the Möbius image)


(i) Start — Domain (circle+grid)
reference circle
grid (h)
grid (v)
p1
p2
p3
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


(i) Translation: z → z + d/c = z + 1.620-0.660j
T1(circle)
T1(grid h)
T1(grid v)
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


(ii) Inversion: w → 1/w
I∘T1(circle)
I∘T1(grid h)
I∘T1(grid v)
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


(iii) Scaling+Rotation: u → s·u with s = -2.100-0.200j
S∘I∘T1(circle)
S∘I∘T1(grid h)
S∘I∘T1(grid v)
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0



Properties of *M*

- Projective Linear Group (Lie Group)
  $PGL(2, \mathbb{C})$

- Preservation of:
  - Circles (lines to circles)
  - Angles (conformal)
  - Symmetry (w.r.t. circles)

Defining M

- Images of 3 points (e.g)

$$(a/b), \quad (b/c), \quad (c/d)$$

- Ratios and Uniqueness

$$\frac{az+b}{cz+d} = M(z) = \frac{kaz+kb}{kcz+kd}$$

- Normalization

$$(ad-bc) = 1$$

Homogeneous Coordinates

- Ratio of 2 complex numbers

$$z = \frac{\delta_1}{\delta_2} = [\delta_1, \delta_2] \neq [0,0]$$

- Two Cases

$$\delta_2 \neq 0$$

$$z = \delta_1/\delta_2$$

$$\delta_2 = 0$$

$$z = \infty$$

Cross Ratio

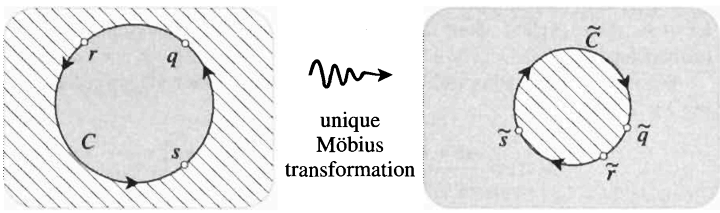
- The *unique*
$$z \mapsto w = M(z)$$
sending
$$q, r, s \mapsto \tilde{q}, \tilde{r}, \tilde{s}$$

$$\frac{(w-\tilde{q})(\tilde{r}-\tilde{s})}{(w-\tilde{s})(\tilde{r}-\tilde{q})} = [w, \tilde{q}, \tilde{r}, \tilde{s}] = [z, q, r, s] = \frac{(z-q)(r-s)}{(z-s)(r-q)} \qquad (\#)$$

- Theorem:
If M maps 4 points $p, q, r, s \mapsto \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$
then, the cross-ratio is invariant.

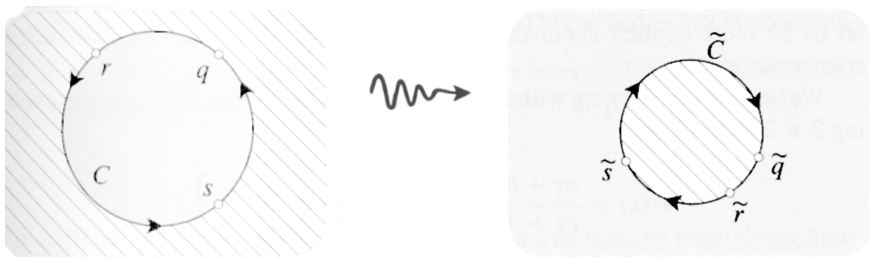
Corollary

Let C be the unique circle through the points q, r, s in the z-plane, oriented so that these points succeed one another in the stated order. Likewise, let $\widetilde{C}$ be the unique oriented circle through $\widetilde{q}$, $\widetilde{r}$, $\widetilde{s}$ in the w-plane. Then the Möbius transformation given by (#) maps C to $\widetilde{C}$, and it maps the region lying to the left of C to the region lying to the left of $\widetilde{C}$.



Orientation Properties

- Maps Oriented Circles to Oriented Circles
  - s.t. Regions are mapped accordingly



Fixed Points

- Solution of

$$z = M(z)$$

- *M* has at most <u>two</u> fixed points
  - except for Id.
- For *M* Normalized

$$\xi_{\pm} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$$

M - Classification

- Fixed Point at Infinity : $c = 0$

  $$M(z) = Az + B$$

- Basic Types
  - Elliptic
  - Hyperbolic
  - *Loxodromic*
  - Parabolic

Elliptic Transform

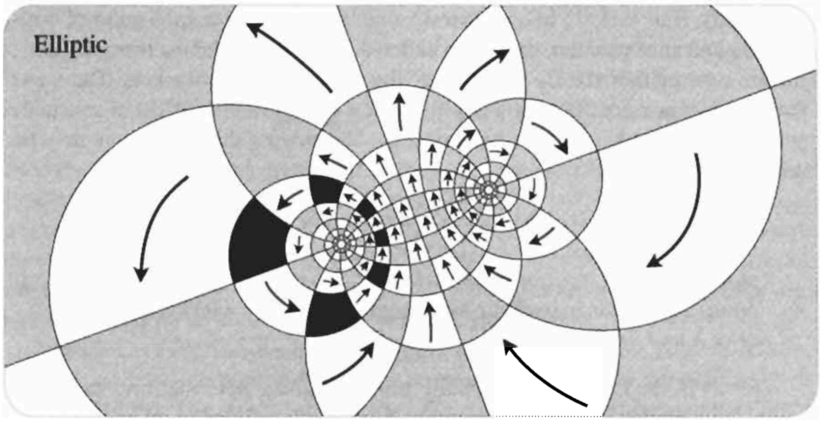
- Rotation

  $$z \mapsto e^{i\alpha} z$$

- *two fixed points*

  $$(0, \infty)$$

Elliptic Transform in $\mathbb{C}$



Hyperbolic Transform

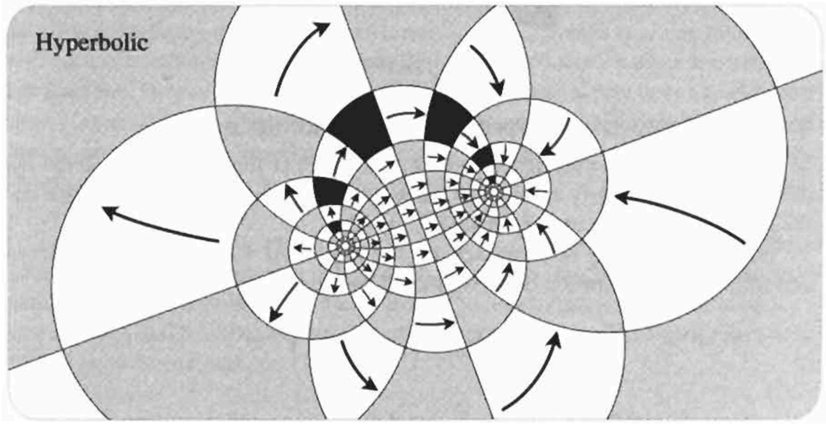
- Scaling

$$z \mapsto \rho z$$

- *two fixed points*

$$(0, \infty)$$

Hyperbolic Transform in $\mathbb{C}$



Loxodromic Transform

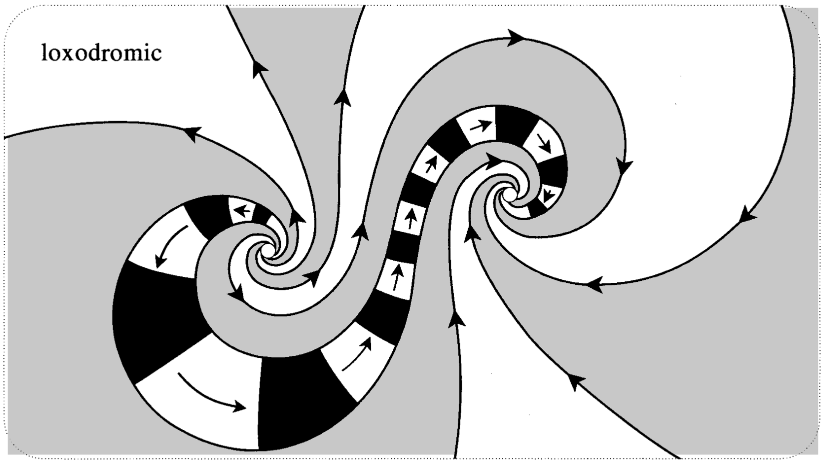
- Rotation and Scaling

$$z \mapsto \rho e^{i\alpha} z$$

- *two fixed points*

(combination of elliptic and hyperbolic)

Loxodromic Transform in $\mathbb{C}$



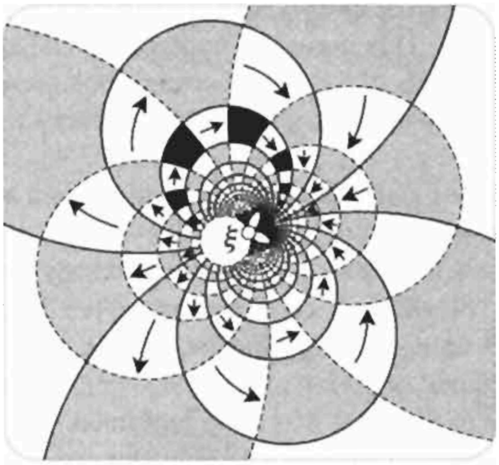
Parabolic Transform

- Translation

$$z \mapsto z + b$$

- *<u>one</u> fixed point at* ∞

Parabolic Transform in $\mathbb{C}$



Applications

360 Cinema

Authoring Issues

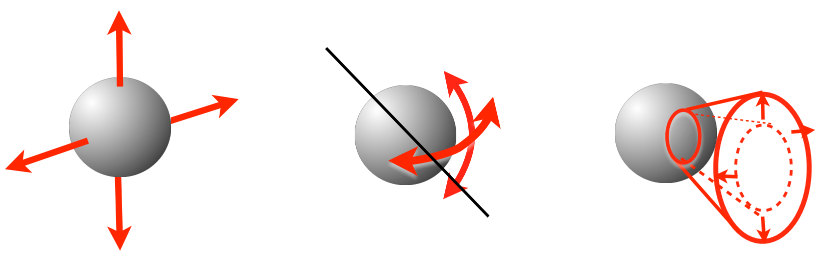
- Passive
  - 360 Movies
- Interactive
  - Google Street View
- Immersive
  - AR Immersive Cinema

Emerging Technologies

360 Camera

- Camera Moves

Track | Pan / Tilt | Zoom



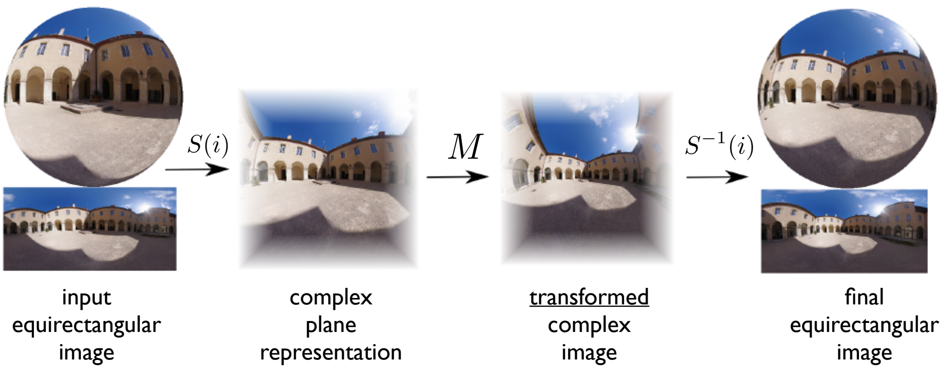
yes | *maybe* | ?

Math of Camera Moves

- Omnidirectional Images + Moebius Transformations
  - Pan / Tilt ⇔ Elliptic Transform
  - Zoom ⇔ Hyperbolic Transform
  - Perspective ⇔ Parabolic Transform ?

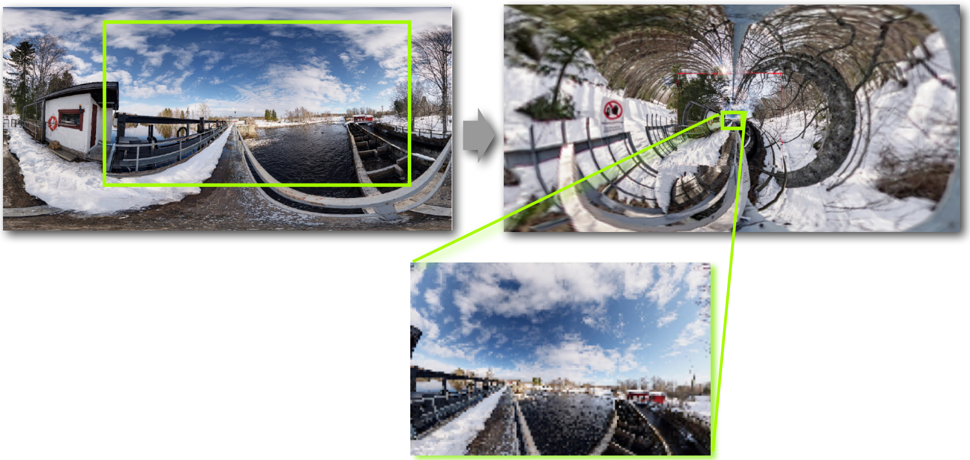
Transformation Pipeline

- Möbius Mapping



Hyperbolic Transform (Example)

- Extreme Zoom



Comparison

- Alternative Projections

input panorama

Control of Perspective

equirectangular

projective

mercator

möbius

Research @ VISGRAF Lab

Moebius Transformations for Manipulation and Visualization of Spherical Panoramas

- Collaboration with
  - Leonardo Koller Sacht
  - Luis Penaranda

Video 1

Different scales applied to an equi-rectangular image

Follow-Up Work

- Preserving Lines
- Perspective Control

Improving Projections of Panoramic Images
with Hyperbolic Möbius Transformations

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IMPA

Tools

Mantra-VR

- Premiere Plug-In

Allows to zoom original sphere to arbitrary point selected on the sphere.
Zoom out point can be also selected arbitrarily.

VFX

- Animated Distortions